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RECIPROCAL CONTINUITY AND COMMON FIXED POINTS OF NONSELF MAPPINGS

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Abstract. We extend the notions of reciprocal continuity and C_q-commutativity to nonself setting besides observing equivalence between compatibility and ϕ -compatibility, and utilize the same to obtain some results on coincidence and common fixed points for two pairs of nonself mappings in metrically convex metric spaces. As an application of our main result, we also prove a common fixed point theorem in Banach spaces besides furnishing several illustrative examples.

1. Introduction

The study of fixed point theorems for nonself mappings in metrically convex metric spaces was initiated by Assad and Kirk [2, 1972]. In practice, there do exist many situations which cannot be described by self mappings and hence the study of nonself mappings is worth investigating. In recent years, inspired by Rhoades [20], several results for single-valued mapping have been proved by various researchers of this domain. In this direction, one may cite [3,4,5,7,8,13,15]. Specifically, Assad [3] proved some results for nonself mappings defined on a closed subset of a complete metrically convex metric space satisfying Kannan type mappings which has subsequentally been generalized by Khan et al. [15] for generalized type contractions. Recently, Imdad and Khan [8,10] and Imdad et al. [9] generalized these results for pairs of mappings. Here one may note that in such results, one often requires all or some of underlying mappings to be continuous (cf. [7, 8, 9, 15]).

On the other hand, Park [18] and Khan et al. [14] used a new technique to prove fixed point theorems in metric spaces by altering distances between the points

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employing suitably equipped continuous control function which has further been pursued by Pathak and Sharma [19], Sastry and Babu [21] and Sastry et al. [22]. Assad [3,4], Abdalla and Zaheer [5], Imdad and Khan [8] and others used this technique in nonself setting.

The control function employed in Sastry et al. [22] to alter distances is indeed a function $\phi : \Re^+ \to \Re^+$ which satisfy the following properties:

(i) ϕ is continuous at origin and monotonically increasing in $\Re^+$,

(ii) $\phi(t) = 0 \Leftrightarrow t = 0$,

(iii) $\phi(2t) \leq 2\phi(t)$.

Remark 1.1. Let $\{t_n\} \subset \Re^+$ satisfying $\phi(t_n) \to 0$ as $n \to \infty$. Then $t_n \to 0$ as $n \to \infty$.

Before proving our results, we collect the relevant definitions and results.

Definition 1.1. [6]. Let K be a nonempty subset of a metric space (X, d) and $G, S : K \to X$. The pair (G, S) is said to be weakly commuting if

$$d(GSx, SGx) \leq d(Gx, Sx)$$

for every $x \in K$ with Gx, $Sx \in K$.

Note that for $K = X$, this definition reduces to that of Sessa [23].

Motivated from [6], we define the ϕ-compatibility for nonself mappings as follows:

Definition 1.2. Let K be a nonempty subset of a metric space (X, d) and $G, S : K \to X$. The pair (G, S) is said to be ϕ-compatible if

$$\lim_{n\to\infty} \phi(d(GSx_n, SGx_n)) = 0$$

whenever there is a sequence $\{x_n\} \subset K$ such that $\lim_{n\to\infty} d(Gx_n, Sx_n) = 0$ with $Gx_n, Sx_n \in K$.

Note that for $K = X$ this definition reduces to ϕ-compatibility due to Sastry et al. [22] and for $\phi = Id.$ and $K = X$, this definition reduces to 'compatibility' for self mappings due to Jungck [11].

In view of Remark 1.1, ϕ-compatibility and compatibility are equivalent. Hence throughout this paper, we use compatibility instead of ϕ-compatibility.

Definition 1.3. [12]. A pair (G, S) of nonself mappings defined on a nonempty subset K of a set X is said to be weakly compatible (or coincidentally commuting) if $Gx = Sx$ for some $x \in K$ with $Gx, Sx \in K \Rightarrow GSx = SGx$.

Definition 1.4. [8]. Let (X, d) be a metric space and K be a nonempty subset of X. Let $F, G, S, T : K \to X$ which satisfy the inequality

$$\begin{aligned} &\phi(d(Fx, Gy)) \\ (1.1.1) \qquad &\leq a \ \max\left\{\frac{1}{2}\phi(d(Tx, Sy)), \phi(d(Tx, Fx)), \phi(d(Sy, Gy))\right\} \\ &+b\ [\phi(d(Tx, Gy)) + \phi(d(Sy, Fx))] \end{aligned}$$

for all distinct $x, y \in K$ with $a, b \geq 0$ such that $a + 2b < 1$, where $\phi : \Re^+ \to \Re^+$ be a function which satisfies (i), (ii) and (iii). Then (F, G) is called generalized (T, S) contraction mappings of K into X.

Definition 1.5. A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$, there exists a point $z \in X$, $x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 1.1. [2]. *Let K be a nonempty closed subset of a metrically convex metric space X. If $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

In this paper, we extend the notions of reciprocal continuity and C_q-commutativity for nonself mappings besides observing equivalence between compatibility and ϕ-compatibility, and proved some common fixed point theorems in metrically convex metric spaces. Our main result generalizes earlier results due to Assad [4], Imdad and Khan [8], Imdad et al. [9], Khan and Bharadwaj [13], Khan et al. [15] and others. We also furnish some illustrative examples.

