

## A NEW EIGENVALUE EMBEDDING APPROACH FOR FINITE ELEMENT MODEL UPDATING

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**Abstract.** This paper concerns the eigenvalue embedding problem (EEP) of updating a symmetric finite-element model so that a few troublesome eigenvalues are replaced by some chosen ones, while the remaining large number of eigenvalues and eigenvectors of the original model do not change. Based on the theory established in [2], by sufficiently utilizing the inherent freedom of the EEP, an expression of the parameterized solution to the EEP is derived. This expression is then used to develop a novel numerical method for solving the EEP, in which the parameters in the solutions are optimized in some sense. This method not only utilizes the freedom of the EEP but also removes the limitation of the method proposed in [6]. The results of our numerical experiments show that the present algorithm is feasible and efficient, and can outperform the iterative method in [3] and the method in [6].

### 1. INTRODUCTION

Consider the system of matrix second-order differential equations of the form

$$(1.1) \quad M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = 0,$$

where  $M$ ,  $C$ , and  $K$  are all real symmetric matrices of  $n$  by  $n$  and are called, respectively, the mass matrix, the damping matrix, and the stiffness matrix. Model (1.1), which is often referred to as a *real symmetric finite element model*, can be obtained from the modeling of vibrating systems, such as bridges, buildings, highways and automobiles using finite-element methods. In many applications, the coefficient matrices enjoy very special properties, such as positive definiteness, sparsity, and so on. If a fundamental solution to (1.1) is represented by

$$(1.2) \quad \mathbf{x}(t) = \mathbf{x}e^{\lambda t},$$

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then the scalar  $\lambda$  and the vector  $\mathbf{x}$  must solve the *quadratic eigenvalue problem* (QEP)

$$(1.3) \quad \mathcal{Q}(\lambda)\mathbf{x} = 0,$$

where

$$(1.4) \quad \mathcal{Q}(\lambda) := M\lambda^2 + C\lambda + K$$

is referred to as the *quadratic matrix polynomial*. The scalars  $\lambda$  and the corresponding nonzero vectors  $\mathbf{x}$  are called, respectively, the *eigenvalues* and the *eigenvectors* of the quadratic matrix polynomial  $\mathcal{Q}(\lambda)$ . Together,  $(\lambda, \mathbf{x})$  is called an *eigenpair* of  $\mathcal{Q}(\lambda)$ . It is known that  $\mathcal{Q}(\lambda)$  has  $2n$  eigenvalues over the complex field, which are the roots of the equation

$$(1.5) \quad \det(\mathcal{Q}(\lambda)) = 0,$$

provided that the leading coefficient matrix  $M$  is nonsingular.

It is well-known that the dynamical behavior of a vibrating system modeled by (1.1) is determined by its natural frequencies and mode shapes, that is, the eigenvalues and eigenvectors of  $\mathcal{Q}(\lambda)$ . The undesired phenomena such as instability and resonance are caused by some “troublesome” eigenvalues and the corresponding eigenvectors of  $\mathcal{Q}(\lambda)$ . Therefore, in order to combat or avoid the undesired phenomena, one way is to update the quadratic model  $\mathcal{Q}(\lambda)$  so that these “troublesome” or unfavorable eigenvalues and eigenvectors are replaced by some suitable ones, which are usually chosen by engineers and designers. Among current developments for finite element model updating, one challenge that is of practical importance is to update the model while retaining the remaining eigenvalues and eigenvectors. Such updating, if possible, is known as *updating with no spill-over*. In this paper, we consider the special model updating with no spill-over, which is known as the *eigenvalue embedding problem* (EEP), stated as follows [3]:

**EEP** Given a real symmetric quadratic matrix polynomial  $\mathcal{Q}(\lambda) = M\lambda^2 + C\lambda + K$  with  $M$  nonsingular and a few of its associated eigenpairs  $\{\lambda_i, \mathbf{x}_i\}_{i=1}^k$  with  $k \leq n$ , assume that the new eigenvalues  $\{\tilde{\lambda}_i\}_{i=1}^k$  have been measured or chosen. Update the quadratic matrix polynomial  $\mathcal{Q}(\lambda)$  to  $\tilde{\mathcal{Q}}(\lambda) = \tilde{M}\lambda^2 + \tilde{C}\lambda + \tilde{K}$ , with  $\tilde{M}, \tilde{C}, \tilde{K}$  real symmetric and  $\tilde{M}$  nonsingular, so that the subset  $\{\lambda_i\}_{i=1}^k$  is replaced by  $\{\tilde{\lambda}_i\}_{i=1}^k$  as  $k$  eigenvalues of  $\tilde{\mathcal{Q}}(\lambda)$  while the remaining  $2n - k$  eigenpairs of  $\tilde{\mathcal{Q}}(\lambda)$ , which are usually unknown, are kept the same as those of the original  $\mathcal{Q}(\lambda)$ . Furthermore, characterize the eigenvectors of  $\tilde{\mathcal{Q}}(\lambda)$  corresponding to  $\{\tilde{\lambda}_i\}_{i=1}^k$ .

Such a problem in control theory is known as the *partial pole assignment problem* and is solved by feedback control. Unfortunately, the use of the feedback control

destroys the symmetry in (1.1); see [7, 8, 10, 14, 15]. Recently, a symmetry preserving iterative scheme that reassigns one real eigenvalue or one complex conjugate pair of eigenvalues at a time was proposed in [3] as a possible numerical method for solving the EEP. But that algorithm suffers from the shortcomings that the iteration may break down before all desired eigenvalues are updated. Based on the spectral decomposition theory, Chu and Xu [6] offered a nice approach for solving the EEP which completely circumvents all inherent troubles of the algorithm suggested in [3]. However, the inherent freedom of the EEP is not sufficiently utilized in this method, since it only simply taken the given eigenvectors  $\{\mathbf{x}_j\}_{j=1}^k$  as the eigenvectors corresponding to the new eigenvalues  $\{\tilde{\lambda}_i\}_{i=1}^k$ , and moreover, it requires that the real eigenvalues are replaced by real ones and the complex eigenvalues are replaced by complex ones.

In this paper, based on the theory established in [2] we sufficiently utilize the inherent freedom of the EEP to derive an expression of the parameterized solution to the EEP. We then use the expression to develop a novel numerical method for solving the EEP, in which the parameters in the solutions are optimized in some sense. This method not only utilizes the freedom of the EEP, but also removes the limitation of the method proposed in [6]. The results of our numerical experiments show that the present algorithm is feasible and efficient, and can outperform the iterative method in [3] and the method in [6].

This paper is organized as follows. In Section 2, we present some notations, definitions, and basic theory, which will be used throughout this paper. In Section 3, the parameterized solution to the EEP is derived, and some necessary and sufficient conditions for the existence of solutions with a positive definite mass matrix are discussed. The numerical approaches and numerical results are presented in Sections 4 and 5, respectively. Finally, we give some conclusion remarks in Section 6.

## 2. PRELIMINARIES

Throughout this paper we adopt the following notations and definitions. The symbol  $\|\cdot\|$  denotes the Euclidean norm of a vector or the spectral norm of a matrix and  $\|\cdot\|_F$  denotes the Frobenius norm of a matrix. For any given square matrix  $A$  of size  $n \times n$ , the spectrum of  $A$  is denoted by  $\lambda(A)$ , and  $A > 0$  denotes a symmetric positive definite matrix. For any given  $m \times n$  real matrix  $B$ , we use  $B^T$  to denote the transpose of  $B$ , and use  $\mathcal{N}(B)$  to denote the *null space* of  $B$ , i.e.,

$$(2.1) \quad \mathcal{N}(B) = \{\mathbf{x} \in \mathbb{R}^n \mid B\mathbf{x} = 0\}.$$

$\dim \mathcal{X}$  denotes the dimension of a subspace  $\mathcal{X}$ , and  $\text{sign}(\xi)$  is employed to denote the sign of  $\xi$  for any real number  $\xi$ . For any given real symmetric matrix  $A$ , the ordered triple

$$i(A) = (i_+(A), i_-(A), i_0(A))$$

denotes the inertia of  $A$ , where  $i_+(A)$  is the number of positive eigenvalues of  $A$ ,  $i_-(A)$  is the number of negative eigenvalues of  $A$ , and  $i_0(A)$  is the number of zero eigenvalues of  $A$ , all counting multiplicity.

