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# HALF LIGHTLIKE SUBMANIFOLDS IN INDEFINITE $\mathcal{S}$-MANIFOLDS

Jae Won Lee and Dae Ho Jin

**Abstract.** In an indefinite metric $g.f.f$-manifold, we study half lightlike submanifolds $M$ tangent to the characteristic vector fields. We discuss the existence of totally umbilical half lightlike submanifolds of an indefinite $\mathcal{S}$-space form.

## 0. Introduction

Sasakian manifolds with semi-Riemannian metric have been considered ([11]), and recently many authors ([2, 3, 8, 9, 10]) study lightlike submanifolds of indefinite Sasakian manifolds. In analogy with the framework of Riemannian geomtery, Brunetti and Pastore [2] introduced indefinite $\mathcal{S}$-manifolds have represented a natural generalization of indefinite Sasakian manifolds. They have studied the geometry of lightlike hypersurfaces of indefinite $\mathcal{S}$-manifolds [3]. In the case of an indefinite Sasakian manifolds, Jin [10] extended lightlike hypersurfaces to half lightlike submanifolds, which is a special case of $r$-lightlike submanifolds [5] such that $r = 1$ and its geometry is more general than that of coisotropic submanifolds. It will be extended to half lightlike submanifolds on an indefinite $\mathcal{S}$-manifold.

We begin with some basic information about half lightlike submanifolds of a semi-Riemannian manifold in Section 1. Afterwards, for an indefinite metric $g.f.f$-manifold we consider a half lightlike submanifold $M$ tangent to the charateristic vector fields, we introduce a particular screen distribution $S(TM)$, using the properties of the indefinite $\mathcal{S}$-manifold. Then we deal with totally umbilical half lightlike submanifolds of an indefinite $\mathcal{S}$-space form in Section 3.

## 1. Lightlike Submanifolds

It is well known that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of half lightlike submanifolds $M$ of a semi-Rimannian manifold $(\bar{M}, \bar{g})$ of codimension 2

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is a vector subbundle of the tangent bundle $TM$ and the normal bundle $TM^\perp$, of rank 1. Thus there exists complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in $TM$ and $TM^\perp$ respectively, which called the *screen* and *coscreen distribution* on $M$, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \; TM^\perp = Rad(TM) \oplus_{orth} S(TM^\perp), \tag{1.1}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle $E$ over $M$. Choose $L \in \Gamma(S(TM^\perp))$ as a unit vector field with $\bar{g}(L, L) = \omega = \pm 1$. Consider the orthogonal complementary distribution $S(TM)^\perp$ to $S(TM)$ in $T\bar{M}$. Certainly $\xi \in \Gamma(Rad(TM))$ and $L$ belong to $\Gamma(S(TM)^\perp)$. Hence we have the following orthogonal decomposition

$$S(TM)^\perp = S(TM^\perp) \oplus_{orth} S(TM^\perp)^\perp,$$

where $S(TM^\perp)^\perp$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)^\perp$. For any null section $\xi$ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(ltr(TM))$ [5] satisfying

$$\bar{g}(\xi, N) = 1, \; \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \; \forall\, X \in \Gamma(S(TM)). \tag{1.2}$$

We call $N$, $ltr(TM)$ and $tr(TM) = S(TM^\perp) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of $M$ with respect to $S(TM)$ respectively. Thus $T\bar{M}$ is decomposed as follows:

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp). \end{aligned} \tag{1.3}$$

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and $P$ the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). The local Gauss and Weingarten formulas of $M$ and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{1.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{1.5}$$

$$\bar{\nabla}_X L = -A_L X + \mu(X)N, \tag{1.6}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{1.7}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{1.8}$$

for all $X,\, Y \in \Gamma(TM)$, where $\nabla$ and $\nabla^*$ are induced linear connections on $TM$ and $S(TM)$ respectively, $B$ and $D$ are called the *local second fundamental forms* of $M$, $C$ is called the *local second fundamental form* on $S(TM)$. $A_N$, $A^*_\xi$ and $A_L$ are linear operators on $TM$ and $\tau$, $\rho$ and $\phi$ are 1-forms on $TM$. We say that $h(X,Y) = B(X,Y)N + D(X,Y)L$ is the *second fundamental tensor* of $M$. Since $\bar{\nabla}$ is torsion-free, $\nabla$ is also torsion-free, and $B$ and $D$ are symmetric. From the facts $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X,Y) = \omega\bar{g}(\bar{\nabla}_X Y, L)$, we know that $B$ and $D$ are independent of the choice of $S(TM)$ and satisfy

$$B(X,\xi) = 0,\ D(X,\xi) = -\omega\mu(X),\ \forall\, X \in \Gamma(TM). \tag{1.9}$$

