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Research Article

On Estimates of Deviation of Functions from Matrix Operators of Their Fourier Series by Some Expressions with r-Differences of the Entries

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We generalize the results of Krasniqi 2012 and Wei and Yu 2012 to the case of r-differences.

1. Introduction

Let $X = L^p$ or $X = C$, where L^p $(1 \le p \le \infty)$ $[C]$ is the class of all 2π–periodic real-valued functions, integrable in the Lebesgue sense with pth power when $p \ge 1$ and essentially bounded when $p = \infty$ [continuous], over $Q = [-\pi, \pi]$ with the norm

$$\|f\|_{L^p} := \|f(\cdot)\|_{L^p} = \begin{cases} \left(\int_Q |f(t)|^p \, dt\right)^{1/p} & \text{when } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t \in Q} |f(t)| & \text{when } p = \infty, \end{cases} \quad (1)$$

$$\|f\|_C := \|f(\cdot)\|_C = \sup_{t \in Q} |f(t)|$$

and consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} \left(a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x\right) \quad (2)$$

with the partial sums $S_k f$.

Let $A := (a_{n,k})$ be an infinite matrix of real numbers such that

$$a_{n,k} \ge 0$$

$$\text{when } k, n = 0, 1, 2, \ldots, \quad \lim_{n \to \infty} a_{n,k} = 0, \quad \sum_{k=0}^{\infty} a_{n,k} = 1. \quad (3)$$

The A-transformation of $(S_k f)$ be given by

$$T_{n,A} f(x) := \sum_{k=0}^{\infty} a_{n,k} S_k f(x) \quad (n = 0, 1, 2, \ldots). \quad (4)$$

In this paper, we study the upper bounds of $\|T_{n,A} f - f\|_X$ by the second modulus of continuity of f in the space X defined by the formula

$$\omega_2(f, \delta)_X := \sup_{0 \le |t| \le \delta} \|\varphi_x(t)\|_X, \quad (5)$$

where

$$\varphi_x(t) := f(x+t) + f(x-t) - 2f(x). \quad (6)$$

The deviation $T_{n,A} f - f$ with lower triangular infinite matrix such that

$$\begin{aligned} a_{n,k} &\ge 0 \quad \text{when } k = 0, 1, 2, \ldots, n, \\ a_{n,k} &= 0 \quad \text{when } k > n, \end{aligned} \quad (7)$$

$$\sum_{k=0}^{n} a_{n,k} = 1, \quad \text{where } n = 0, 1, 2, \ldots, \quad (8)$$

was estimated in the sup norm $\|\cdot\|_C$ by Krasniqi [1, Theorem 3.4, p. 290] (see also [2]) as follows.

Theorem 1. *Let $f \in C$ and $(a_{n,k})$ satisfy (8). Then,*

$$\|T_{n,A}f - f\|_C = O\left(\omega\left(f, \frac{\pi}{n}\right)_C + \sum_{k=1}^{n} \frac{\omega(f,\pi/k)_C}{k} \sum_{l=0}^{k+1} a_{n,l} + \sum_{k=1}^{n} \omega\left(f, \frac{\pi}{k}\right)_C \sum_{l=k}^{n} \left|a_{n,l} - a_{n,l+2}\right|\right), \quad (9)$$

where $\omega(f,\cdot)_C$ denotes the modulus of continuity of f and if $\omega(f,\cdot)_C$ is such that

$$\int_u^{\pi} t^{-2}\omega(f,t)_C\, dt = O(H(u)) \quad \text{when } u \in [0,\pi], \quad (10)$$

where $H(u) \geq 0$, we have

$$\|T_{n,A}f - f\|_C = O\left(\omega\left(f, \frac{\pi}{n}\right)_C + H\left(\frac{\pi}{n}\right) \sum_{k=0}^{n} \left|a_{n,k} - a_{n,k+2}\right|\right). \quad (11)$$

Additionally, if H satisfies the condition

$$\int_0^{u} H(t)\, dt = O(uH(u)) \quad \text{when } u \in [0,\pi], \quad (12)$$

then

$$\|T_{n,A}f - f\|_C = O\left(H\left(\sum_{k=0}^{n} \left|a_{n,k} - a_{n,k+2}\right|\right) \sum_{k=0}^{n} \left|a_{n,k} - a_{n,k+2}\right|\right),$$
$$\|T_{n,A}f - f\|_C = O\left(H\left(\frac{\pi}{n}\right) \sum_{k=0}^{n} \left|a_{n,k} - a_{n,k+2}\right|\right). \quad (13)$$

In our theorems we will consider the r-differences $a_{n,k} - a_{n,k+r}$ instead of $a_{n,k} - a_{n,k+2}$ considered above. We will formulate the general relation for $r \in \mathbb{N}$ like it formulated only for $r = 2$ in [1, Theorem 3.4, p. 290].

2. Statement of the Results

Let us consider a function ω of modulus of continuity type on the interval $[0, 2\pi]$, that is, a nondecreasing continuous function having the following properties: $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$.

Suppose that ω satisfies the condition

$$\int_u^{\pi} t^{-2}\omega(t)\, dt = O(H(u)) \quad \text{when } u \in [0,\pi], \quad (14)$$

where $H(u) \geq 0$ is such that

$$\int_0^{u} H(t)\, dt = O(uH(u)) \quad \text{when } u \in [0,\pi]. \quad (15)$$

Let

$$X_\omega = \left\{f \in X : \omega_2(f,\delta)_X = O(\omega(\delta)) \text{ when } \delta \in [0,2\pi]\right\}. \quad (16)$$

Our main results on the degrees of approximation are the following.