2. Reciprocal Continuity in Nonself Setting

Recently, Pant [17] introduced the notion of reciprocal continuity and used it to prove common fixed point theorems for contraction type self mappings. In what follows, we extend this notion to nonself setting besides furnishing an example and proving a related result.

A natural extension of reciprocal continuity to nonself mappings can be given as follows:

Definition 2.1. Let K be a nonempty subset of a metric space (X, d). A pair of mappings $G, S : K \to X$ is said to be reciprocally continuous if

$\lim_{n\to\infty} GSx_n = Gz$ and $\lim_{n\to\infty} SGx_n = Sz$ whenever there is a sequence $\{x_n\} \subset K$ with $\{Sx_n\},\ \{Gx_n\} \subset K$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Gx_n = z \text{ for some } z \in K.$$

Notice that for $K = X$ this definition reduces to the definition of reciprocal continuity due to Pant [17] given for self mappings. However in the context of above definition the following facts are worth noticing.

(a) If both the component maps G and S are continuous, then they are obviously reciprocally continuous but the converse is not true (see Example 2.1).

(b) There exists a rich class of pairs of discontinuous mappings which are compatible as well as reciprocally continuous and this is why the notion of reciprocal continuity is advantageous in proving results on common fixed points. To substantiate our claim, we furnish the following example.

Example 2.1. Consider $X = [1, \infty)$ equipped with Euclidian metric d and $K = [1, 3]$. Define $G, S : K \to X$ as

$$Gx = \begin{cases} 2x^4 - 1,\ 1 \le x \le 2^{\frac{1}{4}} \\ x^2,\ \text{otherwise,} \end{cases} \qquad Sx = \begin{cases} x^2,\ 1 \le x \le 2^{\frac{1}{4}} \\ 2x^4 - 1,\ \text{otherwise.} \end{cases}$$

Then for any sequence $\{x_n\} \subset K$ with $\{Gx_n\},\ \{Sx_n\} \subset K$, we have

$$d(GSx_n, SGx_n) = |2x_n^4 - x_n^8 - 1| \to 0 \Leftrightarrow x_n \to 1 \Leftrightarrow Gx_n \to 1 \text{ and } Sx_n \to 1.$$

Thus the pair (G, S) is compatible on K. Also notice that for sequence $\{x_n\}$ with $Gx_n \to 1$ and $Sx_n \to 1$, we have $\lim_{n\to\infty} GSx_n = 1 = G1$ and $\lim_{n\to\infty} SGx_n = 1 = S1$ which shows that the pair (G, S) is reciprocally continuous whereas both the components are discontinuous.

Here, we point out that common fixed point theorems for compatible mappings often require the continuity of some or all involved mappings (e.g. [15, Theorem 3.2]). The notion of reciprocal continuity makes it possible to prove common fixed point theorems for compatible mappings under relatively less continuity requirement. In the setting of common fixed point theorems, pair of compatible mappings satisfying a suitable contractive or contraction condition can ensure reciprocal continuity in the presence of continuity of one of the mapping. In the sequel, we prove the following lemma which exhibits that under a contraction condition patterned after Khan et al. [15] in a nonself setting in metrically convex metric spaces compatibility of pairs implies their reciprocal continuity provided one component map of a pair is continuous.

Lemma 2.1. Let K be a subset of a metric space (X, d) and let $F, G, S, T : K \to X$ be four mappings such that (F, G) be a generalized (T, S) contraction mappings of K into X. Suppose that

(a) $GK \cap K \subset TK$ (resp. $FK \cap K \subset SK$),

(b) the pair (G, S) (resp. (F, T)) is compatible,

(c) G or S (resp. F or T) is continuous,

(d) the control function $\phi : \Re^+ \to \Re^+$ is left continuous.

Then the pair (G, S) (resp. (F, T)) is reciprocally continuous.

Proof. Let us assume that the pair (G, S) is compatible and S is continuous. Let $\{x_n\}$ be a sequence in K such that $Gx_n \to z$ and $Sx_n \to z$ with $\{Gx_n\}$, $\{Sx_n\} \subset K$. Since S is continuous, we get $SSx_n \to Sz$. Now compatibility of (G, S) implies that $\lim_{n\to\infty} d(GSx_n, SGx_n) = 0$, that is $GSx_n \to Sz$. Since $GK \cap K \subset TK$ for each x_n there exists some y_n in TK with $GSx_n = Ty_n$. Then $SSx_n \to Sz$, $SGx_n \to Sz$, $GSx_n \to Sz$ and $Ty_n \to Sz$. We assert that $Fy_n \to Sz$. If not, then there exists a subsequence Fy_m, a number $\epsilon > 0$ and a positive integer $n_0 \in N$ such that for each $m \ge n_0$, we have $d(GSx_m, Fy_m) \ge \epsilon$, $d(Fy_m, Sz) \ge \epsilon$. Now using (1.1.1), we have