The induced connection $\nabla$ of $M$ is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{1.10}$$

for all $X,\, Y,\, Z \in \Gamma(TM)$, where $\eta$ is a 1-form on $TM$ such that

$$\eta(X) = \bar{g}(X,N),\ \forall\, X \in \Gamma(TM). \tag{1.11}$$

But the connection $\nabla^*$ on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.12}$$

$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{1.13}$$

$$\omega D(X,PY) = g(A_L X, PY), \qquad \bar{g}(A_L X, N) = \omega\rho(X), \tag{1.14}$$

$$\omega D(X,Y) = g(A_L X, Y) - \mu(X)\eta(Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{1.15}$$

By (1.12) and (1.13), we show that $A^*_\xi$ and $A_N$ are $\Gamma(S(TM))$-valued shape operators related to $B$ and $C$ respectively and $A^*_\xi$ is self-adjoint on $TM$ and

$$A^*_\xi \xi = 0. \tag{1.16}$$

But $A_N$ is not self-adjoint on $S(TM)$. We know that $A_N$ is self-adjoint in $S(TM)$ if and only if $S(TM)$ is an integrable distribution [5]. From (1.15), we show that $A_L$ is not self-adjoint on $TM$. $A_L$ is self-adjoint in $TM$ if and only if $\mu(X) = 0$ for all $X \in \Gamma(S(TM))$ [9]. From (1.4), (1.8) and (1.9), we have

$$\bar{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi - \omega\mu(X)L,\ \forall\, X \in \Gamma(TM). \tag{1.17}$$

Denote by $\bar{R}$ and $R$ the curvature tensors of the connections $\bar{\nabla}$ and $\nabla$ respectively. Using the local Gauss-Weingarten formulas (1.4) $\sim$ (1.6) for $M$, we have the Gauss-Codazzi equations for $M$, for all $X,\ Y,\ Z\in\Gamma(TM)$:

$$
\begin{aligned}
&\bar{R}(X,Y)Z\\
&= R(X,Y)Z + B(X,Z)A_NY - B(Y,Z)A_NX\\
&\quad +D(X,Z)A_LY - D(Y,Z)A_LX + \{(\nabla_X B)(Y,Z)\\
&\quad -(\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\\
&\quad +\mu(X)D(Y,Z) - \mu(Y)D(X,Z)\}N + \{(\nabla_X D)(Y,Z)\\
&\quad -(\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) - \rho(Y)B(X,Z)\}L
\end{aligned}
\tag{1.18}
$$

## 2. Characteristic Half Lightlike Submanifolds of Indefinite $g.f.f$-Manifolds

A manifold $\bar{M}$ is called a *globally framed f-manifold ( or $g.f.f$-manifold)* if it is endowed with a non null $(1,1)$-tensor field $\bar{\phi}$ of constant rank, such that $ker\bar{\phi}$ is parallelizable i.e. there exist global vector fields $\bar{\xi_\alpha}$, $\alpha\in\{1,\cdots,r\}$, with their dual 1- forms $\bar{\eta}^\alpha$, satisfying $\bar{\phi}^2=-I+\sum_{\alpha=1}^r\bar{\eta}^\alpha\otimes\bar{\xi_\alpha}$ and $\bar{\eta}^\alpha(\bar{\xi_\beta})=\delta^\alpha_\beta$.

The $g.f.f$-manifold $(\bar{M}^{2n+r},\bar{\phi},\bar{\xi_\alpha},\bar{\eta}^\alpha)$, $\alpha\in\{1,\cdots,r\}$, is said to be an indefinite metric $g.f.f$-manifold if $\bar{g}$ is a semi-Riemannian metric, with index $\nu$, $0<\nu<2n+r$, satisfying the following compatibility condtion

$$\bar{g}(\bar{\phi}X,\bar{\phi}Y)=\bar{g}(X,Y)-\sum_{\alpha=1}^r\epsilon_\alpha\bar{\eta}^\alpha(X)\bar{\eta}^\alpha(Y)$$