Theorem 2. *If $f \in X_\omega$, where ω satisfies condition (14) such that (15) holds and $r \in \mathbb{N}$, then*

$$\|T_{n,A}f - f\|_X = O\left(H\left(\frac{\pi}{n+1}\right)\left(\frac{\pi}{n+1} + \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|\right)\right). \quad (17)$$

Additionally, if a matrix A is such that

$$\left[\sum_{l=0}^{n} \sum_{k=l}^{r+l-1} a_{n,k}\right]^{-1} = O(1), \quad (18)$$

then

$$\|T_{n,A}f - f\|_X = O\left(H\left(\frac{\pi}{n+1}\right) \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|\right). \quad (19)$$

Theorem 3. *If $f \in X_\omega$, where ω satisfies condition (14) such that (15) holds and $r \in \mathbb{N}$, then*

$$\|T_{n,A}f - f\|_X = O\left(H\left(\sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|\right) \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|\right). \quad (20)$$

Theorem 4. *Let $f \in X_\omega$, where ω satisfies condition (14) such that (15) holds and $r \in \mathbb{N}$. If a matrix A is such that*

$$\sum_{k=0}^{\infty} (k+1)\, a_{n,k} = O(n+1), \quad (21)$$

then

$$\|T_{n,A}f - f\|_X = O\left(\omega\left(\frac{\pi}{n+1}\right) + H\left(\frac{\pi}{n+1}\right) \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|\right). \quad (22)$$

Theorem 5. *If $f \in X$ and a matrix A is such that (21) holds and $r \in \mathbb{N}$, then*

$$\|T_{n,A}f - f\|_X = O\left(\omega_2\left(f, \frac{\pi}{n+1}\right)_X + \sum_{\mu=1}^{n} \frac{\omega_2(f,\pi/\mu)_X}{\mu} \sum_{k=0}^{\mu+1} a_{n,k} + \sum_{\mu=1}^{n} \omega_2\left(f, \frac{\pi}{\mu}\right)_X \sum_{k=\mu}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|\right). \quad (23)$$

Corollary 6. *For a lower triangular infinite matrix A conditions (18) and (21) hold always and therefore for a lower triangular infinite matrix such that conditions (7) and (8) hold one can obtain the results from Theorems 2, 4, and 5 without the assumptions (18) and (21), where the mentioned results of Krasniqi follow as the special cases with $r = 2$. Moreover, one can consider the essentially wider class of sequences than in the mentioned paper with the same degrees of approximation (see, e.g., [3, Theorem 2]).*

Corollary 7. *From Theorem 5 it follows that if $f \in X_\omega$ where ω satisfies condition (14) such that (15) holds and a matrix A is such that (21) and*

$$\sum_{k=m}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| \leq K \sum_{k=m/c}^{\infty} \frac{a_{n,k}}{k}, \tag{24}$$

with some $c > 1$ and $r \in \mathbb{N}$, hold, then

$$\begin{aligned} &\left\|T_{n,A} f - f\right\|_X \\ &\quad = O\left(\frac{H(\pi/(n+1))}{n+1} + \sum_{k=1}^{n} a_{n,k} \frac{H(\pi/(k+1))}{k+1}\right). \end{aligned} \tag{25}$$

Remark 8. The class of sequences defined by condition (24) was introduced by the second author in [4]. The similar classes were considered by Dyachenko and Tikhonov in [5] with $r = 1$ (see also [6]).

Remark 9. We note that instead of (14) and (15) one can use Bari-Stechkin conditions

$$\begin{aligned} &\int_0^u t^{-1}\omega(t)\,dt = O(\omega(u)), \\ &\int_u^{2\pi} t^{-2}\omega(t)\,dt = O(\omega(u)) \\ &\qquad\qquad \text{when } u \in [0, 2\pi]. \end{aligned} \tag{26}$$

Then all results are true for $t^{-1}\omega(t)$ instead of $H(y)$ and Lemmas 12 and 14 are not necessary.

3. Auxiliary Results

We begin this section by some notations from [7]. Let for $r = 1, 2, \ldots$

$$\begin{aligned} D_{k,r}(t) &= \frac{\sin((2k+r)t/2)}{2\sin(rt/2)}, \\ \widetilde{D}_{k,r}(t) &= \frac{\cos((2k+r)t/2)}{2\sin(rt/2)}. \end{aligned} \tag{27}$$

It is clear by [8] that

$$\begin{aligned} S_k f(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_{k,1}(t)\,dt, \\ T_{n,A} f(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{k=0}^{n} a_{n,k} D_{k,1}(t)\,dt. \end{aligned} \tag{28}$$

Hence

$$T_{n,A} f(x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \varphi_x(t) \sum_{k=0}^{n} a_{n,k} D_{k,1}(t)\,dt. \tag{29}$$

Next, we present the known estimates.

Lemma 10 (see [8]). *If $0 < |t| \leq \pi$ then*

$$\begin{aligned} \left|D_{k,1}(t)\right| &\leq \frac{\pi}{2|t|}, \\ \left|\widetilde{D}_{k,1}(t)\right| &\leq \frac{\pi}{2|t|} \end{aligned} \tag{30}$$

and, for any real t, we have

$$\left|D_{k,1}(t)\right| \leq k + \frac{1}{2}. \tag{31}$$

Lemma 11 (see [7]). *Let $r \in N$, $l \in Z$, and $a := (a_n) \subset \mathbb{C}$. If $x \neq 2l\pi/r$ then for every $m \geq n$*

$$\begin{aligned} \sum_{k=n}^{m} a_k \sin kx = &-\sum_{k=n}^{m} (a_k - a_{k+r}) \widetilde{D}_{k,r}(t) \\ &+ \sum_{k=m+1}^{m+r} a_k \widetilde{D}_{k,-r}(t) \\ &- \sum_{k=n}^{n+r-1} a_k \widetilde{D}_{k,-r}(t), \\ \sum_{k=n}^{m} a_k \cos kx = &\sum_{k=n}^{m} (a_k - a_{k+r}) D_{k,r}(t) \\ &- \sum_{k=m+1}^{m+r} a_k D_{k,-r}(t) \\ &+ \sum_{k=n}^{n+r-1} a_k D_{k,-r}(t). \end{aligned} \tag{32}$$

Lemma 12 (see [9]). *If (14) and (15) hold then*

$$\int_0^u t^{-1}\omega(t)\,dt = O(uH(u)) \quad \text{when } u \in [0, 2\pi]. \tag{33}$$

We additionally proved two slight changed estimates.

