

A NOTE ON BAZILEVIČ FUNCTIONS

Yong Chan Kim and Toshiyuki Sugawa

(Dedicated to Professor Yoshihiro Mizuta on the occasion of his 60th birthday)

Abstract. We provide a few sufficient conditions for a normalized analytic function in the unit disk to be a Bazilevič function of prescribed type.

1. INTRODUCTION

Throughout the paper, \mathbf{A} denotes the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized so that $f(0) = 0$ and $f'(0) = 1$.

Let α and β be real numbers with $\alpha > 0$ and set $\gamma = \alpha + i\beta$. An $f \in \mathbf{A}$ is called a *Bazilevič* function of type (α, β) if

$$(1) \quad \begin{aligned} f(z) &= \left[\gamma \int_0^z g(\zeta)^\alpha h(\zeta) \zeta^{i\beta-1} d\zeta \right]^{1/\gamma} \\ &= z \left[\gamma \int_0^1 \left(\frac{g(tz)}{tz} \right)^\alpha h(tz) t^{\gamma-1} dt \right]^{1/\gamma} \end{aligned}$$

for a starlike (univalent) function g in \mathbf{A} and an analytic function h with $h(0) = 1$ satisfying $\operatorname{Re}(e^{i\lambda} h) > 0$ in \mathbb{D} for some $\lambda \in \mathbb{R}$. Here and hereafter, $t^{\gamma-1} = e^{(\gamma-1)\log t}$ with $\log t \in (-\infty, 0)$ for $0 < t < 1$ and, for a non-vanishing analytic function G with $G(0) = 1$, any power G^δ , $\delta \in \mathbb{C}$, will be understood as $\exp(\delta \log G)$, where $\log G$ means the analytic branch in \mathbb{D} with $\log G(0) = 0$. We denote by $\mathbf{B}(\alpha, \beta)$ the class of Bazilevič functions of type (α, β) . If we specify the real number λ in the above definition, we denote by $\mathbf{B}_\lambda(\alpha, \beta)$ the corresponding subclass of $\mathbf{B}(\alpha, \beta)$.

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Let \mathbf{S} , \mathbf{S}^* , \mathbf{K} , \mathbf{C} , and $\mathbf{S}_p(\lambda)$, $-\pi/2 < \lambda < \pi/2$, denote the subclasses of \mathbf{A} of functions univalent, starlike, convex, close-to-convex, and λ -spirallike, respectively. (For these classes, see [3] for instance, though the notation is not same as here.) It is well known that the inclusion relations $\mathbf{K} \subset \mathbf{S}^* \subset \mathbf{C} \subset \mathbf{S}$ are valid. For $\lambda \in \mathbb{R}$, we also denote by \mathbf{P}_λ the class of analytic functions h with $h(0) = 1$ and $\operatorname{Re}(e^{i\lambda}h) > 0$ in \mathbb{D} . Note that \mathbf{P}_0 is known as the Carathéodory class.

Let $\alpha > 0, \beta \in \mathbb{R}$ and $-\pi/2 < \lambda < \pi/2$. In view of (1), for $f \in \mathbf{A}$, we readily see that $f \in \mathbf{B}_\lambda(\alpha, \beta)$ if and only if

$$(2) \quad \operatorname{Re} \left[e^{i\lambda} \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{g(z)} \right)^\alpha \left(\frac{f(z)}{z} \right)^{i\beta} \right] > 0$$

for some $g \in \mathbf{S}^*$. In this way, the definition of $\mathbf{B}(\alpha, \beta)$ can be extended to the case when $\alpha \geq 0$ naturally. By the above description, we have $\mathbf{B}_0(0, 0) = \mathbf{S}^*$, $\mathbf{B}_\lambda(0, 0) = \mathbf{S}_p(\lambda)$ and $\mathbf{B}(1, 0) = \mathbf{C}$.

Bazilevič [1] showed that $\mathbf{B}(\alpha, \beta) \subset \mathbf{S}$ for $\alpha > 0, \beta \in \mathbb{R}$. Later, Sheil-Small [5] extended it to the case $\alpha \geq 0$ and gave a geometric characterization for $\mathbf{B}(\alpha, \beta)$. So far, Bazilevič functions form the largest known class in \mathbf{S} which has concrete expressions. It is, however, not easy to study them because the expression is somewhat complicated.