for any $X,\ Y\in\Gamma(T\bar{M})$, being $\epsilon_\alpha=\pm1$ according to whether $\bar{\xi}_\alpha$ is spacelike or timelike. Then, for any $\alpha\in\{1,\cdots,r\}$, one has $\bar{\eta}^\alpha(X)=\epsilon_\alpha\bar{g}(X,\bar{\xi}_\alpha)$. An indefinite metric g.f.f-manifold is called an *indefinite $\mathcal{S}$-manifold* if it is normal and $d\bar{\eta}^\alpha=\Phi$, for any $\alpha\in\{1,\cdots,r\}$, where $\Phi(X,Y)=\bar{g}(X,\bar{\phi}Y)$ for any $X$, $Y\in\Gamma(T\bar{M})$. The normality condition is expressed by the vanishing of the tensor field $N=N_{\bar{\phi}}+2\sum_{\alpha=1}^r d\bar{\eta}^\alpha\otimes\bar{\xi}_\alpha$, $N_{\bar{\phi}}$ the Nijenhuis torsion of $\bar{\phi}$. Furthermore, as proved in [2], the Levi-Civita connection of an indefinite $\mathcal{S}$-manifold satisfies:

$$(\bar{\nabla}_X\bar{\phi})Y=\bar{g}(\bar{\phi}X,\bar{\phi}Y)\bar{\xi}+\bar{\eta}(Y)\bar{\phi}^2(X),\tag{2.1}$$

where $\bar{\xi}=\sum_{\alpha=1}^r\bar{\xi}_\alpha$ and $\bar{\eta}=\sum_{\alpha=1}^r\epsilon_\alpha\bar{\eta}^\alpha$. We recall that $\bar{\nabla}_X\bar{\xi}_\alpha=-\epsilon_\alpha\bar{\phi}X$ and $ker\bar{\phi}$ is an integrable flat distribution since $\bar{\nabla}_{\bar{\xi}_\alpha}\bar{\xi}_\beta=0$.( more details in [2]).

An indefinite $\mathcal{S}$-manifold $(\bar{M},\bar{\phi},\bar{\xi}_\alpha,\bar{\eta}^\alpha)$ is called an *indefinite $\mathcal{S}$-space form*, denoted by $\bar{M}(c)$, if it has the constant $\bar{\phi}$-sectional curvature $c$ [2]. The curvature tensor $\bar{R}$ of this space form $\bar{M}(c)$ is given by

$$
\begin{aligned}
&4\bar{R}(X,Y,Z,W)\\
=&-(c+3\epsilon)\{\bar{g}(\bar{\phi}Y,\bar{\phi}Z)\bar{g}(\bar{\phi}X,\bar{\phi}W)-\bar{g}(\bar{\phi}X,\bar{\phi}Z)\bar{g}(\bar{\phi}Y,\bar{\phi}W)\}\\
&-(c-\epsilon)\{\Phi(W,X)\Phi(Z,Y)-\Phi(Z,X)\Phi(W,Y)+2\Phi(X,Y)\Phi(W,Z)\}\\
&-\{\bar{\eta}(W)\bar{\eta}(X)\bar{g}(\bar{\phi}Z,\bar{\phi}Y)-\bar{\eta}(W)\bar{\eta}(Y)\bar{g}(\bar{\phi}Z,\bar{\phi}X)\\
&+\bar{\eta}(Y)\bar{\eta}(Z)\bar{g}(\bar{\phi}W,\bar{\phi}X)-\bar{\eta}(Z)\bar{\eta}(X)\bar{g}(\bar{\phi}W,\bar{\phi}Y)\}
\end{aligned}
\tag{2.2}
$$

for any vector fields $X, Y, Z, W \in \Gamma(T\bar{M})$.

**Theorem 2.1.** *Let $M$ be a half lightlike submanifold of an indefinite $\mathcal{S}$-manifold $\bar{M}$ such that all the charateristic vector fields $\bar{\xi}_\alpha$ are tangent to $M$. Then there exist a screen $S(TM)$ such that*

