The Annals of Probability
2010, Vol. 38, No. 1, 439–442
DOI: 10.1214/09-AOP463

CORRECTION

MEASURE CONCENTRATION FOR EUCLIDEAN DISTANCE IN THE CASE OF DEPENDENT RANDOM VARIABLES

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Ann. Appl. Probab. **32** (2004) 2526–2544

We correct an error in the paper identified in the title. The error affects the factor before the square root of entropy in Theorem 1; the same factor appears in Theorem 2.

There is an error in the proof, and also in the statement, of Lemma 1 (page 2537). This error propagates to the Auxiliary Theorem (page 2537) and also affects Theorems 1 and 2. Here we give a proof of Theorem 1 with a correction term.

THEOREM 1 (Theorem 1 corrected).

$$W(p^n, q^n) \le \left(C \cdot \sqrt{\frac{1}{\delta} \cdot \frac{v}{t}} + 1\right) \cdot \sqrt{\frac{2}{\rho} \cdot D(p^n \| q^n)},$$

where

$$C = \min \frac{\sqrt{x}}{1 - \exp(-x)}$$

and the min *is taken on values of x of the form $x = t\delta(M/2N)$ which varies with M by steps smaller than $t/2N$. C is bounded by an absolute constant. In the original paper this result was claimed without the added term* 1.

In the proof we use the following concepts from the original paper:

"Sites" and "patches" are defined on psge 2528. The random sequence of patches $(I_1, I_2, \dots)$ and the Markov chain $(Y^n(0) = Z^n(0), Z^n(1), Z^n(2), \dots)$ are defined in Section 2 (pages 2534–2536). The numbers t and v are defined in Theorem 1 (page 2531).

Received August 2008.
[1]Supported in part by the Grant OTKA T-76088 and T-68258 of the Hungarian Academy of Sciences.
AMS 2000 subject classifications. 60K35, 82C22.
Key words and phrases. Measure concentration, Wasserstein distance, relative entropy, Dobrushin–Shlosman mixing condition, Gibbs sampler.

PROOF OF THEOREM 1 CORRECTED. We have

$$W^2(p^n, q^n) \le \sum_{i=1}^{n} \left(\sum_{l=1}^{\infty} (Z_i(l) - Z_i(l-1)) \right)^2 .$$

For a patch I and $l \ge 1$, define a joint conditional distribution,

$$\operatorname{dist}\big((Z_I(l-1), U_I(l)) | \bar{Z}_I(l-1) = \bar{z}_i\big)$$

as that joining of the marginals $\operatorname{dist}(Z_I(l-1)|\bar{Z}_I(l-1) = \bar{z}_I)$ and $Q_i(\cdot|\bar{z}_I)$, that minimizes the expected squared distance,

$$\mathbb{E}\{|Z_I(l-1) - U_I(l)|^2 | \bar{Z}_I(l-1) = \bar{z}_I\}$$

for each value of $\bar{z}_I$. [$U_I(l)$ cannot be considered as part of some random sequence $U^n(l)$.] We define the joint distribution of the sequence $(Z^n(m))$ and the random variables $U_I(l)$ in such a way that $U_I(l)$ depends on the sequence $(Z^n(m))$ only through $Z^n(l-1)$. Put

$$V_I(l) = Z_I(l-1) - U_I(l)$$

(whether $I_l = I$ or not). Moreover, define

$$\delta_I(l) = 1 \qquad \text{if } I_l = I \quad \text{and} \quad \delta_I(l) = 0 \qquad \text{if } I_l \ne I.$$

For fixed I, the sequence $(\delta_I(l) : l \in [1, M])$ is Bernoulli with $\Pr\{\delta_I(l) = 1\} = 1/N$, and for all $l \in [1, M]$, $\delta_I(l)$ is independent of $\mathcal{B}_{l-1}$, the σ-algebra generated by $(Z^n(0), Z^n(1), \dots, Z^n(l-1))$. Thus for all $m \ge l$,

$$\mathbb{E}\{\delta_I(m) V_{I,i}(m) | \mathcal{B}_{l-1}\} = \mathbb{E}\{V_{I,i}(m) | \mathcal{B}_{l-1}\}/N. \tag{1}$$

We have

$$Z_i(l) - Z_i(l-1) = \sum_{I \ni i} \delta_I(l) V_{I,i}(l),$$

where $V_{I,i}(l)$ denotes the ith coordinate of V_I. Thus

$$W^2(p^n, q^n) \le \sum_{i=1}^{n} \left(\sum_{I \ni i} \sum_{l=1}^{\infty} \delta_I(l) V_{I,i}(l) \right)^2 .$$