$$\begin{aligned}
&\phi(d(Fy_m, GSx_m)) \\
&\le a \max\left\{\frac{1}{2}\phi(d(Ty_m, SSx_m)), \phi(d(Ty_m, Fy_m)), \phi(d(SSx_m, GSx_m))\right\} \\
&\quad + b[\phi(d(Ty_m, GSx_m)) + \phi(d(SSx_m, Fy_m))
\end{aligned}$$

which on letting $m \to \infty$, reduces to

$$\phi(d(Fy_m, Sz)) \le (a + b)\phi(d(Fy_m, Sz)) < \phi(d(Fy_m, Sz))$$

which is a contradiction. Hence $\lim_{n\to\infty} Fy_m = Sz$. If $Gz \ne Sz$, then by (1.1.1), we get

$$\begin{aligned}
\phi(d(Gz, Fy_n)) &\le a \max\left\{\frac{1}{2}\phi(d(Ty_n, Sz)), \phi(d(Ty_n, Fy_n)), \phi(d(Sz, Gz))\right\} \\
&\quad + b[\phi(d(Ty_n, Gz)) + \phi(d(Sz, Fy_n))]
\end{aligned}$$

which on letting $n \to \infty$, reduces to

$$\begin{aligned}
\phi(d(Gz, Sz)) &\le (a + b)\phi(d(Sz, Gz)) \\
&< \phi(d(Sz, Gz))
\end{aligned}$$

a contradiction. Hence $Sz = Gz$. Thus by assuming $Sx_n \to z$ and $Gx_n \to z$ along with continuity of S, we obtain $SGx_n \to Sz$ and $GSx_n \to Gz(= Sz)$ which shows that the pair (G, S) is reciprocally continuous. We arrive at the same conclusion when the pair (G, S) is compatible and G is continuous. The proof for the other pair (F, T) is similar, hence it is omitted. This completes the proof.

3. Results

Our main result runs as follows:

Theorem 3.1. *Let K be a nonempty closed subset of a complete metrically convex metric space X. If (F, G) is a generalized (T, S) contraction mappings of K into X which satisfy*

(a) $\partial K \subset SK \cap TK,\ FK \cap K \subset SK,\ GK \cap K \subset TK,$

(b) $Tx \in \partial K \Rightarrow Fx \in K,\ Sx \in \partial K \Rightarrow Gx \in K,$

(c) *either the pair (G, S) is compatible and reciprocally continuous or the pair (F, T) is compatible and reciprocally continuous,*

then the pair (G, S) as well as (F, T) has a point of coincidence.

Moreover, G, S, F and T have a unique common fixed point provided the pairs (G, S) and (F, T) are weakly compatible.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Let $x \in \partial K$. Since $\partial K \subset TK$, there exists a point $x_0 \in K$ such that $x = Tx_0$. From the implication $Tx_0 \in \partial K \Rightarrow Fx_0 \in K \cap FK \subset SK$. Let $x_1 \in K$ be such that $y_1 = Sx_1 = Fx_0 \in K$. Since $y_1 = Fx_0$, then there exists a point $y_2 = Gx_1$ such that

$$d(y_1, y_2) = d(Fx_0, Gx_1).$$

Suppose $y_2 \in K$. Then $y_2 \in K \cap GK \subset TK$ which implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin K$, then there exists a point $p \in \partial K$ such that

$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2).$$

Since $p \in \partial K \subset TK$, there exists a point $x_2 \in K$ such that $p = Tx_2$ and so

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2).$$

Let $y_3 = Fx_2$ be such that

$$d(y_2, y_3) = d(Gx_1, Fx_2).$$

Thus repeating the forgoing arguments one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{2n} = Gx_{2n-1},\ y_{2n+1} = Fx_{2n}$.

(ii) $y_{2n} \in K \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin K \Rightarrow Tx_{2n} \in \partial K$

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}).$$

(iii) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Sx_{2n+1} \in \partial K$

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}).$$

We denote

$$\begin{aligned} P_0 &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\}, \\ P_1 &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\}, \\ Q_0 &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\}, \\ Q_1 &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}. \end{aligned}$$

Note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$. Similarly, $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now, we distinguish the following three cases:

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_0$, then

$$\begin{aligned} &\phi(d(Tx_{2n}, Sx_{2n+1})) \\ &= \phi(d(Fx_{2n}, Gx_{2n-1})) \\ &\leq a \max\left\{\frac{1}{2}\phi(d(Tx_{2n}, Sx_{2n-1})), \phi(d(Tx_{2n}, Fx_{2n})), \phi(d(Sx_{2n-1}, Gx_{2n-1}))\right\} \\ &\quad + b[\phi(d(Tx_{2n}, Gx_{2n-1})) + \phi(d(Fx_{2n}, Sx_{2n-1}))] \\ &= a \max\{\phi(d(y_{2n}, y_{2n-1})), \phi(d(y_{2n}, y_{2n+1}))\} + b\phi(d(y_{2n-1}, y_{2n+1})) \\ &= a \max\{\phi(d(y_{2n}, y_{2n-1})), \phi(d(y_{2n}, y_{2n+1}))\} + b[\phi(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}))]. \end{aligned}$$