Assume the  $k$  eigenpairs  $\{\lambda_j, \mathbf{x}_j\}_{j=1}^k$  given in the EEP are in the following form

$$\begin{aligned} \lambda_{2j-1} &= \bar{\lambda}_{2j} = \alpha_j + i\beta_j, & \alpha_j \in \mathbb{R}, \beta_j > 0, & \quad j = 1, 2, \dots, \ell, \\ \mathbf{x}_{2j-1} &= \bar{\mathbf{x}}_{2j} = \mathbf{x}_{jR} + i\mathbf{x}_{jI}, & \mathbf{x}_{jR}, \mathbf{x}_{jI} \in \mathbb{R}^n, & \quad j = 1, 2, \dots, \ell, \end{aligned}$$

and

$$\lambda_j \in \mathbb{R}, \quad \mathbf{x}_j \in \mathbb{R}^n, \quad j = 2\ell + 1, \dots, k.$$

Define

$$(2.2) \quad X_1 := [\mathbf{x}_{1R}, \mathbf{x}_{1I}, \dots, \mathbf{x}_{\ell R}, \mathbf{x}_{\ell I}, \mathbf{x}_{2\ell+1}, \dots, \mathbf{x}_k],$$

$$(2.3) \quad \Lambda_1 := \text{diag} \left( \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \dots, \begin{bmatrix} \alpha_\ell & \beta_\ell \\ -\beta_\ell & \alpha_\ell \end{bmatrix}, \lambda_{2\ell+1}, \dots, \lambda_k \right),$$

which are referred to as the *real representations* of  $\{\mathbf{x}_j\}_{j=1}^k$  and  $\{\lambda_j\}_{j=1}^k$ , respectively.

Similarly, let the real representation of the new measured eigenvalues  $\{\tilde{\lambda}_i\}_{i=1}^k$  be

$$(2.4) \quad \tilde{\Lambda}_1 = \text{diag} \left( \begin{bmatrix} \tilde{\alpha}_1 & \tilde{\beta}_1 \\ -\tilde{\beta}_1 & \tilde{\alpha}_1 \end{bmatrix}, \dots, \begin{bmatrix} \tilde{\alpha}_{\tilde{\ell}} & \tilde{\beta}_{\tilde{\ell}} \\ -\tilde{\beta}_{\tilde{\ell}} & \tilde{\alpha}_{\tilde{\ell}} \end{bmatrix}, \tilde{\lambda}_{2\tilde{\ell}+1}, \dots, \tilde{\lambda}_k \right),$$

which means that the new measured eigenvalues just contain  $\tilde{\ell}$  complex conjugate pairs. It is worthwhile to point out that here  $\tilde{\ell}$  is not necessary equal to  $\ell$ , while  $\tilde{\ell} = \ell$  is required in [6].

Also, let the remaining  $2n - k$  eigenpairs of  $\mathcal{Q}(\lambda)$  be denoted by  $\{\lambda_j, \mathbf{x}_j\}_{j=k+1}^{2n}$ , and let the real representations of  $\{\lambda_j\}_{j=k+1}^{2n}$  and  $\{\mathbf{x}_j\}_{j=k+1}^{2n}$  be  $\Lambda_2$  and  $X_2$ , respectively.

Using the notations above, the EEP starts from the known matrix equality

$$(2.5) \quad M[X_1, X_2] \begin{bmatrix} \Lambda_1^2 & 0 \\ 0 & \Lambda_2^2 \end{bmatrix} + C[X_1, X_2] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} + K[X_1, X_2] = 0,$$

and we want to find a real symmetric matrix triplet  $(\tilde{M}, \tilde{C}, \tilde{K})$  with  $\tilde{M}$  nonsingular such that

$$(2.6) \quad \tilde{M}[\tilde{X}_1, X_2] \begin{bmatrix} \tilde{\Lambda}_1^2 & 0 \\ 0 & \Lambda_2^2 \end{bmatrix} + \tilde{C}[\tilde{X}_1, X_2] \begin{bmatrix} \tilde{\Lambda}_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} + \tilde{K}[\tilde{X}_1, X_2] = 0$$

for some  $n \times k$  real matrix  $\tilde{X}_1$ , which serves as the real representation of the eigenvectors corresponding to the new measured eigenvalues.

Let  $\Lambda$  be any  $m \times m$  real matrix. Define

$$(2.7) \quad \mathcal{D}_\Lambda := \left\{ D \in \mathbb{R}^{m \times m} \mid D = D^\top, D\Lambda = (D\Lambda)^\top \right\}.$$

The main theorem and algorithm of this paper are derived based on the following theorem, which has been proved in [2].

**Theorem 2.1.** Let  $(\Lambda, X) \in \mathbb{R}^{2n \times 2n} \times \mathbb{R}^{n \times 2n}$  with

$$(2.8) \quad U(\Lambda, X) := \begin{bmatrix} X \\ X\Lambda \end{bmatrix}$$

being nonsingular. Then there exists a real symmetric matrix triplet  $(M, C, K)$  with  $M$  nonsingular such that

$$(2.9) \quad MX\Lambda^2 + CX\Lambda + KX = 0$$

if and only if there exists a nonsingular  $D \in \mathcal{D}_\Lambda$  such that

$$(2.10) \quad XD^{-1}X^\top = 0,$$

where  $\mathcal{D}_\Lambda$  defined by (2.7). In this case, the triplet  $(M, C, K)$  is given by

$$(2.11) \quad \begin{cases} M = (X\Lambda D^{-1}X^\top)^{-1}, \\ C = -M(X\Lambda^2 D^{-1}X^\top)M, \\ K = -M(X\Lambda^3 D^{-1}X^\top)M + CM^{-1}C. \end{cases}$$

**Remark 2.1.** Notice that when a triplet  $(M, C, K)$  is represented in the form of (2.11), then it is easy to verify that

$$(2.12) \quad D = \begin{bmatrix} X \\ X\Lambda \end{bmatrix}^\top \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X \\ X\Lambda \end{bmatrix}.$$

In addition, it has been proved that the following two statements are equivalent:

- $U(\Lambda, X)$  defined by (2.8) is nonsingular and there exists a nonsingular  $D \in \mathcal{D}_\Lambda$  such that (2.10) holds;
- There exists a nonsingular  $D \in \mathcal{D}_\Lambda$  such that (2.10) holds and  $X\Lambda D^{-1}X^\top$  is nonsingular.

See [2] for more details.

As the end of this section, we make the following hypotheses, which we will not repeated in the rest of this paper.

- $\Lambda_1$  and  $\tilde{\Lambda}_1$  have only simple eigenvalues, i.e.,  $\lambda_i \neq \lambda_j$  and  $\tilde{\lambda}_i \neq \tilde{\lambda}_j$  for  $i \neq j$ ,  $i, j = 1, \dots, k$ .
- $\lambda(\Lambda_1) \cap \lambda(\Lambda_2) = \emptyset$  and  $\lambda(\tilde{\Lambda}_1) \cap \lambda(\Lambda_2) = \emptyset$ .
- $X_1$  defined in (2.2) is of full column rank.

### 3. SOLUTIONS OF THE EEP

In this section, based on Theorem 2.1 we will derive an expression of the parameterized solution to the EEP under some proper assumptions.