$$\bar{\phi}(S(TM)^\perp) \subset S(TM).$$

*Proof.* Since $\bar{\phi}$ is skew symmetric with respect to $\bar{g}$, we have $\bar{g}(\bar{\phi}\xi, \xi) = 0$. Thus $\bar{\phi}\xi$ blongs to $TM \oplus S(TM^\perp)$. If $Rad(TM) \cap \bar{\phi}(Rad(TM)) \neq \{0\}$, then there exists a non-vanishing smooth real valued function $f$ such that $\bar{\phi}\xi = f\xi$. Apply $\bar{\phi}$ to the equation and $\bar{\phi}$-properties, we have $(f^2+1)\xi = 0$. Therefore, we get $f^2+1 = 0$, which is a contradition. Thus $Rad(TM) \cap \bar{\phi}(Rad(TM)) = \{0\}$. Moreover, if $S(TM^\perp) \cap \bar{\phi}(Rad(TM)) \neq \{0\}$, then there exists a non-vanishing smooth real valued function $h$ such that $\bar{\phi}\xi = hL$. In this case, we have $h^2 = \bar{g}(hL, hL) = \bar{g}(\bar{\phi}\xi, \bar{\phi}\xi) = 0$, which is a contradiction to $h \neq 0$. Thus we have $S(TM^\perp) \cap \bar{\phi}(Rad(TM)) = \{0\}$. This enables one to choose a screen distribution $S(TM)$ such that it cotains $\bar{\phi}(Rad(TM))$ as a vector subbundle. From the facts $\bar{g}(\bar{\phi}N, N) = 0$ and $\bar{g}(\bar{\phi}N, \xi) = -\bar{g}(N, \bar{\phi}\xi) = 0$, using the above method, we also show that $\bar{\phi}(ltr(TM))$ is a vector subbundle of $S(TM)$ of rank 1. On the other hand, from the facts $\bar{g}(\bar{\phi}L, L) = 0, \bar{g}(\bar{\phi}L, \xi) = -\bar{g}(L, \bar{\phi}\xi) = 0$ and $\bar{g}(\bar{\phi}L, N) = -\bar{g}(L, \bar{\phi}N) = 0$, we show that $\bar{\phi}(S(TM^\perp))$ is also a vector subbundle of $S(TM)$. ■

**Note 2.** Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM^* = TM/Rad(TM)$ considered by Kupeli [12]. Thus all screens $S(TM)$ are mutually isomorphic. For this reason, we consider only half lightlike submanifolds equipped with a screen $S(TM)$ such that $\bar{\phi}(S(TM)^\perp) \subset S(TM)$. We call such a screen $S(TM)$ the *generic screen* of $M$.

**Definition 1.** Let $M$ be a half lightlike submanifold of $\bar{M}$ such that all the charateristic vector fields $\bar{\xi}_\alpha$ are tangent to $M$. A screen distribution $S(TM)$ is said to be *characteristic* if $ker\,\bar{\phi} \subset S(TM)$ and $\bar{\phi}(S(TM)^\perp) \subset \Gamma(S(TM))$.

**Definition 2.** A half lightlike submanifold $M$ of $\bar{M}$ is said to be *characteristic* if $ker\bar{\phi} \subset TM$ and a characteristic screen distribution $(S(TM))$ is chosen.

**Proposition 2.1.** [3]. *Let* $(M, g, S(TM))$ *be a lightlike hypersurface of an indefinite* $\mathcal{S}$*-manifold* $(\bar{M}, \bar{\phi}, \bar{\xi_\alpha}, \bar{\eta}^\alpha, \bar{g})$. *Then* $M$ *is a characteristic lightlike submanifold of* $\bar{M}$.

By Theorem 2.1, the characteristic screen $S(TM)$ is expressed as follow:

$$S(TM) = \{\bar{\phi}(Rad(TM)) \oplus \bar{\phi}(ltr(TM))\} \oplus_{orth} \bar{\phi}(S(TM^\perp) \oplus_{orth} D_o,$$

where $D_o$ is the uniquely defined non-degenerate ditribution. Then each $\bar{\xi}_\alpha \in D_o$ and the general decompositions (1.1) and (1.3) reduce to

$$TM = D_o \oplus \mathcal{F} \tag{2.3}$$

$$T\bar{M} = D_o \oplus \mathcal{E}, \tag{2.4}$$

$$TM = D \oplus \bar{\phi}(ltr(TM)) \oplus \bar{\phi}(S(TM^\perp)) \tag{2.5}$$

where $D := D_o \oplus \bar{\phi}(Rad(TM)) \oplus Rad(TM)$ and

$$\mathcal{E} := \{\bar{\phi}(Rad(TM)) \oplus \bar{\phi}(ltr(TM))\} \oplus \{Rad(TM) \oplus ltr(TM) \oplus S(TM^\perp)\},$$

$$\mathcal{F} := \{\bar{\phi}(Rad(TM)) \oplus \bar{\phi}(ltr(TM))\} \oplus Rad(TM).$$