If $d(y_{2n}, y_{2n-1}) \geq d(y_{2n}, y_{2n+1})$, then

$$\phi(d(Tx_{2n}, Sx_{2n+1})) \leq (a + 2b)\phi(d(Tx_{2n}, Sx_{2n-1})).$$

Otherwise, if $d(y_{2n}, y_{2n-1}) < d(y_{2n}, y_{2n+1})$, then we have

$$\begin{aligned} \phi(d(Tx_{2n}, Sx_{2n+1})) &\leq a\phi(d(y_{2n}, y_{2n+1})) + 2b\phi(d(y_{2n}, y_{2n+1})) \\ &= (a + 2b)\phi(d(y_{2n}, y_{2n+1})) \\ &< \phi(d(y_{2n}, y_{2n+1})) \end{aligned}$$

which is a contradiction. Hence

$$\phi(d(Tx_{2n}, Sx_{2n+1})) \leq (a+2b)\phi(d(Tx_{2n}, Sx_{2n-1})).$$

Similarly, if $(Sx_{2n-1}, Tx_{2n}) \in Q_0 \times P_0$, then

$$\phi(d(Sx_{2n-1}, Tx_{2n})) \leq (a+2b)\phi(d(Sx_{2n-1}, Tx_{2n-2})).$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_1$, then we have

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}),$$

and hence

$$\phi(d(Tx_{2n}, Sx_{2n+1})) \leq \phi(d(Tx_{2n}, y_{2n+1})) = \phi(d(y_{2n}, y_{2n+1})).$$

Now, as in Case 1, we obtain

$$\phi(d(Tx_{2n}, Sx_{2n+1})) \leq (a+2b)\phi(d(Tx_{2n}, Sx_{2n-1})).$$

In case $(Sx_{2n-1}, Tx_{2n}) \in Q_1 \times P_0$, then

$$\phi(d(Sx_{2n-1}, Tx_{2n})) \leq (a+2b)\phi(d(Sx_{2n-1}, Tx_{2n-2})).$$

Case 3. If $d(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_0$, then $Sx_{2n-1} \in Q_0$ and

$$d(Tx_{2n}, Sx_{2n+1}) = d(Tx_{2n}, y_{2n+1}) \leq d(Tx_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1}).$$

Note that $d(y_{2n}, Sx_{2n+1}) = d(Fx_{2n}, Gx_{2n-1})$, therefore proceeding as in Case 1, we have

$$\begin{aligned}\phi(d(y_{2n}, y_{2n+1})) &= \phi(d(Fx_{2n}, Gx_{2n-1})) \leq (a+2b)\phi(d(Tx_{2n}, Sx_{2n-1})) \\ &< \phi(d(Tx_{2n}, Sx_{2n-1})),\end{aligned}$$

and thus $d(y_{2n}, y_{2n+1}) \leq d(Tx_{2n}, Sx_{2n-1})$ as ϕ is an increasing function, therefore, we can write

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n}) + d(Tx_{2n}, Sx_{2n-1}) = d(Sx_{2n-1}, y_{2n})$$

and hence

$$\begin{aligned}\phi(d(Tx_{2n}, Sx_{2n+1})) &\leq \phi(d(Sx_{2n-1}, y_{2n})) \leq (a+2b)\phi(d(Tx_{2n-2}, Sx_{2n-1})) \\ &= k\phi(d(Tx_{2n-2}, Sx_{2n-1})), \text{ where } k = a+2b.\end{aligned}$$

Thus in all the cases, we have

$$\phi(d(Tx_{2n}, Sx_{2n+1})) \le k \max\{\phi(d(Sx_{2n-1}, Tx_{2n})), \phi(d(Tx_{2n-2}, Sx_{2n-1}))\}$$

whereas

$$\phi(d(Sx_{2n+1}, Tx_{2n+2})) \le k \max\{\phi(d(Sx_{2n-1}, Tx_{2n})), \phi(d(Tx_{2n}, Sx_{2n+1}))\}.$$

It can be shown by induction that for $n \ge 1$,

$$\phi(d(Tx_{2n}, Sx_{2n+1})) \le k^{2n-1} \max\{\phi(d(Tx_0, Sx_1)), \phi(d(Sx_1, Tx_2))\}$$

and

$$\phi(d(Sx_{2n+1}, Tx_{2n+2})) \le k^{2n} \max\{\phi(d(Sx_1, Tx_2)), \phi(d(Tx_2, Sx_3))\}.$$

Now, for any positive integer p, we have

$$\begin{aligned}
&\phi(d(Tx_{2n}, Sx_{2n+p})) \\
&\le \phi(d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, Tx_{2n+2}) + \ldots + d(Tx_{2n+p-1}, Sx_{2n+p})) \\
&\le \phi((1 + k + k^2 + \ldots + k^{n-1})k^{2n} \max\{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}) \\
&= \phi\left(\left(\frac{k^{2n}}{1-k}\right) \max\{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}\right)
\end{aligned}$$

which shows that the sequence $\{Tx_0, Sx_1, Tx_2, \ldots, Sx_{2n-1}, Tx_{2n}, Sx_{2n+1}, \ldots\}$ is Cauchy in K. Then as noted in [6], there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in P_0 and Q_0 respectively and converges to some z in K as K is a closed subset of X.