Let

$$(3.1) \quad \tilde{\Lambda} = \text{diag}(\tilde{\Lambda}_1, \Lambda_2).$$

By Theorem 2.1, from (2.6) we easily see that once we find a nonsingular matrix  $\tilde{D} \in \mathcal{D}_{\tilde{\Lambda}}$  such that

$$(3.2) \quad [\tilde{X}_1, X_2] \tilde{D}^{-1} [\tilde{X}_1, X_2]^\top = 0,$$

for some  $n \times k$  real matrix  $\tilde{X}_1$ , with

$$(3.3) \quad \begin{bmatrix} \tilde{X}_1 & X_2 \\ \tilde{X}_1 \tilde{\Lambda}_1 & X_2 \Lambda_2 \end{bmatrix}$$

being nonsingular, then one solution  $(\tilde{M}, \tilde{C}, \tilde{K})$  to the EEP can be given by (2.11) in terms of  $\tilde{D}$ ,  $\tilde{X}_1$ ,  $\tilde{\Lambda}_1$ ,  $X_2$ , and  $\Lambda_2$ . Noting  $\lambda(\tilde{\Lambda}_1) \cap \lambda(\Lambda_2) = \emptyset$ , we can easily derive that  $\tilde{D} \in \mathcal{D}_{\tilde{\Lambda}}$  if and only if  $\tilde{D} = \text{diag}(\tilde{D}_1, \tilde{D}_2)$ , where  $\tilde{D}_1 \in \mathcal{D}_{\tilde{\Lambda}_1}$  and  $\tilde{D}_2 \in \mathcal{D}_{\Lambda_2}$ . Thus, (3.2) can be rewritten as

$$(3.4) \quad \tilde{X}_1 \tilde{D}_1^{-1} \tilde{X}_1^\top + X_2 \tilde{D}_2^{-1} X_2 = 0.$$

Define

$$(3.5) \quad D_1 = \begin{bmatrix} X_1 \\ X_1 \Lambda_1 \end{bmatrix}^\top \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_1 \Lambda_1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} X_2 \\ X_2 \Lambda_2 \end{bmatrix}^\top \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X_2 \\ X_2 \Lambda_2 \end{bmatrix}.$$

Then, by Theorem 2.1, this, together with (2.5), gives rise to

$$(3.6) \quad X_1 D_1^{-1} X_1^\top + X_2 D_2^{-1} X_2 = 0.$$

Here we take

$$(3.7) \quad \tilde{D}_2 = D_2.$$

The reasons for doing this are threefold. First, we do not want the expression of the solutions to the EEP to involve  $X_2$  and  $\Lambda_2$ , since they are not given or hard to get. Second, according to the spectral decomposition theorem in [6],  $D_2$  can be interpreted as the normalization matrix of the eigenvector matrix  $X_2$ . Equality (3.7) means that the updating maintains the same normalization matrix for  $X_2$ . Third, it makes the problem much easier to handle, and both methods in [3] and [6] are carried out under this assumption.

Combining (3.4), (3.6), and (3.7), we get

$$(3.8) \quad \tilde{X}_1 \tilde{D}_1^{-1} \tilde{X}_1^\top = X_1 D_1^{-1} X_1^\top.$$

Noticing that for any  $\mathbf{z} \in \mathcal{N}(\tilde{X}_1^\top)$ , we have

$$X_1 D_1^{-1} X_1^\top \mathbf{z} = \tilde{X}_1 \tilde{D}_1^{-1} \tilde{X}_1^\top \mathbf{z} = 0,$$

which implies that

$$X_1^\top \mathbf{z} = 0,$$

since  $X_1$  is of full column rank and  $D_1$  is nonsingular. This shows that  $\mathcal{N}(\tilde{X}_1^\top) \subset \mathcal{N}(X_1^\top)$ . Noting that

$$\dim \mathcal{N}(\tilde{X}_1^\top) \geq n - k = \dim \mathcal{N}(X_1^\top),$$

we obtain  $\mathcal{N}(\tilde{X}_1^\top) = \mathcal{N}(X_1^\top)$ , and so, there exists a matrix  $Z \in \mathbb{R}^{k \times k}$  such that

$$(3.9) \quad \tilde{X}_1 = X_1 Z.$$

Substituting (3.9) into (3.8) and using the fact that  $X_1$  is of full column rank, we have

$$(3.10) \quad Z \tilde{D}_1^{-1} Z^\top = D_1^{-1},$$

which implies that  $Z$  is nonsingular. This shows that if (3.7) holds, then there exist matrices  $\tilde{X}_1$  and  $\tilde{D}_1$  such that (3.2) holds is equivalent to there exist matrices  $Z$  and  $\tilde{D}_1$  such that (3.10) holds.

Without loss of generosity, we may assume that  $D_1$  is in the following form

$$(3.11) \quad D_1 = \text{diag} \left( \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\ell}, \varepsilon_{2\ell+1}, \dots, \varepsilon_k \right),$$

where  $\varepsilon_j = 1$  or  $-1$ ,  $j = 2\ell+1, \dots, k$ . In fact, since  $\Lambda_1$  has only simple eigenvalues, we can always normalize the given eigenvectors  $\{\mathbf{x}_j\}_{j=1}^k$  so that (3.11) holds. See Algorithm 4.1 in the next section for more details.

On the other hand, since  $\tilde{\Lambda}_1$  has only simple eigenvalues, it follows immediately that  $\tilde{D}_1 \in \mathcal{D}_{\tilde{\Lambda}_1}$  if and only if the matrix  $\tilde{D}_1$  has the form

$$(3.12) \quad \tilde{D}_1 = \text{diag} \left( \begin{bmatrix} \tilde{\xi}_1 & \tilde{\eta}_1 \\ \tilde{\eta}_1 & -\tilde{\xi}_1 \end{bmatrix}, \dots, \begin{bmatrix} \tilde{\xi}_\ell & \tilde{\eta}_\ell \\ \tilde{\eta}_\ell & -\tilde{\xi}_\ell \end{bmatrix}, \tilde{\xi}_{2\ell+1}, \dots, \tilde{\xi}_k \right).$$

Furthermore, the equality (3.10) means that the matrix  $\tilde{D}_1$  is congruent to the matrix  $D_1$ . By Sylvester's law of inertia, this, together with (3.12), implies that  $\tilde{D}_1$  must have the following form

$$(3.13) \quad \tilde{D}_1 = \Gamma^\top P_1^\top D_1 P_1 \Gamma,$$

where

$$(3.14) \quad \Gamma := \text{diag} \left( \gamma_1 \begin{bmatrix} \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & \cos \theta_1 \end{bmatrix}, \dots, \gamma_\ell \begin{bmatrix} \cos \theta_\ell & \sin \theta_\ell \\ -\sin \theta_\ell & \cos \theta_\ell \end{bmatrix}, \gamma_{2\ell+1}, \dots, \gamma_k \right)$$

with  $\theta_j \in \mathbb{R}$  and  $\gamma_j > 0$ , and  $P_1$  is a  $k \times k$  permutation matrix with the property

$$(3.15) \quad P_1^\top D_1 P_1 = \text{diag} \left( \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\tilde{\ell}}, \tilde{\varepsilon}_{2\ell+1}, \dots, \tilde{\varepsilon}_k \right).$$

Observe that (3.15) implies that  $\tilde{\ell} \leq \min\{i_+(D_1), i_-(D_1)\}$ . This gives us a necessary condition for the existence of solutions to the EEP.

**Theorem 3.1.** If the EEP has a solution, then the number of complex conjugate pairs contained in the new measured eigenvalues  $\{\tilde{\lambda}_j\}_{j=1}^k$  cannot exceed the minimum of both the number of positive eigenvalues of  $D_1$  and the number of negative eigenvalues of  $D_1$ , where  $D_1$  is defined by (3.5) and uniquely determined by the given eigenpairs  $\{\lambda_i, \mathbf{x}_i\}_{i=1}^k$ .

As  $D_1^{-1}$  is in the form of (3.11), we are able to construct a  $k \times k$  permutation matrix  $P_2$  such that

$$(3.16) \quad D_1^{-1} = D_1 = P_2 J P_2^\top,$$

where

$$(3.17) \quad J := \begin{bmatrix} I_r & 0 \\ 0 & -I_s \end{bmatrix}, \quad r := i_+(D_1), \quad s := i_-(D_1).$$

Substituting (3.16) and (3.13) into (3.10) and setting

$$(3.18) \quad W := P_2^\top Z \Gamma^{-1} P_1^\top P_2,$$

we get

$$(3.19) \quad W J W^\top = J.$$

This matrix equation is always solvable and an expression of its general solution can be given, which are summarized below.

**Lemma 3.1.** The general solution of equation (3.19) is given by

$$(3.20) \quad W = \begin{bmatrix} (I_r + Y Y^\top)^{\frac{1}{2}} Q_r^\top & Y \\ Q_s Y^\top Q_r^\top & Q_s (I_s + Y^\top Y)^{\frac{1}{2}} \end{bmatrix},$$

where  $Q_r \in \mathbb{R}^{r \times r}$ ,  $Q_s \in \mathbb{R}^{s \times s}$  are arbitrary orthogonal matrices, and  $Y \in \mathbb{R}^{r \times s}$  is arbitrary.