In this paper, we give a few sufficient conditions for a function in \mathbf{A} to belong to a class of Bazilevič functions. Let $-\pi/2 < \lambda < \pi/2$ and set $\zeta = e^{i\lambda}$. We now define the slit domain U_λ by

$$U_\lambda = \mathbb{C} \setminus \{iy : y \geq A_\lambda \text{ or } y \leq -1/A_\lambda\}, \quad A_\lambda = \frac{\cos \lambda}{1 + \sin \lambda}.$$

Note that U_λ is starlike with respect to the origin. In order to state our result, we also introduce the notation

$$P[\gamma, f](z) = 1 + \frac{zf''(z)}{f'(z)} + (\gamma - 1) \frac{zf'(z)}{f(z)}$$

for $\gamma \in \mathbb{C}$ and $f \in \mathbf{A}$.

Theorem 1. *Let $\alpha > 0, \beta \in \mathbb{R}, -\pi/2 < \lambda < \pi/2$ and $f \in \mathbf{A}$. Suppose that*

$$(3) \quad P[\alpha + i\beta, f](z) - \alpha p(z) - i\beta \in U_\lambda, \quad z \in \mathbb{D},$$

holds true for some $p \in \mathbf{P}_0$. Then $f \in \mathbf{B}_\lambda(\alpha, \beta)$.

We have, in particular, the following consequence.

Corollary 1. *Let $\alpha > 0, \beta \in \mathbb{R}$ and $f \in \mathbf{A}$. Suppose that*

$$\operatorname{Re} P[\alpha + i\beta, f](z) > 0, \quad z \in \mathbb{D}.$$

Then $f \in \mathbf{B}_\lambda(\alpha, \beta)$ for every $\lambda \in (-\pi/2, \pi/2)$.

Indeed, under the assumption of the corollary, $p := (P[\alpha + i\beta, f] - i\beta)/\alpha$ belongs to \mathbf{P}_0 . Thus, the assumption of Theorem 1 is satisfied for all λ and the claim follows.

Remark 1. If $\beta = 0$, the condition $\operatorname{Re} P[\alpha, f] > 0$ means that f is $(1/\alpha)$ -convex (cf. [3, p. 10]). We note that Sakaguchi [4] obtained the stronger result that the condition $\operatorname{Re} P[\gamma, f] > -1/2$ in \mathbb{D} for some $\gamma \in \mathbb{C}$ with $\operatorname{Re} \gamma > -1/2$ is enough to ensure the univalence of f .

Our second result gives a way of constructing many Bazilevič functions from several known functions.

Theorem 2. Let m be a positive integer, $\alpha_1, \dots, \alpha_m \in (0, +\infty)$, $\beta \in \mathbb{R}$ and set $\gamma = \alpha_1 + \dots + \alpha_m + i\beta$. If $h \in \mathbf{P}_\lambda$ for some $\lambda \in \mathbb{R}$ and $g_1, \dots, g_m \in \mathbf{S}^*$, then the function f defined by

$$f(z) = \left[\gamma \int_0^z g_1(\zeta)^{\alpha_1} \dots g_m(\zeta)^{\alpha_m} h(\zeta) \zeta^{i\beta-1} d\zeta \right]^{1/\gamma}, \quad z \in \mathbb{D},$$

belongs to $\mathbf{B}_\lambda(\alpha_1 + \dots + \alpha_m, \beta)$.

2. PRELIMINARIES

For analytic functions g and h in \mathbb{D} , g is said to be *subordinate* to h if there exists an analytic function ω in \mathbb{D} such that

$$\omega(0) = 0, \quad |\omega(z)| < 1 \quad \text{and} \quad g(z) = h(\omega(z)) \quad (z \in \mathbb{D}).$$

This subordination will be denoted by $g \prec h$ or, conventionally, $g(z) \prec h(z)$. In particular, when h is univalent in \mathbb{D} , $g \prec h$ if and only if

$$g(0) = h(0) \quad \text{and} \quad g(\mathbb{D}) \subset h(\mathbb{D}).$$

The following result is a key ingredient of the proof of Theorem 1.

Lemma 3. ([3, Cor. 3.1d.1]). Let φ be a non-vanishing analytic function in \mathbb{D} such that $\varphi(0) = 1$ and $z\varphi'(z)/\varphi(z)$ is starlike. Suppose that a non-vanishing analytic function h in \mathbb{D} with $h(0) = 1$ satisfies

$$\frac{zh'(z)}{h(z)} \prec \frac{z\varphi'(z)}{\varphi(z)}.$$

Then $h \prec \varphi$.