Since the pair (G, S) is reciprocally continuous as well as compatible, therefore ($Tx_{2n_k} = Gx_{2n_k-1}$ and $Sx_{2n_k-1} \in K$) $SGx_{2n_k-1} \to Sz$, $GSx_{2n_k-1} \to Gz$ and $Gz = Sz$.

Since $GK \cap K \subset TK$, there exists a point w in K such that $Gz = Tw = Sz$. Now we show that $Gz = Fw$. Suppose that $Gz \ne Fw$. Using (1.1.1), we have

$$\phi(d(Fw, Gz)) \le (a + b)\phi(d(Fw, Gz))$$

which is a contradiction and hence $Gz = Sz = Fw = Tw$. Thus both the pairs have points of coincidence.

Since the pair (G, S) is weakly compatible, we have

$$GGz = GSz = SGz = SSz.$$

We show that $GGz = Gz$. Suppose that it is not so, then again using (1.1.1), we obtain

$$\begin{aligned}\phi(d(Gz, GGz)) &= \phi(d(Fw, GGz))\\ &\leq (a+2b)\phi(d(Gz, GGz)) < \phi(d(Gz, GGz))\end{aligned}$$

a contradiction and hence $Gz = GGz$.

Also $GGz = G(Sz) = SGz$, therefore Gz is a common fixed point of the pair (G, S).

Also, suppose that $Fw \neq FFw$, then again as above

$$\begin{aligned}\phi(d(Fw, FFw)) &= \phi(d(FFw, Gz))\\ &\leq (a+2b)\phi(d(Fw, FFw)) < \phi(d(Fw, FFw))\end{aligned}$$

which is a contradiction and hence $FFw = Fw$ and $FFw = FTw = TFw$. This shows that Fw is a common fixed point of the pair (F, T). Hence Gz is a unique common fixed point of G, S, F and T. The uniqueness of Gz is easily follows from (1.1.1).

If we assume that (F, T) is compatible pair of reciprocally continuous mappings, then proceeding on similar lines one can establish the earlier conclusions. This completes the proof.

Remark 3.1. Theorem 3.1 generalizes earlier results due to Assad [4], Imdad and Khan [8], Imdad et al. [9], Khan and Bharadwaj [13], Khan et al. [15] and others as we never require continuity of involved mappings.

Theorem 3.2. *Let K be a nonempty closed subset of a complete metrically convex metric space X. If (F, G) be a generalized (T, S) contraction mappings of K into X which satisfy*

- (a) $\partial K \subset SK \cap TK,\ FK \cap K \subset SK,\ GK \cap K \subset TK,$
- (b) $Tx \in \partial K \Rightarrow Fx \in K,\ Sx \in \partial K \Rightarrow Gx \in K,$
- (c) *either pair (G, S) is compatible and G (or S) continuous or pair (F, T) is compatible and F (or T) continuous,*
- (d) *the control function $\phi : \Re^+ \to \Re^+$ is left continuous.*

Then the pair (G, S) as well as (F, T) has a point of coincidence.

Proof. Suppose the pair (G, S) is compatible and G continuous, then in view of Lemma 2.1, the pair (G, S) is reciprocally continuous. Similarly if we assume the compatibility of the pair (G, S) and continuity of S, then the pair (G, S) again reciprocally continuous. Hence the proof follows from Theorem 3.1.

Finally, we prove a result when ‘closedness of K’ is replaced by ‘compactness of K’.

Theorem 3.3. *Let (X, d) be a complete metrically convex metric space and K be a nonempty compact subset of X. Let $F, G, T : K \to X$ satisfying*

(a) $\partial K \subset TK,\ (FK \cup GK) \cap K \subseteq TK,$
(b) $Tx \in \partial K \Rightarrow Fx, Gx \in K,$
(c) $\phi(d(Fx, Gy)) < M(x, y)$ *with* $M(x, y) > 0$ *for* $x, y \in K$ *where*

$$M(x,y) = a\max\left\{\frac{1}{2}\phi(d(Tx,Ty)), \phi(d(Tx,Fx)), \phi(d(Ty,Gy))\right\} + b[\phi(d(Tx,Gy)) + \phi(d(Ty,Fx))]. \tag{3.3.1}$$

Then F, G and T have points of coincidence provided the pair (F, T) or (G, T) is compatible and reciprocally continuous.