Lemma 13. *If (14) and (15) hold, then, for $c \geq 1$ and $\beta > \alpha > 0$,*

$$\begin{aligned} &\int_\alpha^\beta t^{-1}\omega(t)\,dt = O((\beta-\alpha) H(c(\beta-\alpha))) \\ &\qquad\qquad \text{when } (\beta-\alpha) \in [0, 2\pi]. \end{aligned} \tag{34}$$

Proof. By substitution of $t = u+\alpha$ and monotonicity of $\omega(u)/u$ (see Lemma 15) we obtain, for $c \geq 1$,

$$\begin{aligned} \int_\alpha^\beta t^{-1}\omega(t)\,dt &= \int_0^{\beta-\alpha} \frac{\omega(u+\alpha)}{u+\alpha}du \leq 2\int_0^{\beta-\alpha} \frac{\omega(u)}{u}du \\ &\leq 2\int_0^{c(\beta-\alpha)} \frac{\omega(u)}{u}du \end{aligned} \tag{35}$$

and the desired result follows from the Lemma 12. □

Lemma 14. *If (14) and (15) hold, then, for $b \ge 1$,*

$$\int_u^\pi t^{-2}\omega(t)\,dt = O(H(bu)) \quad \text{when } u \in [0,\pi]. \tag{36}$$

Proof. Using Lemma 13 with $c = 1$, $\alpha = 0$ and (14) our result follows

$$\begin{aligned}\int_u^\pi t^{-2}\omega(t)\,dt &= \left(\int_u^{bu} + \int_{bu}^{\pi}\right) t^{-2}\omega(t)\,dt\\ &\le \frac{1}{u}\int_u^{bu}\frac{\omega(t)}{t}dt + O(H(bu))\\ &\le \frac{1}{u}\int_0^{bu}\frac{\omega(t)}{t}dt + O(H(bu))\\ &= \frac{1}{u}O(buH(bu)) + O(H(bu))\\ &= O((b+1)H(bu)),\end{aligned} \tag{37}$$

for $b \ge 1$. □

Finally, we present very useful property of the modulus of continuity.

Lemma 15 (see [8]). *A function ω of modulus of continuity type on the interval $[0, 2\pi]$ satisfies the following condition:*

$$\delta_2^{-1}\omega(\delta_2) \le 2\delta_1^{-1}\omega(\delta_1) \quad \text{for } \delta_2 \ge \delta_1 > 0. \tag{38}$$

4. Proofs of the Results

Proof of Theorem 2. It is clear that for an odd r

$$\begin{aligned}&T_{n,A}f(x) - f(x)\\ &= \frac{1}{\pi}\sum_{m=0}^{[r/2]}\int_{2m\pi/r}^{2m\pi/r+\pi/r}\varphi_x(t)\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\,dt\\ &\quad + \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r}\varphi_x(t)\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\,dt\\ &= I_1(x) + I_2(x)\end{aligned} \tag{39}$$

and for an even r

$$\begin{aligned}&T_{n,A}f(x) - f(x) = \frac{1}{\pi}\\ &\cdot\sum_{m=0}^{[r/2]-1}\left(\int_{2m\pi/r}^{2m\pi/r+\pi/r} + \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r}\right)\varphi_x(t)\\ &\cdot\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\,dt = I_1'(x) + I_2(x).\end{aligned} \tag{40}$$

Then,

$$\|T_{n,A}f - f\|_X \le \|I_1\|_X + \|I_1'\|_X + \|I_2\|_X. \tag{41}$$

Since, by Lemma 11,

$$\begin{aligned}\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t) &= \sum_{k=0}^{\infty}a_{n,k}\frac{\sin((2k+1)t/2)}{2\sin(t/2)}\\ &= \frac{1}{2\sin(t/2)}\left(\sum_{k=0}^{\infty}a_{n,k}\sin kt\cos(t/2)\right.\\ &\quad\left.+ \sum_{k=0}^{\infty}a_{n,k}\cos kt\sin(t/2)\right)\\ &= \frac{\cos(t/2)}{2\sin(t/2)}\left(-\sum_{k=0}^{\infty}(a_{n,k} - a_{n,k+r})\widetilde{D}_{k,r}(t)\right.\\ &\quad\left.- \sum_{k=0}^{r-1}a_{n,k}\widetilde{D}_{k,-r}(t)\right) + \frac{1}{2}\left(\sum_{k=0}^{\infty}(a_{n,k} - a_{n,k+r})D_{k,r}(t)\right.\\ &\quad\left.+ \sum_{k=0}^{r-1}a_{n,k}D_{k,-r}(t)\right),\end{aligned} \tag{42}$$

then

$$\begin{aligned}&\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\right|\\ &\le \frac{1}{2|\sin(t/2)\sin(rt/2)|}\left(\sum_{k=0}^{\infty}|a_{n,k} - a_{n,k+r}|\right.\\ &\quad\left.+ \sum_{k=0}^{r-1}a_{n,k}\right) \le \frac{1}{|\sin(t/2)\sin(rt/2)|}\sum_{k=0}^{\infty}|a_{n,k} - a_{n,k+r}|.\end{aligned} \tag{43}$$