It is convenient to translate the condition (2) into one in terms of the quantity $P[\gamma, f]$ for the present aim.

Lemma 4. *Let $\alpha > 0, \beta \in \mathbb{R}, \lambda \in (-\pi/2, \pi/2), f \in \mathbf{A}$ and set $\gamma = \alpha + i\beta$. Then $f \in \mathbf{B}_{\lambda}(\alpha, \beta)$ if and only if*

$$P[\gamma, f] = \alpha p + i\beta + \frac{zh'}{h}$$

for some $p \in \mathbf{P}_0$ and $h \in \mathbf{P}_{\lambda}$.

Proof. First assume that f is given by (1). Then

$$zf(z)^{\gamma-1}f'(z) = g(z)^{\alpha}h(z)z^{i\beta}.$$

Taking logarithmic derivatives of both sides and multiplying with z , we obtain the relation

$$1 + \frac{zf''(z)}{f'(z)} + (\gamma - 1)\frac{zf'(z)}{f(z)} = \alpha\frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} + i\beta.$$

Since $p = zg'/g \in \mathbf{P}_0$, we have the required condition. We can easily trace back the above procedure by taking $g \in \mathbf{S}^*$ so that $p = zg'/g$.

3. PROOF OF THEOREMS

First, we prove the following lemma for the Möbius transformation

$$\psi_{\lambda}(z) = \frac{1 + \bar{\zeta}z}{1 - \zeta z} = \frac{1 + e^{-i\lambda}z}{1 - e^{i\lambda}z}.$$

Lemma 5. *Let $\lambda \in (-\pi/2, \pi/2)$. Then the above ψ_{λ} maps the unit disk \mathbb{D} conformally onto the half-plane $\operatorname{Re}(e^{i\lambda}w) > 0$. Moreover, the function $Q_{\lambda}(z) = z\psi'_{\lambda}(z)/\psi_{\lambda}(z)$ maps \mathbb{D} conformally onto the domain U_{λ} .*

Proof. For brevity, we set $\psi = \psi_{\lambda}$ and $Q = Q_{\lambda}$ for a while. Since $\psi(-\zeta) = 0, \psi(\bar{\zeta}) = \infty$ and $\psi(i) = i\bar{\zeta}$, the image of $\partial\mathbb{D}$ under ψ is the line $\operatorname{Re}(\zeta w) = 0$. Thus, we see that ψ satisfies $\psi(0) = 1$ and maps the unit disk conformally onto the half-plane $\operatorname{Re}(e^{i\lambda}w) > 0$.

We next consider the function Q . Since

$$Q(z) = \frac{2z \cos \lambda}{(1 + \bar{\zeta}z)(1 - \zeta z)},$$

we have the expression

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{\zeta z}{1 - \zeta z} + \frac{-\bar{\zeta}z}{1 + \zeta z}.$$

Since $\operatorname{Re} [z/(1 - z)] > -1/2$ for $|z| < 1$, we see that $\operatorname{Re} (zQ'(z)/Q(z)) > 0$, equivalently, Q is a starlike univalent function in \mathbb{D} .

Finally, for $z = e^{i\theta}$, we have

$$Q(z) = \frac{2 \cos \lambda}{\bar{z} - z + \bar{\zeta} - \zeta} = \frac{i \cos \lambda}{\sin \theta + \sin \lambda}.$$

Since

$$-1 + \sin \lambda \leq \sin \theta + \sin \lambda \leq 1 + \sin \lambda,$$

the boundary values of Q form the set

$$\{iy : y \geq \cos \lambda / (1 + \sin \lambda)\} \cup \{\infty\} \cup \{iy : y \leq -\cos \lambda / (1 - \sin \lambda)\}.$$

Therefore, Q maps \mathbb{D} onto the domain U_λ as required. The proof is now complete. ■

The function Q_λ is a variant of the so-called “open door mapping” (see [3, §2.5]).