*Proof.* For any matrix  $A \in \mathbb{R}^{m \times n}$ , it is easy to verify that the matrix equality

$$(3.21) \quad (I_m + A A^\top)^{\frac{1}{2}} A = A (I_n + A^\top A)^{\frac{1}{2}}$$

always holds. Using this equality, we can easily check that  $W$  in the form of (3.20) is a solution of (3.19) for any given  $Q_r$ ,  $Q_s$ , and  $Y$ . Therefore, it suffices, if we can prove that any solution  $W$  of equation (3.19) must have the form of (3.20).

Partition  $W$  as

$$(3.22) \quad W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$

where  $W_{11} \in \mathbb{R}^{r \times r}$  and  $W_{22} \in \mathbb{R}^{s \times s}$ , then it follows immediately from (3.19) that

$$(3.23) \quad \begin{cases} W_{11} W_{11}^\top - W_{12} W_{12}^\top = I_r, \\ W_{21} W_{21}^\top - W_{22} W_{22}^\top = -I_s, \\ W_{11} W_{21}^\top - W_{12} W_{22}^\top = 0. \end{cases}$$

Using the pole decomposition of matrices, it is easy to derive from the first two equations of (3.23) that there exist two orthogonal matrices  $Q_r \in \mathbb{R}^{r \times r}$  and  $Q_s \in \mathbb{R}^{s \times s}$  such that

$$(3.24) \quad W_{11} = (I_r + W_{12} W_{12}^\top)^{\frac{1}{2}} Q_r^\top,$$

$$(3.25) \quad W_{22} = (I_s + W_{21} W_{21}^\top)^{\frac{1}{2}} Q_s.$$

Then substituting (3.24) and (3.25) into the last equation of (3.23), we get

$$(I_r + W_{12}W_{12}^\top)^{\frac{1}{2}}Q_r^\top W_{21}^\top = W_{12}Q_s^\top (I_s + W_{21}W_{21}^\top)^{\frac{1}{2}},$$

or

$$(3.26) \quad Q_r(I_r + W_{12}W_{12}^\top)^{-\frac{1}{2}}W_{12} = W_{21}^\top (I_s + W_{21}W_{21}^\top)^{-\frac{1}{2}}Q_s.$$

Let the SVD decompositions of  $W_{12}$  and  $W_{21}^\top$  be

$$W_{12} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top \quad \text{and} \quad W_{21}^\top = \tilde{U} \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \tilde{V}^\top,$$

respectively, where  $U, \tilde{U} \in \mathbb{R}^{r \times r}$  and  $V, \tilde{V} \in \mathbb{R}^{s \times s}$  are orthogonal matrices,  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_t)$  with  $t = \text{rank}(W_{12})$  and  $\sigma_1 \geq \dots \geq \sigma_t > 0$ ,  $\tilde{\Sigma} = \text{diag}(\hat{\sigma}_1, \dots, \hat{\sigma}_{\hat{t}})$  with  $\hat{t} = \text{rank}(W_{21})$  and  $\hat{\sigma}_1 \geq \dots \geq \hat{\sigma}_{\hat{t}} > 0$ . Substituting them into (3.26) yields

$$(3.27) \quad Q_r U \begin{bmatrix} \Sigma(I_t + \Sigma^2)^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} V^\top = \tilde{U} \begin{bmatrix} \tilde{\Sigma}(I_{\hat{t}} + \tilde{\Sigma}^2)^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \tilde{V}^\top Q_s,$$

which implies  $t = \hat{t}$  and

$$\Sigma(I_t + \Sigma^2)^{-\frac{1}{2}} = \tilde{\Sigma}(I_t + \tilde{\Sigma}^2)^{-\frac{1}{2}},$$

and so  $\Sigma = \tilde{\Sigma}$ . Then, by the properties of SVD decompositions, we know that  $U, V, \tilde{U}, \tilde{V}$  can be chosen so that  $Q_r U = \tilde{U}$  and  $V^\top = \tilde{V}^\top Q_s$ . Thus we have

$$Q_r U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top = \tilde{U} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tilde{V}^\top Q_s,$$

which leads to

$$(3.28) \quad W_{21} = Q_r^\top W_{12}^\top Q_s.$$

Then substituting it into (3.25), we get  $W_{22} = Q_s(I_s + W_{12}^\top W_{12})^{\frac{1}{2}}$ , which completes the proof of the lemma.  $\blacksquare$

In summary, the arguments lead to the parameterized expressions of the required matrices  $\tilde{D}_1$  and  $\tilde{X}_1$  in (3.4) as follows:

$$(3.29) \quad \tilde{D}_1 = \Gamma^\top P_1^\top D_1 P_1 \Gamma, \quad \tilde{X}_1 = X_1 Z = X_1 P_2 W P_2^\top P_1 \Gamma,$$

where the matrices  $\Gamma$ ,  $W$ , and  $P_1$  can be chosen, while the matrices  $X_1$ ,  $D_1$ , and  $P_2$  are determined by given data.

Thus, applying Theorem 2.1, we can now state our main theorem concerning the solutions to the EEP below.

**Theorem 3.2.** Let the notations be defined as the above, assume that  $D_1$  defined by (3.5) has the form as in (3.11) and  $\tilde{\ell} \leq \min\{i_+(D_1), i_-(D_1)\}$ . For any given  $W$  in the form of (3.20) and  $P_1$  with the property (3.15), if the matrix  $M^{-1} + \check{X}_1(W\tilde{\Omega}_1JW^\top - \Omega_1J)\check{X}_1^\top$  is nonsingular, where  $\check{X}_1 = X_1P_2$ ,  $\Omega_1 = P_2^\top\Lambda_1P_2$ ,  $\tilde{\Omega}_1 = P_2^\top P_1\tilde{\Lambda}_1P_1^\top P_2$ , then one solution to the EEP is given by

$$(3.30) \quad \begin{aligned} \tilde{M} &= [M^{-1} + \check{X}_1(W\tilde{\Omega}_1JW^\top - \Omega_1J)\check{X}_1^\top]^{-1}, \\ \tilde{C} &= \tilde{M}[M^{-1}CM^{-1} - \check{X}_1(W\tilde{\Omega}_1^2JW^\top - \Omega_1^2J)\check{X}_1^\top]\tilde{M}, \\ \tilde{K} &= \tilde{M}[M^{-1}(K - CM^{-1}C)M^{-1} \\ &\quad - \check{X}_1(W\tilde{\Omega}_1^3JW^\top - \Omega_1^3J)\check{X}_1^\top]\tilde{M} + \tilde{C}\tilde{M}^{-1}\tilde{C}. \end{aligned}$$

*Proof.* Using (3.29) and (3.16) and noting that  $\Gamma\tilde{\Lambda}_1\Gamma^{-1} = \tilde{\Lambda}_1$ , we have

$$(3.31) \quad \begin{aligned} &\tilde{X}_1\tilde{\Lambda}_1\tilde{D}_1^{-1}\tilde{X}_1^\top \\ &= (X_1P_2WP_2^\top P_1\Gamma)\tilde{\Lambda}_1(\Gamma^{-1}P_1^\top D_1P_1\Gamma^{-\top})(\Gamma^\top P_1^\top P_2W^\top P_2^\top X_1^\top) \\ &= X_1P_2WP_2^\top P_1\tilde{\Lambda}_1P_1^\top D_1P_2W^\top P_2^\top X_1^\top \\ &= X_1P_2WP_2^\top P_1\tilde{\Lambda}_1P_1^\top P_2JW^\top P_2^\top X_1^\top \\ &= \check{X}_1W\tilde{\Omega}_1JW^\top\check{X}_1^\top. \end{aligned}$$

Applying Theorem 2.1, we have

$$(3.32) \quad M^{-1} = X_1\Lambda_1D_1^{-1}X_1^\top + X_2\Lambda_2D_2^{-1}X_2^\top,$$

$$(3.33) \quad \tilde{M}^{-1} = \tilde{X}_1\tilde{\Lambda}_1\tilde{D}_1^{-1}\tilde{X}_1^\top + X_2\Lambda_2D_2^{-1}X_2^\top,$$

provided that the matrix

$$(3.34) \quad \tilde{X}_1\tilde{\Lambda}_1\tilde{D}_1^{-1}\tilde{X}_1^\top + X_2\Lambda_2D_2^{-1}X_2^\top$$

is nonsingular. Combining (3.33), (3.32), and (3.31) yields

$$(3.35) \quad \begin{aligned} \tilde{M}^{-1} - M^{-1} &= \tilde{X}_1\tilde{\Lambda}_1\tilde{D}_1^{-1}\tilde{X}_1^\top - X_1\Lambda_1D_1^{-1}X_1^\top \\ &= \check{X}_1(W\tilde{\Omega}_1JW^\top - \Omega_1J)\check{X}_1^\top, \end{aligned}$$

from which the first equality of (3.30) follows, and moreover, we can easily see that the assumption implies that the matrix defined by (3.34) is nonsingular.