Hence, by Lemma 10,

$$\begin{aligned}&\|I_1\|_X \le \frac{1}{\pi}\sum_{m=0}^{[r/2]}\int_{2m\pi/r}^{2m\pi/r+\pi/r}\|\varphi_x(t)\|_X\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\right|dt = \frac{1}{\pi}\\ &\cdot\sum_{m=0}^{[r/2]}\left(\int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} + \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r}\right)\|\varphi_x(t)\|_X\\ &\cdot\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\right|dt \le \frac{1}{2}\sum_{m=0}^{[r/2]}\int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)}\frac{O(\omega(t))}{t}dt\\ &+ \frac{1}{\pi}\\ &\cdot\sum_{m=0}^{[r/2]}\int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r}\frac{O(\omega(t))}{|\sin(t/2)\sin(rt/2)|}\sum_{k=0}^{\infty}|a_{n,k} - a_{n,k+r}|\,dt.\end{aligned} \tag{44}$$

Using Lemmas 13 and 14 with $c = b = r$ and the estimates $|\sin(t/2)| \ge |t|/\pi$ for $t \in [0,\pi]$ and $|\sin(rt/2)| \ge rt/\pi - 2m$ for $t \in [2m\pi/r, 2m\pi/r + \pi/r]$, where $m \in \{0,\ldots,[r/2]\}$, we obtain

$$\begin{aligned}&\|I_1\|_X \le O(1)\left(\left[\frac{r}{2}\right] + 1\right)\frac{\pi}{r(n+1)}H\left(\frac{\pi}{n+1}\right)\\ &\quad + \sum_{k=0}^{\infty}|a_{n,k} - a_{n,k+r}|\end{aligned}$$

$$\cdot \sum_{m=0}^{[r/2]} \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r} \frac{O(\omega(t))}{t(rt/\pi - 2m)} dt = O(1) \frac{\pi}{n+1} \cdot H\left(\frac{\pi}{n+1}\right) + \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| \cdot \sum_{m=0}^{[r/2]} \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r} \frac{O(\omega(t))}{(rt/\pi)(t - 2m\pi/r)} dt = O(1) \cdot \frac{\pi}{n+1} H\left(\frac{\pi}{n+1}\right) + \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| \cdot \sum_{m=0}^{[r/2]} \int_{\pi/r(n+1)}^{\pi/r} \frac{O(\omega(t+2m\pi/r))}{(rt/\pi)(t+2m\pi/r)} dt \le O(1) \cdot \left[\frac{\pi}{n+1} H\left(\frac{\pi}{n+1}\right) + \left(\left[\frac{r}{2}\right]+1\right)\frac{2\pi}{r} \cdot \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| \int_{\pi/r(n+1)}^{\pi/r} \frac{\omega(t)}{t^2} dt\right] = O(1) \cdot \left[\frac{\pi}{n+1} H\left(\frac{\pi}{n+1}\right) + \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| H\left(\frac{\pi}{n+1}\right)\right]. \tag{45}$$

Analogously

$$\left\|I_1'\right\|_X = O(1)\left[\frac{r}{2}\right]\left[\frac{\pi}{r(n+1)} H\left(\frac{\pi}{n+1}\right) + \frac{2\pi}{r}\sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| H\left(\frac{\pi}{n+1}\right)\right]. \tag{46}$$

Similarly, by Lemmas 10, 13, and 14 with $c = b = r$ and the estimates $|\sin(t/2)| \ge |t|/\pi$ for $t \in [0,\pi]$ and $|\sin(rt/2)| \ge 2(m+1) - rt/\pi$ for $t \in [2(m+1)\pi/r - \pi/r, 2(m+1)\pi/r - \pi/r(n+1)]$, where $m \in \{0, \ldots, [r/2]-1\}$, we get

$$\left\|I_2\right\|_X \le \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r} \left\|\varphi_x(t)\right\|_X \left|\sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\right| dt = \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \left(\int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} + \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r}\right) \cdot \left\|\varphi_x(t)\right\|_X \left|\sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\right| dt \le \frac{1}{2} \cdot \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{O(\omega(t))}{t} dt + \frac{1}{\pi} \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| \cdot \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} \frac{O(\omega(t))}{\left|\sin(t/2)\sin(rt/2)\right|} dt \le \frac{1}{2} \cdot \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{O(\omega(t))}{t} dt + \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| \cdot \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} \frac{O(\omega(t))}{(rt/\pi)\left[2(m+1)\pi/r - t\right]} dt = \frac{1}{2} \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{O(\omega(t))}{t} dt + \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| \sum_{m=0}^{[r/2]-1} \int_{\pi/r(n+1)}^{\pi/r} \frac{O(\omega(-t+2(m+1)\pi/r))}{(r/\pi)t(-t+2(m+1)\pi/r)} dt \le \frac{1}{2} \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{O(\omega(t))}{t} dt + \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| \left[\frac{r}{2}\right] \frac{2\pi}{r} \int_{\pi/r(n+1)}^{\pi/r} \frac{O(\omega(t))}{t^2} dt. \tag{47}$$

Thus

$$\left\|I_2\right\|_X = O(1)\left[\frac{\pi}{n+1} H\left(\frac{\pi}{n+1}\right) + \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right| H\left(\frac{\pi}{n+1}\right)\right]. \tag{48}$$

Collecting these estimates we obtain the first result.