Since $\psi_\lambda(0) = 1$, we observe that $\varphi \in \mathbf{P}_\lambda$ if and only if $\varphi \prec \psi_\lambda$. Hence we obtain the proof of Theorem 1 as follows :

Proof of Theorem 1. Let $\lambda \in (-\pi/2, \pi/2)$ and suppose that $f \in \mathbf{A}$ satisfies (3) for some $p \in \mathbf{P}_0$. Take a function $g \in \mathbf{S}^*$ so that $zg'/g = p$. If we put

$$h = \left(\frac{zf'}{f}\right) \left(\frac{f}{g}\right)^\alpha \left(\frac{f}{z}\right)^{i\beta},$$

then the same computation as in the proof of Lemma 4 gives the relation

$$\frac{zh'}{h} = P[\alpha + i\beta, f] - \alpha p - i\beta.$$

Thus, by the assumption and Lemma 5, we have $h(z) \neq 0$ for $z \in \mathbb{D}$ and

$$\frac{zh'}{h} \prec Q_\lambda = \frac{z\psi'_\lambda}{\psi_\lambda}.$$

Now it follows from Lemma 3 that $h \prec \psi_\lambda$, i.e., $h \in \mathbf{P}_\lambda$. Hence Lemma 4 implies that $f \in \mathbf{B}_\lambda(\alpha, \beta)$.

In order to prove Theorem 2, we need the following simple observation.

Lemma 6. Let $\mu_1, \dots, \mu_m \in (0, 1)$ with $\mu_1 + \dots + \mu_m = 1$ and $g_1, \dots, g_m \in \mathbf{S}^*$. Then the function $g \in \mathbf{A}$ defined by

$$g(z) = z \left(\frac{g_1(z)}{z} \right)^{\mu_1} \dots \left(\frac{g_m(z)}{z} \right)^{\mu_m}, \quad z \in \mathbb{D},$$

belongs to \mathbf{S}^* .

Proof. By taking the logarithmic derivative of g and multiplying with z , we have

$$\frac{zg'(z)}{g(z)} = \mu_1 \frac{zg'_1(z)}{g_1(z)} + \dots + \mu_m \frac{zg'_m(z)}{g_m(z)}.$$

Since $\operatorname{Re} \{zg'_j(z)/g_j(z)\} > 0$ for each j , we conclude that $\operatorname{Re} \{zg'(z)/g(z)\} > 0$.

Remark 2. For a convex subdomain V of $\mathbb{C} \setminus \{0\}$ with $1 \in V$, we let $\mathbf{S}^*(V) = \{f \in \mathbf{A} : zf'(z)/f(z) \in V \ (z \in \mathbb{D})\}$. (For instance, if we choose the half-planes $\operatorname{Re} w > \alpha (\geq 0)$ and $\operatorname{Re}(e^{i\lambda}w) > 0$ and the sector $|\arg w| < \pi\alpha/2$ as V , then we have the classes of functions which are starlike of order α , λ -spirallike, and strongly starlike of order α , respectively.) Then, the above proof tells us that $g \in \mathbf{S}^*(V)$ whenever $g_1, \dots, g_m \in \mathbf{S}^*(V)$.

Remark 3. It is well known that, for $g, k \in \mathbf{A}$ with $g(z) = zk'(z)$, the condition $g \in \mathbf{S}^*$ is equivalent to the condition $k \in \mathbf{K}$. Through this transformation, we understand that the above lemma turns to the known fact that the class \mathbf{K} is convex in the sense of Hornich operations (see Cima and Pfaltzgraff [2, Theorem 6.1]).

Finally, as an immediate consequence of the above lemma, we can prove Theorem 2.

Proof of Theorem 2. By letting $\mu_j = \alpha_j/(\alpha_1 + \dots + \alpha_m)$ and g be as in Lemma 6, we see that $g \in \mathbf{S}^*$ by Lemma 6. Now if we put $\alpha = \alpha_1 + \dots + \alpha_m$ in the expression (1), we have the following expression

$$f(z) = \left[\gamma \int_0^z g_1(\zeta)^{\alpha_1} \dots g_m(\zeta)^{\alpha_m} h(\zeta) \zeta^{i\beta-1} d\zeta \right]^{1/\gamma} = \left[\gamma \int_0^z g(\zeta)^\alpha h(\zeta) \zeta^{i\beta-1} d\zeta \right]^{1/\gamma}.$$

Hence, under the assumptions of Theorem 2, we see that f belongs to $\mathbf{B}_\lambda(\alpha, \beta) = \mathbf{B}_\lambda(\alpha_1 + \dots + \alpha_m, \beta)$.

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Yong Chan Kim
Department of Mathematics Education,
Yeungnam University,
214-1 Daedong Gyongsan 712-749,
Korea
E-mail: kimyc@yumail.ac.kr

Toshiyuki Sugawa
Graduate School of Information Sciences,
Tohoku University,
Sendai 980-8579,
Japan
E-mail: sugawa@math.is.tohoku.ac.jp