Similarly, we have

$$\begin{aligned} \widetilde{M}^{-1}\widetilde{C}\widetilde{M}^{-1} - M^{-1}CM^{-1} &= -\check{X}_1(W\widetilde{\Omega}_1^2JW^\top - \Omega_1^2J)\check{X}_1^\top, \\ \widetilde{M}^{-1}(\widetilde{K} - \widetilde{C}\widetilde{M}^{-1}\widetilde{C})\widetilde{M}^{-1} - M^{-1}(K - CM^{-1}C)M^{-1} \\ &= -\check{X}_1(W\widetilde{\Omega}_1^3JW^\top - \Omega_1^3J)\check{X}_1^\top, \end{aligned}$$

from which the last two equalities of (3.30) can be derived. This completes the proof. ■

**Remark 3.1.** It is critically important to note that the update formula (3.30) from  $Q(\lambda)$  to  $\widetilde{Q}(\lambda)$  does not need the information about  $(\Lambda_2, X_2)$ . In addition, if  $\ell = \widetilde{\ell}$  and we take  $W = I_k$  and  $P_1 = I_k$  in (3.30), then  $(\widetilde{M}, \widetilde{C}, \widetilde{K})$  defined by (3.30) is exactly the same as the update formula gave in [6]. However, since  $W$  and  $P_1$  in (3.30) can be chosen, we can properly choose the two matrices so that some other requirements may be satisfied. Sometimes even in the case the method of [6] fails to solve the EEP, we can still solve the EEP using (3.30) (see Example 5.1 in Section 5).

**Remark 3.2.** When  $n$  is large, directly using (3.30) to calculate  $(\widetilde{M}, \widetilde{C}, \widetilde{K})$  is numerically expensive, since the inverse of  $M$  is required there. As a matter of fact, using the Sherman-Morrison-Woodbury formula, we can get the following more useful equivalence form:

$$\begin{aligned} \widetilde{M} &= R_1M, \\ \widetilde{C} &= R_1R_2R_1^\top, \\ \widetilde{K} &= R_1(K - MS_3M + R_2S_1R_2 \\ &\quad - MS_2C - CS_2M + MS_2MS_2M)R_1^\top, \end{aligned} \tag{3.36}$$

where  $S_1, S_2, S_3, R_1$ , and  $R_2$  are given by

$$\begin{aligned} S_1 &:= -\check{X}_1(W\widetilde{\Omega}_1JW^\top - \Omega_1J) \\ &\quad [I_k + \check{X}_1^\top M\check{X}_1(W\widetilde{\Omega}_1JW^\top - \Omega_1J)]^{-1}\check{X}_1^\top, \\ S_2 &:= \check{X}_1(W\widetilde{\Omega}_1^2JW^\top - \Omega_1^2J)\check{X}_1^\top, \quad R_1 := I + MS_1, \\ S_3 &:= \check{X}_1(W\widetilde{\Omega}_1^3JW^\top - \Omega_1^3J)\check{X}_1^\top, \quad R_2 := C - MS_2M. \end{aligned} \tag{3.37}$$

Notice that here only the inverse of a  $k \times k$  matrix is needed, and so, when  $kl - n$ , using (3.36) to compute the solution to the EEP is far much cheaper than using (3.30).

**Remark 3.3.** From (3.36) it is not difficult to derive that

- (a)  $M^{-1} + \check{X}_1(W\check{\Omega}_1JW^\top - \Omega_1J)\check{X}_1^\top$  is nonsingular if and only if  $I_k + \check{X}_1^\top M\check{X}_1(W\check{\Omega}_1JW^\top - \Omega_1J)$  is nonsingular;
- (b)  $\widetilde{M}$  is positive definite if and only if all the eigenvalues of  $I_k + \check{X}_1^\top M\check{X}_1(W\check{\Omega}_1JW^\top - \Omega_1J)$  are positive, provided that  $M$  is positive definite.

#### 4. ALGORITHMS

In this section we consider how to use the theory established in Section 3 to develop a feasible and efficient algorithm for solving the EEP. In fact, Theorem 3.1 has given us a recipe for doing this. In Summary, the computation of the solution to the EEP requires three major steps:

**Step 1.** Normalize the given eigenvectors  $\{\mathbf{x}_j\}_{j=1}^k$  so that the matrix  $D_1$  defined by (3.5) has the form of (3.11), and find the numbers  $r, s$  and the permutation matrix  $P_2$  in (3.16).

**Step 2.** Select the permutation matrix  $P_1$  with the property (3.15) and the matrix  $W$  in the form of (3.20) so that some requirements are satisfied.

**Step 3.** Compute the symmetric matrices  $\widetilde{M}, \widetilde{C}$ , and  $\widetilde{K}$  by (3.36).

Notice that the assumption on  $\Lambda_1$  implies that the matrix  $D_1$  must have the following form

$$D_1 = \text{diag} \left( \begin{bmatrix} \xi_1 & \eta_1 \\ \eta_1 & -\xi_1 \end{bmatrix}, \dots, \begin{bmatrix} \xi_\ell & \eta_\ell \\ \eta_\ell & -\xi_\ell \end{bmatrix}, \xi_{2\ell+1}, \dots, \xi_k \right),$$

where  $\xi_i, \eta_i \in \mathbb{R}$ . It is easy to derive that Step 1 can be carried out by the following algorithm.

#### Algorithm 4.1.

**for**  $j = 1 : \ell$

$$\xi_j = \mathbf{x}_{jR}^\top C \mathbf{x}_{jR} + 2\mathbf{x}_{jR}^\top M(\alpha_j \mathbf{x}_{jR} - \beta_j \mathbf{x}_{jI})$$

$$\eta_j = \mathbf{x}_{jR}^\top C \mathbf{x}_{jI} + 2\alpha_j \mathbf{x}_{jR}^\top M \mathbf{x}_{jI} + \beta_j(\mathbf{x}_{jR}^\top M \mathbf{x}_{jR} + \mathbf{x}_{jI}^\top M \mathbf{x}_{jI})$$

$$\omega_j = \sqrt{\xi_j^2 + \eta_j^2}, \quad \zeta_j = \frac{\xi_j}{\omega_j}, \quad \cos \theta_j = -\text{sign}(\eta_j) \sqrt{\frac{1 + \zeta_j}{2}}, \quad \sin \theta_j = \sqrt{\frac{1 - \zeta_j}{2}}$$

$$[\mathbf{x}_{jR}, \mathbf{x}_{jI}] := \omega_j^{-\frac{1}{2}} [\mathbf{x}_{jR}, \mathbf{x}_{jI}] \begin{bmatrix} \cos \theta_j & \sin \theta_j \\ -\sin \theta_j & \cos \theta_j \end{bmatrix}$$