Applying condition (18) we have

$$\left[(n+1)\sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|\right]^{-1} = \left[\sum_{l=0}^{n}\sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|\right]^{-1} \le \left[\sum_{l=0}^{n}\sum_{k=l}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|\right]^{-1} \le \left[\sum_{l=0}^{n}\left|\sum_{k=l}^{\infty} \left(a_{n,k} - a_{n,k+r}\right)\right|\right]^{-1} \le \left[\sum_{l=0}^{n}\sum_{k=l}^{r+l-1} a_{n,k}\right]^{-1} = O(1) \tag{49}$$

and the second result also follows. □

Proof of Theorem 3. Analogously, as in the proof of Theorem 2, we consider an odd r and an even r. Then,

$$T_{n,A} f(x) - f(x) = I_1(x) + I_2(x) \tag{50}$$

or

$$T_{n,A}f(x) - f(x) = I_1'(x) + I_2(x), \tag{51}$$

respectively, and

$$\|T_{n,A}f - f\|_X \leq \|I_1\|_X + \|I_1'\|_X + \|I_2\|_X. \tag{52}$$

Since $A_{n,r} = \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \leq 2$

$$\begin{aligned}
\|I_1\|_X &\leq \frac{1}{\pi}\sum_{m=0}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r+\pi/r} \|\varphi_x(t)\|_X \left|\sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\right| dt \\
&= \frac{1}{\pi}\sum_{m=0}^{[r/2]} \left(\int_{2m\pi/r}^{2m\pi/r+(1/r)A_{n,r}} + \int_{2m\pi/r+(1/r)A_{n,r}}^{2m\pi/r+\pi/r}\right) \\
&\cdot \|\varphi_x(t)\|_X \left|\sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\right| dt, \\
\|I_2\|_X &\leq \frac{1}{\pi} \\
&\cdot \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r} \|\varphi_x(t)\|_X \left|\sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\right| dt \leq \frac{1}{\pi} \\
&\cdot \sum_{m=0}^{[r/2]-1} \left(\int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-(1/r)A_{n,r}} + \int_{2(m+1)\pi/r-(1/r)A_{n,r}}^{2(m+1)\pi/r}\right) \\
&\cdot \|\varphi_x(t)\|_X \left|\sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\right| dt.
\end{aligned} \tag{53}$$

Therefore, in the terms I_1, I_1', and I_2 we can estimate analogously as in the proof of Theorem 2 and thus we obtain the desired estimate. □

Proof of Theorem 4. Similarly, as in the proof of Theorem 2, we consider an odd r and an even r. Then,

$$\begin{aligned}
&T_{n,A}f(x) - f(x) \\
&\quad = \frac{1}{\pi}\int_0^{\pi/r(n+1)} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\, dt \\
&\quad + \frac{1}{\pi}\int_{\pi/r(n+1)}^{\pi/r} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\, dt \\
&\quad + \frac{1}{\pi}\sum_{m=1}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r+\pi/r} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\, dt \\
&\quad + \frac{1}{\pi}\sum_{m=0}^{[r/2]-1} \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\, dt \\
&\quad = J_1(x) + J_2(x) + I_1''(x) + I_2(x)
\end{aligned} \tag{54}$$

or

$$\begin{aligned}
&T_{n,A}f(x) - f(x) = J_1(x) + J_2(x) \\
&\quad + \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]-1} \int_{2m\pi/r}^{2m\pi/r+\pi/r} + \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r}\right) \varphi_x(t) \\
&\quad \cdot \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\, dt = J_1(x) + J_2(x) + I_1'''(x) \\
&\quad + I_2(x),
\end{aligned} \tag{55}$$

respectively. Therefore,

$$\begin{aligned}
\|T_{n,A}f - f\|_X &\leq \|J_1\|_X + \|J_2\|_X + \|I_1''\|_X + \|I_1'''\|_X \\
&\quad + \|I_2\|_X.
\end{aligned} \tag{56}$$

By Lemma 10 and (18),

$$\begin{aligned}
\|J_1\|_X &\leq \frac{1}{\pi}\int_0^{\pi/r(n+1)} \|\varphi_x(t)\|_X \left|\sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\right| dt \\
&\leq \frac{1}{\pi}\sum_{k=0}^{\infty} (k+1)\, a_{n,k} \int_0^{\pi/r(n+1)} \omega(t)\, dt \\
&= O(1)(n+1)\int_0^{\pi/r(n+1)} \omega(t)\, dt \\
&\leq O(1)\frac{\pi}{r}\omega\left(\frac{\pi}{r(n+1)}\right) = O\left(\omega\left(\frac{\pi}{n+1}\right)\right).
\end{aligned} \tag{57}$$

Further, by the same lemmas and conditions as above and Lemma 15, we obtain, with

$$\kappa = \begin{cases} 1 & \text{when } r \text{ is even}, \\ 0 & \text{when } r \text{ is odd}, \end{cases} \tag{58}$$