Proof. We assert that $M(x, y) = 0$ for some $x, y \in K$. Otherwise $M(x, y) \neq 0$ for any $x, y \in K$. Define

$$f(x,y) = \frac{\phi(d(Fx,Gy))}{M(x,y)}.$$

Then f is continuous and satisfies $f(x, y) < 1$ for all $(x, y) \in K \times K$. Since $K \times K$ is compact, there exists $(u, v) \in K \times K$ such that $f(x, y) \leq f(u, v) = c < 1$ for $x, y \in K$ which in turn yields $\phi(d(Fx, Gy)) \leq c\ M(x, y)$ for $x, y \in K$ and $0 < c < 1$. Therefore using $(3.3.1)$, one obtains $ca + 2cb < 1$. Now by Theorem 3.1 (with restriction S = T), one gets $Tz = Fz$ and $Tw = Gw$ for some $z, w \in K$. Consequently $M(z, w) = 0$, contradicting the fact $M(x, y) > 0$. Therefore $M(x, y) = 0$ for some $x, y \in K$ which implies $Tx = Fx$ and $Tx = Ty = Gy$. If $M(x, x) = 0$ then $Tx = Gx$ and if $M(x, x) \neq 0$ then using $(3.3.1)$, one infers that $d(Tx, Gx) \leq 0$ yielding thereby $Tx = Gx$. Similarly, in either of the cases $M(y, y) = 0$ or $M(y, y) > 0$, one concludes that $Ty = Fy$. Thus we have shown that F, G and T have a common point of coincidence. For fixed point, the proof is identical to that of Theorem 3.1, and hence omitted. This completes the proof.

Now, we furnish an example which demonstrates the validity of the hypotheses of Theorem 3.1 besides establishing the genuineness of our extension over several other relevant results of the existing literature.

Example 3.1. Let $X = \Re$ with the usual metric and $K = [0, 3]$. Define $\phi : \Re^+ \to \Re^+$ as $\phi(t) = t$ and $F, G, S, T : K \to X$ as

$$Fx = \begin{cases} x^2 \text{ if } 0 \leq x \leq 2 \\ \frac{1}{2} \text{ if } 2 < x \leq 3, \end{cases} \qquad Tx = \begin{cases} 2x^4 \text{ if } 0 \leq x \leq 2 \\ 3 \text{ if } 2 < x \leq 3, \end{cases}$$

$$Gx = \begin{cases} x^3 \text{ if } 0 \leq x \leq 2 \\ \frac{1}{2} \text{ if } 2 < x \leq 3 \end{cases} \qquad \text{and } Sx = \begin{cases} 2x^6 \text{ if } 0 \leq x \leq 2 \\ 3 \text{ if } 2 < x \leq 3. \end{cases}$$

Since $\partial K = \{0,3\}$ and $TK \cap SK = [0,32] \cap [0,128] = [0,32]$, hence $\partial K = \{0,3\}$ $\subset TK \cap SK$. Further, $FK \cap K = [0,4] \cap [0,3] = [0,3] \subset SK$ and $GK \cap K = [0,8] \cap [0,3] \subset TK$. Also

$$T0 = 0 \in \partial K \Rightarrow F0 = 0 \in K,\ T3 = 3 \in \partial K \Rightarrow F3 = \frac{1}{2} \in K,$$

$$S0 = 0 \in \partial K \Rightarrow G0 = 0 \in K,\ S3 = 3 \in \partial K \Rightarrow G3 = \frac{1}{2} \in K.$$

Moreover, if for $x \in [0,2]$ and $y \in (2,3]$, then

$$\begin{aligned} d(Fx, Gy) &= |x^2 - \frac{1}{2}| \\ &\le \frac{2}{3}\ \max\{\frac{1}{2}d(Tx,Sy), d(Tx,Fx), d(Sy,Gy)\} \\ &\quad + \frac{1}{7}\{d(Fx,Sy) + d(Tx,Gy)\}. \end{aligned}$$

Next, if $x, y \in (2,3]$, then

$$\begin{aligned} d(Fx, Gy) &= 0 = \frac{1}{2}d(Tx,Sy), \\ &\le \frac{2}{3}\ \max\{\frac{1}{2}d(Tx,Sy), d(Tx,Fx), (Sy,Gy)\} \\ &\quad + \frac{1}{7}\{d(Fx,Sy) + d(Tx,Gy)\}. \end{aligned}$$

Finally, if $x, y \in [0,2]$, then

$$\begin{aligned} d(Fx, Gy) &= |x^2 - y^3| \\ &\le \frac{2}{3}\ \max\{\frac{1}{2}d(Tx,Sy), d(Tx,Fx), (Sy,Gy)\} \\ &\quad + \frac{1}{7}\{d(Fx,Sy) + d(Tx,Gy)\}, \end{aligned}$$

which shows that the contraction condition (1.1.1) is satisfied for every distinct $x, y \in K$.