**end**

$$r := \ell, \quad s := \ell$$

```

for  $j = 2\ell + 1 : k$ 
     $\xi_j = \mathbf{x}_j^\top C \mathbf{x}_j + 2\lambda_j \mathbf{x}_j^\top M \mathbf{x}_j$ 
     $\mathbf{x}_j := |\xi_j|^{-\frac{1}{2}} \mathbf{x}_j$ 
    if  $\xi_j > 0$ ,  $r := r + 1$ , else  $s := s + 1$  end
end
Reorder the given real eigenpairs  $\{\lambda_j, \mathbf{x}_j\}_{j=2\ell+1}^k$  into  $\{\lambda_{i_j}, \mathbf{x}_{i_j}\}_{j=2\ell+1}^k$  so that
     $\xi_{i_{2j-1}} > 0, \quad \xi_{i_{2j}} < 0, \quad j = \ell + 1, \dots, \tau := \min\{r, s\}$ 
     $\text{sign}(\xi_{i_j}) = \text{sign}(r - s), \quad j = 2\tau + 1, \dots, k$ 
 $\lambda_j := \lambda_{i_j}, \quad \mathbf{x}_j := \mathbf{x}_{i_j}, \quad j = 2\ell + 1, \dots, k$ 
if  $r > s$ 
     $P_2 := [e_1, e_3, \dots, e_{2\tau-1}, e_{2\tau+1}, e_{2\tau+2}, \dots, e_k, e_2, e_4, \dots, e_{2\tau}]$ 
else
     $P_2 := [e_1, e_3, \dots, e_{2\tau-1}, e_2, e_4, \dots, e_{2\tau}, e_{2\tau+1}, e_{2\tau+2}, \dots, e_k]$ 
end
 $D_1 := \text{diag} \left( \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\tau}, \text{sign}(r - s) I_{k-2\tau} \right)$ 

```

This algorithm not only normalizes the given eigenvectors, but also reorders the given real eigenvalues, which leads to the matrix  $D_1$  defined by (3.5) has the form as shown in Algorithm 4.1. Of course, there is more than meets the requirement of (3.11). However, it is convenient for us to carry out the following steps.

In the rest of this section, we will always assume that the given eigenvectors have been normalized and the given real eigenvalues have been reordered by Algorithm 4.1.

Step 2 is the key step of this method. Let us first consider how to select the permutation matrix  $P_1$ . In fact, from the computation formula (3.30), we can see that the role of  $P_1$  played in (3.30) is essentially to specify a correspondence between the set  $\{\lambda_i\}_{i=1}^k$  and  $\{\tilde{\lambda}_i\}_{i=1}^k$ , that is, to specify a permutation  $\{i_1, \dots, i_k\}$  of  $\{1, \dots, k\}$  such that  $\lambda_i$  is replaced by  $\tilde{\lambda}_{i_j}$  in the updated model for  $i = 1, \dots, k$ . Conversely, if there is a permutation  $\{i_1, \dots, i_k\}$  of  $\{1, \dots, k\}$  such that

- $\tilde{\lambda}_{i_{2j-1}}$  and  $\tilde{\lambda}_{i_{2j}}$  are complex conjugate or both real for  $j = 1, 2, \dots, \tau$  and
- $\tilde{\lambda}_{i_j}$  is real for all  $j = 2\tau + 1, \dots, k$ ,

the permutation matrix  $P_1$  is defined by

$$(4.1) \quad P_1^\top := [e_{i_1}, e_{i_2}, \dots, e_{i_k}]$$

must satisfy (3.15). The permutation with the above properties is referred to as *compatible permutation*. Clearly, a compatible permutation exists if and only if  $\tilde{\ell} \leq \tau$ .

Thus, selecting a permutation matrix  $P_1$  with the property (3.15) amounts to choosing a compatible permutation  $\{i_1, \dots, i_k\}$  of  $\{1, \dots, k\}$ . In practice, such a permutation is usually known, which is usually given by engineers or designers. In summary, if a compatible permutation is given, the matrix  $P_1 \tilde{\Lambda}_1 P_1^\top$  in the computation formula (3.30) can be determined by the following algorithm.

**Algorithm 4.2.**

```

for  $j = 1 : \tau$ 
  if  $\tilde{\lambda}_{i_{2j-1}} = \tilde{\alpha}_{i_j} + i\tilde{\beta}_{i_j}$  with  $\tilde{\beta}_{i_j} > 0$ 
     $\tilde{\Lambda}_j^{[2]} := \begin{bmatrix} \tilde{\alpha}_{i_j} & \tilde{\beta}_{i_j} \\ -\tilde{\beta}_{i_j} & \tilde{\alpha}_{i_j} \end{bmatrix}$ 
  else
     $\tilde{\Lambda}_j^{[2]} := \text{diag}(\tilde{\lambda}_{i_{2j-1}}, \tilde{\lambda}_{i_{2j}})$ 
  end
end
 $\tilde{\Lambda}_1 := P_1 \tilde{\Lambda}_1 P_1^\top = \text{diag}(\tilde{\Lambda}_1^{[2]}, \dots, \tilde{\Lambda}_\tau^{[2]}, \tilde{\lambda}_{i_{2\tau+1}}, \dots, \tilde{\lambda}_{i_k})$ 

```

We next consider how to select the matrix  $W$ . In practice, we prefer the adjustments made to the coefficient matrices as small as possible. Let

$$\kappa := \|\check{X}_1(W\tilde{\Omega}_1 J W^\top - \Omega_1 J)\check{X}_1^\top M\|.$$

Similar to the discussion in [12], we can prove that if  $\kappa < 1$ , then

$$(4.2) \quad \frac{\|\tilde{M} - M\|}{\|M\|} \leq \frac{\kappa}{1 - \kappa},$$

and moreover,  $\|\tilde{C} - C\| = O(\kappa)$ ,  $\|\tilde{K} - K\| = O(\kappa)$  as  $\kappa \rightarrow 0$ . Consequently, it is nature to select  $W$  in the form of (3.20) so that

$$(4.3) \quad \|\check{X}_1(W\tilde{\Omega}_1 J W^\top - \Omega_1 J)\check{X}_1^\top M\|_F^2 = \min.$$

Of course, other optimization problems can be proposed to minimize the adjustments made to the coefficient matrices, but this one is noteworthy and important. However, it seems difficult to get a globally optimal solution to the above optimization problem. Consequently, we only develop a numerical method for computing a sub-optimal solution to the optimization problem (4.3).

Notice that, in (3.20), if the orthogonal matrices  $Q_r$  and  $Q_s$  are diagonal matrices, and  $Y$  has only nonzero elements on its main diagonal, then we have

$$(4.4) \quad \widetilde{W} := P_2 W P_2^\top = \text{diag}(W_1^{[2]}, \dots, W_\tau^{[2]}, E_{k-2\tau}),$$

where  $P_2$  is generated by Algorithm 4.1,  $E_{k-2\tau}$  is a diagonal matrix with only 1 or  $-1$  on its diagonal, and

$$(4.5) \quad W_j^{[2]} = \begin{bmatrix} r_j \sqrt{1 + \sigma_j^2} & \sigma_j \\ r_j s_j \sigma_j & s_j \sqrt{1 + \sigma_j^2} \end{bmatrix},$$

with  $r_j = \pm 1$ ,  $s_j = \pm 1$ , and  $\sigma_j$  being any real numbers for  $j = 1, \dots, \tau$ .

Also, if  $W$  takes in this special form, (3.37) can be rewritten as

$$(4.6) \quad \begin{aligned} S_1 &= -X_1 (\widetilde{W} \widetilde{\Lambda}_1 D_1 \widetilde{W}^\top - \Lambda_1 D_1) \\ &\quad \left[ I_k + X_1^\top M X_1 (\widetilde{W} \widetilde{\Lambda}_1 D_1 \widetilde{W}^\top - \Lambda_1 D_1) \right]^{-1} X_1^\top, \\ S_2 &= X_1 (\widetilde{W} \widetilde{\Lambda}_1^2 D_1 \widetilde{W}^\top - \Lambda_1^2 D_1) X_1^\top, \\ S_3 &= X_1 (\widetilde{W} \widetilde{\Lambda}_1^3 D_1 \widetilde{W}^\top - \Lambda_1^3 D_1) X_1^\top, \end{aligned}$$

where  $D_1$  and  $\widetilde{\Lambda}_1$  are generated by Algorithm 4.1 and 4.2, respectively.