that

$$\begin{aligned}
\|I_1''\|_X + \|J_2\|_X + \|I_1'''\|_X &\leq \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]-\kappa} \int_{2m\pi/r}^{2m\pi/r+\pi/r} + \int_{\pi/r(n+1)}^{\pi/r}\right) \|\varphi_x(t)\|_X \left|\sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\right| dt \\
&= \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]-\kappa} \int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} + \sum_{m=0}^{[r/2]-\kappa} \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r}\right) \|\varphi_x(t)\|_X \left|\sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t)\right| dt
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\pi}\sum_{m=1}^{[r/2]-\kappa}\int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)}\frac{O(\omega(t))}{2|\sin(t/2)|}dt\\
&\quad+\frac{1}{\pi}\sum_{m=0}^{[r/2]-\kappa}\int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r}\frac{O(\omega(t))}{|\sin(t/2)\sin(rt/2)|}\sum_{k=0}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|dt\\
&\le \frac{1}{2}\sum_{m=1}^{[r/2]-\kappa}\int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)}\frac{O(\omega(t))}{t}dt+\sum_{k=0}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\sum_{m=0}^{[r/2]-\kappa}\int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r}\frac{O(\omega(t))}{t(rt/\pi-2m)}dt\\
&\le \sum_{m=1}^{[r/2]-\kappa}\frac{O(\omega(2m\pi/r))}{2m\pi/r}\int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)}dt\\
&\quad+\sum_{k=0}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\sum_{m=0}^{[r/2]-\kappa}\int_{\pi/r(n+1)}^{\pi/r}\frac{O(\omega(t+2m\pi/r))}{(rt/\pi)(t+2m\pi/r)}dt\\
&\le 2\sum_{m=1}^{[r/2]-\kappa}\frac{O(\omega(2\pi/r))}{2\pi/r}\frac{\pi}{r(n+1)}+\frac{2\pi}{r}\left(\left[\frac{r}{2}\right]+1\right)\sum_{k=0}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\int_{\pi/r(n+1)}^{\pi/r}\frac{O(\omega(t))}{t^{2}}dt\\
&=O(1)\left[\omega\left(\frac{\pi}{n+1}\right)+\sum_{k=0}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|H\left(\frac{\pi}{n+1}\right)\right],\\
\left\|I_2\right\|_X &\le \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\left(\int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)}+\int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r}\right)\left\|\varphi_x(t)\right\|_X\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\right|dt\\
&\le \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)}\frac{O(\omega(t))}{|\sin(t/2)\sin(rt/2)|}\sum_{k=0}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|dt\\
&\quad+\frac{1}{2}\sum_{m=0}^{[r/2]-1}\int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r}\frac{O(\omega(t))}{t}dt\\
&\le \sum_{m=0}^{[r/2]-1}\sum_{k=0}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)}\frac{O(\omega(t))}{(rt/\pi)[2(m+1)\pi/r-t]}dt\\
&\quad+\sum_{m=0}^{[r/2]-1}\frac{O(\omega(2(m+1)\pi/r-\pi/r(n+1)))}{2(m+1)\pi/r-\pi/r(n+1)}\int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r}dt\\
&\le\left[\frac{r}{2}\right]\left[\frac{2\pi}{r}\sum_{k=0}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\int_{\pi/r(n+1)}^{\pi/r}\frac{O(\omega(t))}{t^{2}}dt+2\frac{O(\omega(\pi/r))}{\pi/r}\frac{\pi}{r(n+1)}\right]\\
&=O(1)\left[\sum_{k=0}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|H\left(\frac{\pi}{n+1}\right)+\omega\left(\frac{\pi}{n+1}\right)\right].
\end{aligned}
\tag{59}
$$

Thus our proof is complete. □

Proof of Theorem 5. Let as above

$$
\begin{aligned}
&\left\|T_{n,A}f-f\right\|_X\\
&\quad\le\left\|J_1\right\|_X+\left\|J_2\right\|_X+\left\|I_1''\right\|_X+\left\|I_1'''\right\|_X+\left\|I_2\right\|_X,
\end{aligned}
$$

$$
\begin{aligned}
\left\|J_1\right\|_X &\le \frac{1}{\pi}\int_0^{\pi/r(n+1)}\left\|\varphi_x(t)\right\|_X\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\right|dt\\
&\le\frac{1}{\pi}\sum_{k=0}^{\infty}(k+1)a_{n,k}\int_0^{\pi/r(n+1)}\omega_2(f,t)_X\,dt\\
&=O(n+1)\int_0^{\pi/r(n+1)}\omega_2(f,t)_X\,dt
\end{aligned}
$$

$$\leq O(1)\frac{\pi}{r}\omega_2\left(f,\frac{\pi}{r(n+1)}\right)_X = O(1)\,\omega_2\left(f,\frac{\pi}{n+1}\right)_X. \tag{60}$$

Further, taking $\tau_m^1 = [\pi/(rt-2m\pi)]$ and $\tau = [\pi/rt]$, using Lemma 15, we obtain, with

$$\kappa = \begin{cases} 1 & \text{when } r \text{ is even,} \\ 0 & \text{when } r \text{ is odd,} \end{cases} \tag{61}$$

that

$$\begin{aligned}
&\left\|I_1''\right\|_X + \left\|J_2\right\|_X + \left\|I_1'''\right\|_X \leq \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]-\kappa}\int_{2m\pi/r}^{2m\pi/r+\pi/r} + \int_{\pi/r(n+1)}^{\pi/r}\right)\left\|\varphi_x(t)\right\|_X\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\right|dt \\
&= \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]-\kappa}\int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} + \sum_{m=0}^{[r/2]-\kappa}\int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r}\right)\left\|\varphi_x(t)\right\|_X \cdot\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\right|dt \leq \frac{1}{\pi} \\
&\cdot\sum_{m=1}^{[r/2]-\kappa}\int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)}\frac{\omega_2(f,t)_X}{2\left|\sin(t/2)\right|}\sum_{k=0}^{\infty}a_{n,k}\,dt + \frac{1}{\pi} \\
&\cdot\sum_{m=0}^{[r/2]-\kappa}\int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r}\left(\frac{\omega_2(f,t)_X}{2\left|\sin(t/2)\right|}\sum_{k=0}^{\tau_m^1}a_{n,k} + \frac{\omega_2(f,t)_X}{\left|\sin(t/2)\sin(rt/2)\right|}\sum_{k=\tau_m^1}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\right)dt \\
&\leq \frac{1}{2}\sum_{m=1}^{[r/2]-\kappa}\int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)}\frac{\omega_2(f,t)_X}{t}dt + \frac{1}{2}\left(\left[\frac{r}{2}\right]+1\right)\int_{\pi/r(n+1)}^{\pi/r}\frac{\omega_2(f,t)_X}{t}\sum_{k=0}^{\tau}a_{n,k}\,dt + \frac{2\pi}{r}\left(\left[\frac{r}{2}\right]+1\right) \\
&\cdot\int_{\pi/r(n+1)}^{\pi/r}\frac{\omega_2(f,t)_X}{t^2}\sum_{k=\tau}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|dt \\
&\leq \sum_{m=1}^{[r/2]-\kappa}\frac{\omega_2(f,2m\pi/r)_X}{2m\pi/r}\int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)}dt + \frac{1}{2}\left(\left[\frac{r}{2}\right]+1\right)\sum_{\mu=1}^{n}\int_{\mu}^{\mu+1}\frac{\omega_2(f,\pi/rt)_X}{\pi/rt}\sum_{k=0}^{[t]}a_{n,k}\frac{\pi dt}{rt^2} \\
&+ \frac{2\pi}{r}\left(\left[\frac{r}{2}\right]+1\right)\cdot\sum_{\mu=1}^{n}\int_{\mu}^{\mu+1}\frac{\omega_2(f,\pi/rt)_X}{(\pi/rt)^2}\sum_{k=[t]}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\frac{\pi dt}{rt^2} \\
&\leq O(1)\,\omega_2\left(f,\frac{\pi}{n+1}\right)_X + O(1)\cdot\sum_{\mu=1}^{n}\frac{\omega_2(f,\pi/\mu)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k} + O(1)\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\cdot\sum_{k=\mu}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|.
\end{aligned} \tag{62}$$