Similar to the definition of $\bar{\phi}$-invariant submanifold([1], p122), we adopt the condtion $\bar{\phi}(\mathcal{V}) \subseteq \mathcal{V}$ for the $\bar{\phi}$-invariance of a distribution $\mathcal{V}$. Then $D_0$ and $D$ are $\bar{\phi}$-invariant. Obviously, considering the orthogoanl decompositions $D_o = D'_o \perp ker\phi$ and $D = D' \perp ker\phi$, we get $\bar{\phi}(D'_o) = D'_o$, $\bar{\phi}(D') = D'$, and the decompositions in (2.3)∼(2.5) are reduced. For example,

$$TM = D'_o \oplus ker\phi \oplus \mathcal{F}.$$

Now, Consider null vector fields $U$ and $V$, and a non-null vector field $W$ such that

$$U = -\bar{\phi}N, \quad V = -\bar{\phi}\xi, \quad W = -\bar{\phi}L. \tag{2.6}$$

Denote by $S$ the projection morphism of $TM$ on $D$. From (3.3) any vector field $X$ on $M$ is expressed as follows

$$X = SX + u(X)U + w(X)W, \quad \bar{\phi}X = \phi X + u(X)N + w(X)L, \tag{2.7}$$

where $u$, $v$ and $w$ are 1-forms locally defined on $M$ by

$$u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = \epsilon\, g(X, W) \tag{2.8}$$

and $\phi$ is a tensor field of type $(1,1)$ globally defined on $M$ by

$$\phi X = \bar{\phi} SX, \quad \forall X \in \Gamma(TM).$$

We note that if $X \in \Gamma(TM)$, then $SX \in C$, $\phi X = \bar{\phi}(SX) \in D$, so that $S(\phi X) = \phi X$. Furthermore, since $\bar{\phi}(\phi X) = \bar{\phi}(S\phi X) = \bar{\phi} S(\phi X) = \phi^2 X$, we can write $\phi^2 X = -X + \bar{\eta}^\alpha(X)\bar{\xi}_\alpha + u(X)U + w(X)W$ by applying $\bar{\phi}$ to the second equation in (2.7). Finally, since $U \in \bar{\phi}(ltr(TM))$ and $W \in \bar{\phi}(S(TM^\perp))$, we have $\phi U = 0$ , $\phi W = 0$, $\bar{\eta}^\alpha \circ \phi = 0$, and $u(\phi X) = 0$, $w(\phi X) = 0$ for any $X \in \Gamma(TM)$. Thus we can state the following:

**Theorem 2.2.** *Let $(\bar{M}, \bar{\phi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha)$ be an indefinite $\mathcal{S}$-manifold, and let $(M, g, S(TM))$ be a characteristic half lightlike submanifold of $\bar{M}$ such that $\xi$ and $N$ are globally defined on $M$. Then $(\bar{M}, \bar{\phi}, \bar{\xi}_\alpha, U, L, \bar{\eta}^\alpha, u, w)$ is a $g.f.f$-manifold.*

For any $X$, $Y \in \Gamma(TM)$, we compute the field $(\nabla_X \phi)Y$. Using (1.4), (1.5) and (2.7), we get

$$\begin{aligned}(\bar{\nabla}_X \bar{\phi})Y &= (\nabla_X \phi)Y - u(Y)A_N X - w(Y)A_L X \\ &\quad + \{B(X, \phi Y) + (\nabla_X u)(Y) + u(Y)\tau(X) + w(Y)\varphi(X)\}N \\ &\quad + \{D(X, \phi Y) + (\nabla_X w)Y + u(Y)\phi(X)\}L \\ &\quad + B(X, Y)U + D(X, Y)W\end{aligned}$$

then, from (2.1), comparing the components along $TM$, $ltr(TM)$ and $S(TM^\perp)$, we have:

$$\begin{aligned}(\nabla_X \phi)Y &= u(Y)A_N X + w(Y)A_L X - B(X, Y)U \\ &\quad - D(X, Y)W + \bar{g}(\bar{\phi}X, \bar{\phi}Y)\bar{\xi} + \bar{\eta}(Y)\bar{\phi}^2(X)\end{aligned} \tag{2.9}$$

$$(\nabla_X u)(Y) = -B(X, \phi Y) - u(Y)\tau(X) - w(Y)\varphi(X) \tag{2.10}$$

$$(\nabla_X w)Y = -D(X, \phi Y) - u(Y)\phi(X) \tag{2.11}$$

**Definition 3.** Let $(\bar{M}, \bar{\phi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ be an indefinite $g.f.f$-manifold and $(M, g, S(TM))$ a half lightlike submanifold of $\bar{M}$. Then $M$ is called *totally geodesic* if any geodesic of $M$ with respect to the induced connection $\nabla$ is a geodesic of $M$ with respect to $\bar{\nabla}$.