Based on those facts, here we will only select  $W$  in the form of (4.4) so that (4.3) holds. Notice that the matrices  $S_1, S_2, S_3$  are independent of the choice of the matrix  $E_{k-2\tau}$  in (4.4), so we here simply take  $E_{k-2\tau} = I_{k-2\tau}$ . Thus, under this limitation, the optimization problem (4.3) can be reduced into the following  $\tau$  optimization problems:

$$(4.7) \quad \begin{aligned} f(\sigma_j; r_j, s_j) &:= \left\| X_j^{[2]} (W_j^{[2]} \widetilde{\Lambda}_j^{[2]} J_2 W_j^{[2]\top} \right. \\ &\quad \left. - \Lambda_j^{[2]} J_2) X_j^{[2]\top} M \right\|_F^2 = \min, \quad j = 1, \dots, \tau, \end{aligned}$$

where  $\widetilde{\Lambda}_j^{[2]}$  is given in Algorithm 4.2,  $W_j^{[2]}$  is defined by (4.5), and

$$J_2 := \text{diag}(1, -1), \quad X_j^{[2]} := X_1(:, 2j-1 : 2j), \quad \Lambda_j^{[2]} := \Lambda_1(2j-1 : 2j, 2j-1 : 2j).$$

Notice that, for each  $j = 1, 2, \dots, \tau$ , if we have found four real numbers  $\sigma_j^{(i)}$ ,  $i = 1, 2, 3, 4$ , so that

$$(4.8) \quad \begin{aligned} f(\sigma_j^{(1)}; 1, 1) &= \min_{\sigma_j} f(\sigma_j; 1, 1), \quad f(\sigma_j^{(2)}; -1, 1) = \min_{\sigma_j} f(\sigma_j; -1, 1), \\ f(\sigma_j^{(3)}; 1, -1) &= \min_{\sigma_j} f(\sigma_j; 1, -1), \quad f(\sigma_j^{(4)}; -1, -1) = \min_{\sigma_j} f(\sigma_j; -1, -1), \end{aligned}$$

and

$$(\hat{\sigma}_j, \hat{r}_j, \hat{s}_j) \in \left\{ (\sigma_j^{(1)}, 1, 1), (\sigma_j^{(2)}, -1, 1), (\sigma_j^{(3)}, 1, -1), (\sigma_j^{(4)}, -1, -1) \right\}$$

such that

$$(4.9) \quad f(\hat{\sigma}_j, \hat{r}_j, \hat{s}_j) = \min \left\{ f(\sigma_j^{(1)}; 1, 1), f(\sigma_j^{(2)}; -1, 1), f(\sigma_j^{(3)}; 1, -1), f(\sigma_j^{(4)}; -1, -1) \right\},$$

then  $(\hat{\sigma}_j, \hat{r}_j, \hat{s}_j)$  must solve the optimization problem (4.7). For each of the four optimization problems in (4.8), it is to minimize an univariate function, which is twice continuously differentiable for  $\sigma_j$ , and hence its local minimum near zero can be obtained by Newton's method (see Program 50-newton in [13]) with the initial value being zero.

Our discussions above are summarized in the algorithm below.

**Algorithm 4.3.**

Set  $u_1 = u_2 = v_1 = v_3 = 1, u_3 = u_4 = v_2 = v_4 = -1$

**for**  $j = 1 : \tau$

$\sigma_j := 0, r_j := 1, s_j := 1, f := f(0; 1, 1)$

**for**  $i = 1 : 4$

$\sigma := 0$

**while**  $|f'(\sigma; u_i, v_i)| > \epsilon$  (*tolerance*)

$\sigma := \sigma - f'(\sigma; u_i, v_i) / f''(\sigma; u_i, v_i)$

**end**

**if**  $f(\sigma; u_i, v_i) < f$

$r_j := u_i, s_j := v_i, \sigma_j := \sigma, f := f(\sigma; u_i, v_i)$

**end**

**end**

$$W_j^{[2]} = \begin{bmatrix} r_j \sqrt{1 + \sigma_j^2} & \sigma_j \\ r_j s_j \sigma_j & s_j \sqrt{1 + \sigma_j^2} \end{bmatrix}$$

**end**

$$W := \text{diag}(W_1^{[2]}, \dots, W_\tau^{[2]}, I_{k-2\tau})$$

Step 3 only involves matrix-matrix operations, which can be carried out using standard matrix computation subroutines.

In summary, we have the following algorithm for the computation of a sub-optimal solution to the EEP.

**Algorithm 4.4.** (Compute a Sub-Optimal Solution to the EEP)

**Input:** The given data  $M, C, K, \Lambda_1, X_1, \tilde{\Lambda}_1$  as described in Section 2, the given permutation  $\{i_1, \dots, i_k\}$  of  $\{1, \dots, k\}$ , and a tolerance  $\epsilon$ .

**Output:** A sub-optimal solution  $(\tilde{M}, \tilde{C}, \tilde{K})$  to the EEP.

1. Apply Algorithm 4.1 to generate  $D_1$  and the reordered  $\Lambda_1$  and  $X_1$ .
2. Apply Algorithm 4.2 to generate the ordered  $\tilde{\Lambda}_1$ .
3. Apply Algorithm 4.3 to generate  $W$ .
4. Compute  $A = I_k + X_1^\top M X_1 (W \tilde{\Lambda}_1 D_1 W^\top - \Lambda_1 D_1)$ .
5. Solve  $AV = X_1^\top$  for  $V$  using Gaussian elimination with partial pivoting.
6. Compute
 
$$S_1 = -X_1 (W \tilde{\Lambda}_1 D_1 W^\top - \Lambda_1 D_1) V, S_2 = X_1 (W \tilde{\Lambda}_1^2 D_1 W^\top - \Lambda_1^2 D_1) X_1^\top,$$

$$S_3 = X_1 (W \tilde{\Lambda}_1^3 D_1 W^\top - \Lambda_1^3 D_1) X_1^\top, R_1 = I + M S_1, R_2 = C - M S_2 M.$$
7. Compute  $(\tilde{M}, \tilde{C}, \tilde{K})$  by (3.36).

Observe that this algorithm is rich in Basic Linear Algebra Subroutine-3 (BLAS-3) level operations, and so it can be implemented using high-performance software packages such as LAPACK on today's high-speed computers. Moreover, our numerical experiments show that this algorithm is feasible and efficient.

## 5. NUMERICAL EXAMPLES

In this section, we illustrate the the feasibility and efficiency of the present algorithm using some numerical examples. All numerical examples were performed on a Pentium(R) 4.0/2.40G Hz computer using MATLAB 6.5.1, and the tolerance  $\epsilon$  is taken to be  $10^{-13}$ .

**Example 5.1.** Consider a spring system with

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 10 & -2 \\ -2 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}.$$

Direct calculation gives rises to all 4 eigenpairs  $\{\lambda_j, \mathbf{x}_j\}_{j=1}^4$  of  $Q(\lambda) = M\lambda^2 + C\lambda + K$  as follows:

$$\lambda_1 = -1, \quad \lambda_2 = -3, \quad \lambda_3 = \bar{\lambda}_4 = -1 + i,$$

$$\mathbf{x}_1 = (1, 1)^\top, \quad \mathbf{x}_2 = (1, 0)^\top, \quad \mathbf{x}_3 = \bar{\mathbf{x}}_4 = (1, 2)^\top + i(-1, 0)^\top.$$

Next we shall use Algorithm 4.4 described in Section 4 to update  $Q(\lambda)$  to  $\tilde{Q}(\lambda)$  such that the eigenvalues of  $\Lambda_1$  are replaced by the eigenvalues of  $\tilde{\Lambda}_1$ , where  $\Lambda_1$  and  $\tilde{\Lambda}_1$  are given in the following 4 different cases.

**Case 1.** Assume that  $\Lambda_1 = \text{diag}(-1, -3)$  and  $\tilde{\Lambda}_1 = \text{diag}(-1.5, -4)$ .

In this case, if the eigenvector matrix  $\tilde{X}_1$  corresponding to  $\tilde{\Lambda}_1$  is kept the same as the one corresponding to  $\Lambda_1$ , i.e.,  $\tilde{X}_1 = X_1$ , then direct calculation shows that  $M^{-1} + X_1(\tilde{\Lambda}_1 - \Lambda_1)D_1X_1^T$  is singular. Consequently, for those given data the method proposed in [6] fails.

If the method proposed in [3] is used, when  $-1$  is replaced by  $-1.5$  in the first step of the updating, it leads to the loss of positivity of the resulting stiffness matrix, which causes the updating to break down prematurely.