Next, taking $\tau_m^2 = [\pi/(-rt+2(m+1)\pi)]$, we obtain

$$\begin{aligned}
\left\|I_2\right\|_X &\leq \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r}\left\|\varphi_x(t)\right\|_X\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\right|dt \\
&\leq \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\left(\int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} + \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r}\right)\left\|\varphi_x(t)\right\|_X\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}(t)\right|dt \\
&\leq \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)}\left(\frac{\omega_2(f,t)_X}{2\left|\sin(t/2)\right|}\sum_{k=0}^{\tau_m^2}a_{n,k} + \frac{\omega_2(f,t)_X}{\left|\sin(t/2)\sin(rt/2)\right|}\sum_{k=\tau_m^2}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\right)dt \\
&\quad + \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r}\frac{\omega_2(f,t)_X}{2\left|\sin(t/2)\right|}\sum_{k=0}^{\infty}a_{n,k}\,dt \\
&\leq \frac{1}{2}\sum_{m=0}^{[r/2]-1}\int_{\pi/r(n+1)}^{\pi/r}\frac{\omega_2(-t+2(m+1)\pi/r)_X}{-t+2(m+1)\pi/r}\sum_{k=0}^{\tau}a_{n,k}\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=0}^{[r/2]-1}\int_{\pi/r(n+1)}^{\pi/r}\frac{\omega_2\left(-t+2\left(m+1\right)\pi/r\right)_X}{\left(r/\pi\right)t\left(-t+2\left(m+1\right)\pi/r\right)}\sum_{k=\tau}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|dt+\frac{1}{2}\sum_{m=0}^{[r/2]-1}\int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r}\frac{\omega_2\left(t\right)_X}{t}dt\\
&\leq\left[\frac{r}{2}\right]\int_{\pi/r(n+1)}^{\pi/r}\frac{\omega_2\left(f,t\right)_X}{t}\sum_{k=0}^{\tau}a_{n,k}+\frac{2\pi}{r}\left[\frac{r}{2}\right]\int_{\pi/r(n+1)}^{\pi/r}\frac{\omega_2\left(f,t\right)_X}{t^2}\sum_{k=\tau}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|dt\\
&+\sum_{m=0}^{[r/2]-1}\frac{\omega_2\left(f,2\left(m+1\right)\pi/r-\pi/r\left(n+1\right)\right)_X}{2\left(m+1\right)\pi/r-\pi/r\left(n+1\right)}\int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r}dt\\
&\leq O\left(1\right)\sum_{\mu=1}^{n}\frac{\omega_2\left(f,\pi/\mu\right)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}+O\left(1\right)\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\sum_{k=\mu}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|+O\left(1\right)\omega_2\left(f,\frac{\pi}{n+1}\right)_X.
\end{aligned}
\tag{63}
$$

Thus the result follows. □

Proof of Corollary 7. Theorem 5 implies that

$$
\begin{aligned}
\left\|T_{n,A}f-f\right\|_X=O\Bigg(&\omega_2\left(f,\frac{\pi}{n+1}\right)_X\\
&+\sum_{\mu=1}^{n}\frac{\omega_2\left(f,\pi/\mu\right)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}\\
&+\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\sum_{k=\mu}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\Bigg).
\end{aligned}
\tag{64}
$$

Since in (24)

$$
\begin{aligned}
&\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\sum_{k=\mu}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|=O\left(1\right)\\
&\cdot\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\left(\sum_{k=\mu/c}^{\mu-1}\frac{a_{n,k}}{k}+\sum_{k=\mu}^{n}\frac{a_{n,k}}{k}+\sum_{k=n+1}^{\infty}\frac{a_{n,k}}{k}\right)\\
&\leq O\left(1\right)\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\left(\sum_{k=\mu/c}^{\mu-1}\frac{a_{n,k}}{k}\right)+O\left(1\right)\\
&\cdot\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\left(\sum_{k=\mu}^{n}\frac{a_{n,k}}{k}+\sum_{k=n+1}^{\infty}\frac{a_{n,k}}{k}\right)\leq O\left(1\right)\\
&\cdot c\sum_{\mu=1}^{n}\frac{\omega_2\left(f,\pi/\mu\right)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}+O\left(1\right)\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\\
&\cdot\left(\sum_{k=\mu}^{n}\frac{a_{n,k}}{k}+\sum_{k=n+1}^{\infty}\frac{a_{n,k}}{k}\right)
\end{aligned}
\tag{65}
$$