It is easy to see that $M$ is totally geodesic if and only if the local second fundamental forms $B$, $D$ vanish identically.(i.e., $B \equiv 0$ and $D \equiv 0$)

**Theorem 2.3.** *Let $(\bar{M}, \bar{\phi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ be an indefinite $\mathcal{S}$-manifold and $(M, g, S(TM))$ a half lightlike submanifold of $\bar{M}$. Then $M$ is totally geodesic if and only if for any $X \in \Gamma(TM)$ and for any $Y \in \Gamma(D)$,*

$$(\nabla_X\phi)Y = \bar{g}(\bar{\phi}X, \bar{\phi}Y)\bar{\xi} + \bar{\eta}(Y)\bar{\phi}^2(X) \tag{2.12}$$

$$A_N X = -(\nabla_X\phi)U + \bar{g}(X, U)\bar{\xi} \tag{2.13}$$

$$A_L X = -(\nabla_X\phi)W + \bar{g}(X, W)\bar{\xi} \tag{2.14}$$

*Proof.* We assume that $M$ is totally geodesic, that is for all $X, Y \in \Gamma(TM)$, $B(X, Y) \equiv 0$ and $D(X, Y) \equiv 0$. In (2.9), for any $Y \in \Gamma(D)$, we have $u(Y) = 0$ and $w(Y)$, and hence $(\nabla_X\phi)Y = \bar{g}(\bar{\phi}X, \bar{\phi}Y)\bar{\xi} + \bar{\eta}(Y)\bar{\phi}^2(X)$. Again, replacing $Y$ in (2.9) by $U$, we have $(\nabla_X\phi)U = A_N X + \bar{g}(\bar{\phi}X, \bar{\phi}U)\bar{\xi} + \bar{\eta}(U)\bar{\phi}^2(X)$, from which we obtain $A_N X = -(\nabla_X\phi)U + \bar{g}(X, U)\bar{\xi}$ In analogy with (2.13), we have $A_L X = -(\nabla_X\phi)W + \bar{g}(X, W)\bar{\xi}$

Conversely, we suppose that the conditions (2.12), (2.13), and (2.14) hold. If $Y \in \Gamma(TM)$, using decompositon (2.5), there exists locally smooth functions $f$ and $h$ such that $Y = Y_d + fU + hW$, and for any $X \in \Gamma(TM)$, we obtain $B(X, Y) = B(X, Y_d) + fB(X, U) + hB(X, W)$ and $D(X, Y) = D(X, Y_d) + fD(X, U) + hD(X, W)$ . Using (2.9) and (2.12) with $Y = Y_d$, we find $B(X, Y_d)U + D(X, Y_d)W = u(Y_d)A_N X + w(Y_d)A_L X = 0$, which implies $B(X, Y_d) = D(X, Y_d) = 0$. From (2.9), putting $Y = U$ and using (2.13), we get $B(X, U)U + D(X, U)W = 0$, which implies $B(X, U) = D(X, U) = 0$. Again, from (2.9), putting $Y = W$ and using (2.14), we get $B(X, W)U + D(X, W)W = 0$, which implies $B(X, W) = D(X, W) = 0$. The proof is complete. ∎

## 3. Totally Umbilical Half Lightlike Submanifolds of an Indefinite $\mathcal{S}$-Manifold

**Definition 4.** Let $(\bar{M}, \bar{\phi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ be an indefinite $\mathcal{S}$-manifold and $(M, g, S(TM))$ a half lightlike submanifold of $\bar{M}$. We say that $M$ is *totally umbilical* [6] if, on any coordinate neighborhood $\mathcal{U}$, there is a smooth vector field $\mathcal{H} \in \Gamma(tr(TM))$ such that

$$h(X, Y) = \mathcal{H}\, g(X, Y),\ \forall\, X,\, Y \in \Gamma(TM).$$

In case $\mathcal{H} = 0$ on $\mathcal{U}$, we say that $M$ is *totally geodesic*.