Now applying Algorithm 4.4 proposed in Section 4, with  $\{i_1, i_2\} = \{1, 2\}$ , we get

$$\tilde{M} = \begin{bmatrix} 2.0762 & 1.5091 \\ 1.5091 & 3.0538 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 16.3409 & 5.0496 \\ 5.0496 & -2.0181 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 32.5923 & -9.1704 \\ -9.1704 & 4.0762 \end{bmatrix},$$

where the mass matrix  $\tilde{M}$  is not only nonsingular, but also positive definite. However, the solution  $(\tilde{M}, \tilde{C}, \tilde{K})$  above is infeasible for the spring system since  $\tilde{C}(2, 2) = -2.0181 < 0$ . But if  $\tilde{W}$  in (4.6) is taken as

$$\tilde{W} = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix},$$

we can obtain a feasible solution given by

$$\tilde{M} = \begin{bmatrix} 0.4000 & -0.6000 \\ -0.6000 & 1.2333 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 1.2000 & -1.1333 \\ -1.1333 & 2.2000 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 1.7333 & -1.6000 \\ -1.6000 & 2.4000 \end{bmatrix}.$$

This shows that we can properly choose the parameters in the solution (3.36) so that some other requirements may be satisfied.

**Case 2.** Assume that  $\Lambda_1 = \text{diag}(-1, -3)$  and  $\tilde{\Lambda}_1 = \text{diag}(-1.05, -3.05)$ .

Using the method proposed in [6], we have

$$\tilde{M} = \begin{bmatrix} 2.1170 & 0.1114 \\ 0.1114 & 1.0585 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 10.9186 & -1.7772 \\ -1.7772 & 0.7772 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 13.5933 & -6.4568 \\ -6.4568 & 4.1170 \end{bmatrix}.$$

Notice that adjustments made to the coefficient matrices are moderately small when the updated eigenvalues are “near” the original ones. As a matter of fact, direct calculation gives

$$\|\tilde{M} - M\| = 0.2029, \quad \|\tilde{C} - C\| = 0.9558, \quad \|\tilde{K} - K\| = 1.7232.$$

While using Algorithm 4.4, with  $\{i_1, i_2\} = \{1, 2\}$ , we get

$$\tilde{M} = \begin{bmatrix} 1.9331 & 0.0576 \\ 0.0576 & 1.0538 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 9.9630 & -1.7102 \\ -1.7102 & 0.8848 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 12.4088 & -5.9815 \\ -5.9815 & 3.9331 \end{bmatrix},$$

and

$$\|\widetilde{M} - M\| = 0.0899, \quad \|\widetilde{C} - C\| = 0.3685, \quad \|\widetilde{K} - K\| = 0.4095.$$

Observe that our results are much better.

**Case 3.** Assume that  $\Lambda_1 = \text{diag}(-1, -3)$  and  $\widetilde{\Lambda}_1 = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$ , which has a pair of complex conjugate eigenvalues  $-2 \pm i$ .

In this case, the updating requires that two real eigenvalues are replaced by a complex conjugate pair of eigenvalues, so the methods proposed in [3] and [6] cannot be used. While applying Algorithm 4.4, with  $\{i_1, i_2\} = \{1, 2\}$ , we get

$$\widetilde{M} = \begin{bmatrix} 1.5624 & 0.9236 \\ 0.9236 & 6.1908 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} 11.9440 & 10.6666 \\ 10.6666 & -0.8473 \end{bmatrix}, \quad \widetilde{K} = \begin{bmatrix} 38.4019 & -6.9720 \\ -6.9720 & 3.5624 \end{bmatrix}.$$

Observe that  $\widetilde{M}$  and  $\widetilde{K}$  are positive definite.

**Case 4.** Assume that  $\Lambda_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$  and  $\widetilde{\Lambda}_1 = \text{diag}(-0.5, -1.5)$ .

In this case, the updating requires that a complex conjugate pair of eigenvalues are replaced by two real eigenvalues. Of course, for those given data, the methods proposed in [3] and [6] cannot be used. Applying Algorithm 4.4, with  $\{i_1, i_2\} = \{1, 2\}$ , we get

$$\widetilde{M} = \begin{bmatrix} 0.8905 & -0.3422 \\ -0.3422 & 1.1150 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} 3.3431 & -1.1499 \\ -1.1499 & 2.5989 \end{bmatrix}, \quad \widetilde{K} = \begin{bmatrix} 2.0147 & -0.3698 \\ -0.3698 & 1.0460 \end{bmatrix}.$$

Here  $\widetilde{M}$ ,  $\widetilde{C}$ , and  $\widetilde{K}$  are all positive definite.

**Example 5.2.** Consider the quadratic model  $(M, C, K)$ , where  $M \in \mathbb{R}^{66 \times 66}$  and  $K \in \mathbb{R}^{66 \times 66}$  come from the statically condensed oil rig model of the Harwell-Boeing Collection BCSSTRUC1 [17]. The damping matrix here is defined by  $C = 1.55I_{66}$ . By applying the standard MATLAB code `polyeig.m` to  $(M, C, K)$ , we obtain its 132 eigenpairs, out of which are four real eigenvalues and two pairs of complex conjugate eigenvalues, given by

$$\begin{aligned} \lambda_1 = \bar{\lambda}_2 &= -6.2357 \pm 121.9449i, & \lambda_3 = \bar{\lambda}_4 &= -6.4014 \pm 159.0833i, \\ \lambda_5 &= -5.3584, & \lambda_6 &= -9.2761, & \lambda_7 &= -3.4628, & \lambda_8 &= -13.1972. \end{aligned}$$

Let

$$\begin{aligned} \tilde{\lambda}_1 = \tilde{\lambda}_2 &= -6.5000 \pm 125.0000i, & \tilde{\lambda}_3 = \tilde{\lambda}_4 &= -6.8000 \pm 165.0000i, \\ \tilde{\lambda}_5 &= -5.6000, & \tilde{\lambda}_6 &= -10.0000, & \tilde{\lambda}_7 &= -3.8000, & \tilde{\lambda}_8 &= -13.9000. \end{aligned}$$

Using Algorithm 4.4, with  $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$ , we get  $(\widetilde{M}, \widetilde{C}, \widetilde{K})$  and  $\widetilde{X}_1$ . Moreover, we have

$$\begin{aligned}\|\widetilde{M}\widetilde{X}_1\widetilde{\Lambda}_1^2 + \widetilde{C}\widetilde{X}_1\widetilde{\Lambda}_1 + \widetilde{K}\widetilde{X}_1\| &= 3.2556 \times 10^{-9}, \\ \|\widetilde{M}X_2\Lambda_2^2 + \widetilde{C}X_2\Lambda_2 + \widetilde{K}X_2\| &= 1.0055 \times 10^{-7},\end{aligned}$$

and

$$\frac{\|\widetilde{M} - M\|}{\|M\|} = 0.0483, \quad \frac{\|\widetilde{C} - C\|}{\|C\|} = 0.2783, \quad \frac{\|\widetilde{K} - K\|}{\|K\|} = 0.0155.$$

This shows that the present algorithm is reliable.

## 6. CONCLUSION

In this paper, we discussed the symmetry eigenvalue embedding problem (EEP), in which the troublesome or unwanted eigenvalues are replaced by chosen ones, while the remaining large number of eigenpairs do not change. Based on the theory established in [2], we sufficiently utilize the inherent freedom of the EEP to derive an expression of the parameterized solution to the EEP. Then using the expression, we develop a novel numerical method for solving the EEP, in which the parameters in the solutions are optimized in some sense. This method not only utilizes the freedom of the EEP, but also removes the limitation of the method proposed in [6]. The results of our numerical experiments show that the present algorithm is feasible and efficient, and can outperform the iterative method in [3] and the method in [6]. In addition, the present algorithm is rich in Basic Linear Algebra Subroutine-3 (BLAS-3) level operations, and so it can be implemented using high-performance software packages such as LAPACK on today's high-speed computers.

Finally, it is worthwhile to point out that we only obtain a sub-optimal solution of the optimization problem (4.3). How to get its global optimum remains unsolved, which needs further investigation.

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