We claim that for fixed i and I,

$$\mathbb{E}\left(\sum_{l=1}^{\infty} \delta_I(l) V_{I,i}(l) \right)^2 \le 1/N^2 \cdot \mathbb{E}\left(\sum_{l=1}^{\infty} V_{I,i}(l) \right)^2 + 1/N \cdot \mathbb{E} \sum_{l=1}^{\infty} V_{I,i}^2(l). \tag{2}$$

Indeed, we have

$$\sum_{l=1}^{\infty} \delta_I(l) V_{I,i}(l) = 1/N \sum_{l=1}^{\infty} V_{I,i}(l) + \sum_{l=1}^{\infty} \big(\delta_I(l) V_{I,i}(l) - 1/N V_{I,i}(l)\big). \tag{3}$$

By (1), the terms of the second sum on the right-hand side are uncorrelated with each other and also with the terms of the first sum. Thus (3) implies (2).

It follows that

$$W^2(p^n, q^n) \le 1/N^2 \sum_i \mathbb{E}\left(\sum_{I \ni i} \sum_{l=1}^{\infty} |V_{I,i}(l)|\right)^2 + 1/N \cdot \sum_I \sum_{i \in I} \sum_{l=1}^{\infty} \mathbb{E} V_{I,i}^2(l).$$

We apply the Cauchy–Schwarz inequality to the first term. Since each i belongs to, at most, v patches, we get

$$\begin{aligned} W^2(p^n, q^n) &\le v/N^2 \sum_i \sum_{I \ni i} \mathbb{E}\left(\sum_{l=1}^{\infty} |V_{I,i}(l)|\right)^2 + 1/N \cdot \sum_I \sum_{i \in I} \sum_{l=1}^{\infty} \mathbb{E} V_{I,i}^2(l) \\ &= v/N^2 \sum_I \mathbb{E}\left(\sum_{l=1}^{\infty} |V_I(l)|\right)^2 + 1/N \cdot \sum_I \sum_{l=1}^{\infty} \mathbb{E} V_I^2(l). \end{aligned} \tag{4}$$

By the Cauchy–Schwarz inequality and Proposition 2 of the original paper, (4) implies that for every $M \ge 1$

$$\begin{aligned} &\left(\sum_I \mathbb{E}\left(\sum_{l=1}^{\infty} |V_I(l)|\right)^2\right)^{1/2} \\ &\quad \le \sqrt{M} \cdot \sum_I \sum_{k=1}^{\infty} \left(\sum_{l=(k-1)M+1}^{kM} \mathbb{E} V_{I,i}^2(l)\right)^{1/2} \\ &\quad \le \sqrt{M} \sum_I \left(\sum_{l=1}^{M} \mathbb{E} V_{I,i}^2(l)\right)^{1/2} \cdot \frac{1}{1-(1-t\delta/N)^{M/2}} \\ &\quad \le \sqrt{M} \sum_I \left(\sum_{l=1}^{M} \mathbb{E} V_{I,i}^2(l)\right)^{1/2} \cdot \frac{1}{1-\exp(-(Mt\delta/2N))}. \end{aligned} \tag{5}$$

Substituting (5) into (4),

$$W(p^n, q^n) \le \left(\frac{\sqrt{vM/N}}{1-\exp(-(Mt\delta/2N))} + 1\right) \cdot \left(1/N \sum_I \sum_{l=1}^{\infty} \mathbb{E} V_I^2(l)\right)^{1/2}. \tag{6}$$

As in the original paper (page 2540), we have

$$\begin{aligned} 1/N \sum_{l=1}^{\infty} E V_I^2(l) &= \sum_{l=1}^{\infty} \mathbb{E}\{\delta_I(l) V_I^2(l)\} = \sum_{l=1}^{\infty} \mathbb{E}\big(Z_{I_l}(l) - Z_{I_l}(l-1)\big)^2 \\ &\le \frac{2}{\rho} \cdot D(p^n || q^n). \end{aligned}$$

So (6) implies

$$W(p^n, q^n) \le \left(\frac{\sqrt{vM/N}}{(1 - \exp(-(Mt\delta/2N)))} + 1 \right) \cdot \sqrt{\frac{2}{\rho} \cdot D(p^n||q^n)}. \tag{7}$$

Now the argument in the last paragraph on page 2542 of the original paper is used to get the corrected Theorem 1 from (7) . □

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