one has

$$
\begin{aligned}
&\left\|T_{n,A}f-f\right\|_X=O\left(1\right)\omega_2\left(f,\frac{\pi}{n+1}\right)_X+O\left(1\right)\left(1+c\right)\\
&\cdot\sum_{\mu=1}^{n}\frac{\omega_2\left(f,\pi/\mu\right)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}+O\left(1\right)\\
&\cdot\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\sum_{k=\mu}^{n}\frac{a_{n,k}}{k}+O\left(1\right)\\
&\cdot\sum_{k=n+1}^{\infty}\frac{a_{n,k}}{k}\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\leq O\left(1\right)\omega_2\left(f,\frac{\pi}{n+1}\right)_X\\
&+O\left(1\right)\left(1+c\right)\left\{\sum_{\mu=1}^{n}\frac{\omega_2\left(f,\pi/\mu\right)_X}{\mu}\sum_{k=0}^{\mu-1}a_{n,k}\right.\\
&\left.+2\sum_{\mu=1}^{n}\frac{\omega_2\left(f,\pi/\mu\right)_X}{\mu+1}a_{n,\mu+1}+\sum_{\mu=1}^{n}\frac{\omega_2\left(f,\pi/\mu\right)_X}{\mu}a_{n,\mu}\right\}\\
&+O\left(1\right)\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\sum_{k=\mu}^{n}\frac{a_{n,k}}{k}+O\left(1\right)\\
&\cdot\sum_{k=n+1}^{\infty}\frac{a_{n,k}}{k}\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\leq O\left(1\right)\omega_2\left(f,\frac{\pi}{n+1}\right)_X\\
&+O\left(1\right)\left(1+c\right)\sum_{\mu=0}^{n}\frac{\omega_2\left(f,\pi/\left(\mu+1\right)\right)_X}{\mu+1}\sum_{k=0}^{\mu}a_{n,k}\\
&+O\left(1\right)\left[3\left(1+c\right)+1\right]\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X\sum_{k=\mu}^{n}\frac{a_{n,k}}{k}\\
&+O\left(1\right)\sum_{k=n+1}^{\infty}\frac{a_{n,k}}{k}\sum_{\mu=1}^{n}\omega_2\left(f,\frac{\pi}{\mu}\right)_X=O\left(1\right)\\
&\cdot\omega_2\left(f,\frac{\pi}{n+1}\right)_X+O\left(1\right)\\
&\cdot\sum_{k=0}^{n}a_{n,k}\sum_{\mu=k}^{n}\frac{\omega_2\left(f,\pi/\left(\mu+1\right)\right)_X}{\mu+1}+O\left(1\right)\\
&\cdot\sum_{k=1}^{n}\frac{a_{n,k}}{k}\sum_{\mu=1}^{k}\omega_2\left(f,\frac{\pi}{\mu}\right)_X+O\left(1\right)
\end{aligned}
$$

$$\cdot \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \sum_{\mu=1}^{n} \omega_2 \left(f, \frac{\pi}{\mu} \right)_X, \tag{66}$$

If (14) and (15) hold, then

$$\begin{aligned} \omega_2 \left(f, \frac{\pi}{n+1} \right)_X &\leq \frac{1}{n} \sum_{\mu=1}^{n} \omega_2 \left(f, \frac{\pi}{\mu} \right)_X \\ &\leq O(1)\, 4\pi \frac{H(\pi/(n+1))}{n+1}, \\ \sum_{\mu=k}^{n} \frac{\omega_2 (f, \pi/(\mu+1))_X}{\mu+1} &\leq 8 \int_{\pi/(n+2)}^{\pi/(k+1)} \frac{\omega_2 (f,t)_X}{t} dt \\ &\leq O(1)\, 8\pi \frac{H(\pi/(k+1))}{k+1} \end{aligned} \tag{67}$$

and therefore

$$\begin{aligned} \left\| T_{n,A} f - f \right\|_X &= O\left(\frac{H(\pi/(n+1))}{n+1} \right) \\ &\quad + O\left(\sum_{k=1}^{n} a_{n,k} \frac{H(\pi/(k+1))}{k+1} \right) \\ &\quad + O\left(H\left(\frac{\pi}{n+1} \right) \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \right). \end{aligned} \tag{68}$$

Since

$$\sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \leq \frac{1}{n+1} \sum_{k=n+1}^{\infty} a_{n,k} = O\left(\frac{1}{n+1} \right) \tag{69}$$

the result follows (cf. [6]). □

Competing Interests

The authors declare that they have no competing interests.

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References

[1] X. Z. Krasniqi, "Some further results on the degree of approximation of continuous functions," *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Computatorica*, vol. 38, pp. 279–294, 2012.

[2] B. Wei and D. Yu, "On the degree of approximation of continuous functions by means of Fourier series," *Mathematical Communications*, vol. 17, no. 1, pp. 211–219, 2012.

[3] B. Szal, "A new class of numerical sequences and its applications to uniform convergence of sine series," *Mathematische Nachrichten*, vol. 284, no. 14-15, pp. 1985–2002, 2011.

[4] B. Szal, "On weighted L^p integrability of functions defined by trigonometric series," *Journal of Inequalities and Applications*, vol. 2010, Article ID 485705, 19 pages, 2010.

[5] M. Dyachenko and S. Tikhonov, "Integrability and continuity of functions represented by trigonometric series: coefficients criteria," *Studia Mathematica*, vol. 193, no. 3, pp. 285–306, 2009.

[6] S. Tikhonov, "Trigonometric series with general monotone coefficients," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 721–735, 2007.

[7] B. Szal, "On L-convergence of trigonometric series," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 2, pp. 449–463, 2011.

[8] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, UK, 2002.

[9] P. Chandra, "A note on the degree of approximation of continuous functions," *Acta Mathematica Hungarica*, vol. 62, no. 1-2, pp. 21–23, 1993.