Notice that the pair (G,S) (also (F,T)) is compatible and reciprocally continuous (e.g. $x_n = \frac{1}{n}$). Moreover, 0 is a point of common coincidence as $T0 = F0$ and $S0 = G0$ with $TF0 = 0 = FT0$ and $SG0 = 0 = GS0$ which shows that the pairs (F,T) and (G,S) are weakly compatible. Thus all the conditions of Theorem 3.1 are satisfied and '0' is the unique common fixed point of F, G, S and T.

Here, it may be pointed out that all the four involved mappings are discontinuous which establishes the utility of our results over the ones hypothesizing continuity requirement.

4. An Application

Recently, Al-Thagafi and Shahzad [1] introduced the concept of C_q-commuting mappings and proved some common fixed point theorems along with results on invariant approximations. We extend the notion of C_q-commutativity to a pair of nonself mappings. As an application of Theorem 3.1, we prove a common fixed point theorem for the C_q-commuting mappings in Banach spaces.

Definition 4.1. Let (G, S) be a pair of nonself mappings defined on q-starshaped subset K of a normed space X with $q \in F(S)$ and $C_q(G,S) := \cup\{C(G_k, S) : 0 \leq k \leq 1\}$ where $G_k x = kGx + (1-k)q$. Then the pair (G, S) is said to be C_q-commuting if $GSx = SGx$ for all $x \in C_q(G,S)$ provided $Gx, Sx \in K$.

Clearly, C_q-commuting mappings are weakly compatible but converse need not be true in general. The following simple example illustrates the situation better.

Example 4.1. Let $X = \Re$ with usual norm and $K = [0, 3]$. Define $G, S : K \to X$ by $Gx = x^2$ for all $x \neq 2$ and $G2 = 1$, and $Sx = 2x$ for all $x \in K$. Then K is q-starshaped with $q = 0 \in F(S)$, $C(G,S) = \{0\}$ and $C_q(G,S) = K\backslash\{2\}$. It is easy to verify that (G, S) is a pair of weakly compatible nonself mappings but not C_q-commuting.

In an attempt to generalize the notion of weak commutativity due to Sessa [23], Pant [16] introduced the notions of pointwise R-weak commutativity and R-weak commutativity for self mappings. Imdad and Kumar [8] extended these notions for a pair of nonself mappings and proved some coincidence and fixed point theorems in metrically convex metric spaces.

Definition 4.2. [8]. Let K be a nonempty subset of a normed space $(X, \|.\|)$ and $G, S : K \to X$. Then the pair (G, S) is said to be pointwise R-weakly commuting on K if for a given $x \in K$ there exists a real number $R > 0$ such that

$$\|GSx - SGx\| \leq R\|Gx - Sx\|$$

provided $Gx, Sx \in K$.

If above inequality holds for all $x \in K$, then the pair of mappings is said to be R-weakly commuting on K. R-weakly commuting mappings are pointwise R-weakly commuting but converse need not be true (see Example 4.4). Notice that for $K = X$ and $R = 1$, R-weak commutativity reduces to weak commutativity for self mappings due to Sessa [23]. Here it may be noted that R-weak commutativity implies weak compatibility at the points of coincidence and remains a minimal condition to obtain results on common fixed points.

The classes of C_q-commuting and R-weakly commuting mappings are different. The following examples demonstrate the situation better.

The pair of mappings in Example 4.1 is R-weakly commuting but not C_q-commuting.

Example 4.2. Let $X = \Re$ with usual norm and $K = [1, 2]$. Define $G, S : K \to X$ as

$$Gx = \begin{cases} x^2, \text{ if } 1 < x \leq 2 \\ \\ 1, \text{ if } x = 1 \end{cases} \quad \text{and } Sx = \begin{cases} 2x, \text{ if } 1 < x < 2 \\ \\ 1, \text{ if } x \in \{1, 2\}. \end{cases}$$

K is q-starshaped set with $q = 1 \in F(S)$ and $C_q(G, S) = \{1\}$. It is easy to verify that the pair (G, S) is C_q-commuting but not R-weakly commuting.

However, there do exist pair of mappings which possesses both the properties at the same time and otherwise as well.

Example 4.3. Let $X = \Re$ with usual norm and $K = [0, \frac{5}{4}]$. Define $G, S : K \to X$ as

$$Sx = \begin{cases} 1, \text{ if } 0 \leq x \leq 1 \\ \\ \dfrac{2}{3}, \text{ if } 1 < x \leq \dfrac{5}{4} \end{cases} \quad \text{and } Gx = \begin{cases} 1 + \dfrac{x^2}{4}, \text{ if } x \in \{0\} \cup (1, \dfrac{5}{4}] \\ \dfrac{2}{3}, \text{ if } 0 < x < 1 \\ 1, \text{ if } x = 1. \end{cases}$$

Clearly, K is q-starshaped set with $q = 1 \in F(S)$ and $C_q(G, S) = \{0, 1\}$. It is easy to verify that the pair (G, S) is C_q-commuting as well as R-weakly commuting.