It is easy to see that $M$ is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions $\beta$ and $\delta$ such that

$$B(X, Y) = \beta g(X, Y),\ D(X, Y) = \delta g(X, Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{3.1}$$

**Theorem 3.1.** *Let $M$ be a totally umbilical half lightlike submanifold of an indefinite $\mathcal{S}$-manifold $\bar{M}$. Then $M$ is totally geodesic.*

*Proof.* Apply the operator $\bar{\nabla}_X$ to $\bar{g}(\bar{\phi}\xi, L) = 0$ with $X \in \Gamma(TM)$ and use (1.5), (1.6), (1.12), (1.14) and (1.17), we have

$$B(X, \bar{\phi}L) = D(X, \bar{\phi}\xi), \ \forall\, X \in \Gamma(TM).$$

As $M$ is totally umbilical, from the last equation and (3.1), we have

$$\beta g(X, \bar{\phi}L) = \omega\delta g(X, \bar{\phi}\xi), \ \forall\, X \in \Gamma(TM).$$

Replace $X$ by $\bar{\phi}N$ and $\bar{\phi}L$ in this equation by turns, we have

$$0 = \omega\delta, \qquad \omega\beta = 0 \tag{3.2}$$

Thus we have $\mathcal{H} = 0$. ∎

**Theorem 3.2.** *Let* $(\bar{M}(c), \bar{\phi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ *be an indefinite* $\mathcal{S}$*-space form and* $(M, g, S(TM))$ *a half lightlike submanifold of* $\bar{M}$. *If* $(M, g, S(TM))$ *is totally umbilical, then* $c = \epsilon = \sum_{\alpha=1}^{r} \epsilon_\alpha$.

*Proof.* Since $\bar{\eta}(\xi) = 0$ and $\bar{g}(\bar{\phi}\xi, \bar{\phi}X) = 0$ for any $X \in \Gamma(TM)$, $\bar{M}(c)$ is an indefinite $\mathcal{S}$-space form implies the Riemannian curvature $\bar{R}$ in (2.2) is given by

$$\begin{aligned} &4\bar{R}(X, Y, Z, \xi) \\ &= -(c-\epsilon)\{\Phi(\xi, X)\Phi(Z, Y) - \Phi(Z, X)\Phi(\xi, Y) + 2\Phi(X, Y)\Phi(\xi, Z)\} \\ &= -(c-\epsilon)\{\bar{g}(V, X)\Phi(Z, Y) - \Phi(Z, X)\bar{g}(V, Y) + 2\Phi(X, Y)\bar{g}(V, Z)\}, \end{aligned} \tag{3.3}$$

for any $X, Y, Z, \in \Gamma(TM)$. So, replacing $X$, $Y$, $Z$ by $PX$, $\xi$, $PZ$ in (3.3), we find

$$\begin{aligned} &4\bar{R}(X, Y, Z, \xi) \\ &= -(c-\epsilon)\{-\bar{g}(V, PX)\bar{g}(PZ, V) - 2\bar{g}(X, V)\bar{g}(V, Z)\} \\ &= 3(c-\epsilon)u(PZ)u(PX) \end{aligned} \tag{3.4}$$

On the other hand, from (1.18), we have

$$\begin{aligned} &\bar{R}(X, Y, Z, \xi) \\ &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) \\ &\quad -\tau(Y)B(X, Z) + \mu(X)D(Y, Z) - \mu(Y)D(X, Z) \end{aligned} \tag{3.5}$$

Theorem 3.1, $\bar{R}(X, Y, Z, \xi) = 0$ and therefore, we have $4\bar{R}(X, Y, Z, \xi) = 3(c-\epsilon)u(PZ)u(PX)$. Choosing $X = Z = U \in \Gamma(S(TM))$, we obtain $c = \epsilon$. ∎

**Corollary 3.3.** *There is no totally umbilical characteristic half lightlike submanifolds of an indefinite* $\mathcal{S}$*-space form* $\bar{M}(c)$ *with* $c \neq \epsilon$.

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Jae Won Lee
Department of Mathematics
Sogang University
Sinsu-dong, Mapo-gu 121-742
Republic of Korea
E-mail: leejaewon@sogang.ac.kr

Dae Ho Jin
Department of Mathematics
Dongguk University
Gyeongju 780-714
Republic of Korea
E-mail: jindh@dongguk.ac.kr