Example 4.4. Let $X = \Re$ with usual norm and $K = [0, 1]$. Define $G, S : [0, 1] \to [\frac{3}{4}, \frac{5}{4}) \subset \Re$ as

$$Sx = \left\{ \dfrac{3}{4}, \text{ if } 0 \leq x \leq 1 \right. \quad \text{and} Gx = \begin{cases} \dfrac{3}{4} + \dfrac{x^2}{2}, \text{ if } 0 \leq x < 1 \\ \dfrac{3}{4}, \text{ if } x = 1. \end{cases}$$

First, note that if $x \in [0, \frac{1}{\sqrt{2}}]$ then $Gx, Sx \in [0, 1] = K$. One can show that (G,S) is not R-weakly commuting on $[0, \frac{1}{\sqrt{2}}]$, for one cannot find $R > 0$ satisfying the definition of R-weak commutativity. But, for some $x \in [0, \frac{1}{\sqrt{2}}]$, one can always find some $R > 0$ satisfying the definition of pointwise R-weak commutativity. For instance if we take $x = \frac{1}{2\sqrt{2}}$ then

$$\left|SG\Big(\frac{1}{2\sqrt{2}}\Big) - GS\Big(\frac{1}{2\sqrt{2}}\Big)\right| \leq R \ \left|S\Big(\frac{1}{2\sqrt{2}}\Big) - G\Big(\frac{1}{2\sqrt{2}}\Big)\right|$$

holds for all $R \geq \frac{9}{2}$.

Notice that K is q-starshaped with $q = \frac{3}{4} \in F(S)$ and $C(G, S) = \{0, 1\}$ but the pair (G, S) is not weak compatible and hence not C_q-commuting on K.

Remark 4.1. It is straightforward to note that the pair (G, S) of C_q-commuting mappings is R-weakly commuting on $C_q(G, S)$ and hence pointwise R-weakly commuting. But converse need not be true (see Example 4.4).

Now we state and prove the following theorem in Banach spaces as an application of our main theorem.

Theorem 4.1. *Let K be a nonempty weakly compact q-starshaped subset of a Banach space X and (F, G) be a generalized S-nonexpansive mappings (with $\phi(t) = t$) of K into X satisfying*

(a) $\partial K \subset SK,\ (FK \cup GK) \cap K \subset SK,$

(b) $Sx \in \partial K \Rightarrow Fx, Gx \in K,$

(c) *the pair (G, S) is compatible and C_q-commuting.*

Moreover, if $(I - G)$ is demiclosed and S is affine, then the mappings F, G and S have a common fixed point in K provided S is continuous.

Proof. Notice that (due to Lemma 2.1) the pair (G, S) is reciprocally continuous. Since K is q-starshaped subset of X, then $(1 - t)q + tx \in K$ for all $x \in K$. Define a mapping $G_n : K \to X$ by $G_n x = (1 - k_n)q + k_n Gx$ for all $x \in K$, where $\{k_n\}$ is a sequence in [0,1] such that $k_n \to 1$. Then it is straightforward to verify that the pair (F, G_n) is generalized S-contraction mappings of K into X and G_n also satisfying conditions $(a) - (c)$. Since weak topology is Hausdorff and K is weakly compact, therefore K is weakly closed and hence strongly closed. Now by Theorem 3.1 (with $T = S$) for each $n \geq 2$, the mappings F, G_n and S have a unique common fixed point, say z_n. By the weak compactness of K, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ and $z \in K$ such that $z_{n_i} \to z$ weakly. Since weakly convergent sequences are bounded, therefore, $\{z_{n_i}\}$ is also bounded, i.e. there is a constant $\rho > 0$ such that $\|z_{n_i}\| < \rho$ for all $n \geq 2$. For each $n \geq 2$, we have

$$\begin{aligned}(I - G)z_{n_i} &= z_{n_i} - k_{n_i}^{-1}[G_{n_i} z_{n_i} - (1 - k_{n_i})q] \\ &= (1 - k_{n_i}^{-1})z_{n_i} + (k_{n_i}^{-1} - 1)q\end{aligned}$$

and hence

$$\|(I-G)z_{n_i}\| \leq |k_{n_i}^{-1} - 1|(\rho + \|q\|).$$

Since $k_{n_i}^{-1} \to 1$ as $n \to \infty$, we have $(I-G)z \to 0 \in K$. Also $z_{n_i} \rightharpoonup z \in K$ and $(I-G)$ is demiclosed, it follows that $(I-G)z = 0$ yielding thereby $Gz = z$. As z_{n_i} is a fixed point of S and S is continuous, then $Sz = z$. Suppose $Fz \neq z$, then using inequality (1.1.1)(for $T = S$)

$$\begin{aligned} d(Fz,z) = d(Fz,Gz) &\leq a\max\{\frac{1}{2}d(Sz,Sz), d(Sz,Fz), d(Sz,Gz)\} \\ &\quad +b[d(Sz,Gz)+d(Sz,Fz)]d(Fz,z) \\ &\leq (a+b)d(Fz,Sz) < d(Fz,Sz) = d(Fz,z) \end{aligned}$$

which is a contradiction. Hence z is a common fixed point of the mappings F, G and S. This completes the proof.

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