Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2016, Article ID 7210540, 8 pages
http://dx.doi.org/10.1155/2016/7210540

Research Article

Local Hypoellipticity by Lyapunov Function

E. R. Aragão-Costa

Instituto de Ciências Matemáticas e de Computaçao, Universidade de São Paulo, Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil

Correspondence should be addressed to E. R. Aragão-Costa; ritis@icmc.usp.br

Received 7 July 2015; Accepted 20 December 2015

Academic Editor: Maria Grazia Naso

We treat the local hypoellipticity, in the first degree, for a class of abstract differential operators complexes; the ones are given by the following differential operators: $L_j = \partial/\partial t_j + (\partial\phi/\partial t_j)(t, A)A$, $j = 1, 2, \ldots, n$, where $A : D(A) \subset H \rightarrow H$ is a self-adjoint linear operator, positive with $0 \in \rho(A)$, in a Hilbert space H, and $\phi = \phi(t, A)$ is a series of nonnegative powers of A^{-1} with coefficients in $C^{\infty}(\Omega)$, Ω being an open set of $\mathbb{R}^n$, for any $n \in \mathbb{N}$, different from what happens in the work of Hounie (1979) who studies the problem only in the case $n = 1$. We provide sufficient condition to get the local hypoellipticity for that complex in the elliptic region, using a Lyapunov function and the dynamics properties of solutions of the Cauchy problem $t'(s) = -\nabla\mathrm{Re}\phi_0(t(s))$, $s \geq 0$, $t(0) = t_0 \in \Omega, \phi_0 : \Omega \rightarrow \mathbb{C}$ being the first coefficient of $\phi(t, A)$. Besides, to get over the problem out of the elliptic region, that is, in the points $t^* \in \Omega$ such that $\nabla\mathrm{Re}\phi_0(t^*) = 0$, we will use the techniques developed by Bergamasco et al. (1993) for the particular operator $A = 1 - \Delta : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$.

1. Introduction

In this work, we want to lay down sufficient condition for the local hypoellipticity, in the first degree, of the differential complex given by the following operators:

$$L_j = \frac{\partial}{\partial t_j} + \frac{\partial \phi}{\partial t_j}(t, A)\, A, \quad j = 1, 2, \ldots, n, \tag{1}$$

where $A : D(A) \subset H \rightarrow H$ is a self-adjoint linear operator, positive with $0 \in \rho(A)$, in a Hilbert space H, and $\phi(t, A)$ is a series of nonnegative powers of A^{-1} with coefficients in $C^{\infty}(\Omega)$, Ω being an open set of $\mathbb{R}^n$.

The map $\phi = \phi(t, A)$ is given by

$$\phi(t, A) = \sum_{k=0}^{\infty} \phi_k(t)\, A^{-k}, \tag{2}$$

with convergence in $\mathscr{L}(H)$, uniform in compacts of Ω, and $\phi_k \in C^{\infty}(\Omega) = C^{\infty}(\Omega; \mathbb{C})$ for every $k \in \mathbb{N} \cup \{0\}$.

We will observe, using a method from [1–3], that the local hypoellipticity of the differential complex generated by the operators above is equivalent to the local hypoellipticity of a simpler complex, namely, the one generated by the differential operators

$$L_{j,0} := \frac{\partial}{\partial t_j} + \frac{\partial \mathrm{Re}\phi_0}{\partial t_j}(t)\, A, \quad j = 1, 2, \ldots, n. \tag{3}$$

The local solvability of the transpose of this complex in top degree was firstly studied in [3]. There, the authors consider a method, a result from [4] we might add, to get the local solvability and they assume that the leading coefficient is analytic. Here, we will just assume that the leading coefficient is C^{∞} and use dynamic property to obtain the local hypoellipticity in the elliptic region and, after that, use some of the techniques developed in [5] to study the problem in the nonelliptic one, the only case we suppose the analyticity of ϕ_0.

To be more specific, we are going to explore the properties of the gradient system generated by the C^{∞} function $\mathrm{Re}\phi_0$, that is, the system

$$\begin{aligned} t'(s) &= -\nabla\mathrm{Re}\phi_0(t(s)), \quad s \geq 0, \\ t(0) &= t_0 \in \Omega, \end{aligned} \tag{4}$$

to get that for every point $t_0 \in \Omega \setminus \mathscr{E}$, where $\mathscr{E} := \{t^* \in \Omega : \nabla \mathrm{Re}\phi_0(t^*) = 0\}$, there exists an open set $U \subset \Omega$ with $t_0 \in U$ and $U \cap \mathscr{E} = \varnothing$, such that for each $u \in C^\infty(U; H^{-\infty})$ which fulfill

$$\sum_{j=1}^{n} L_{j,0} u dt_j = f \text{ in } U, \tag{5}$$

with $f \in \Lambda^1 C^\infty(U; H^\infty)$, then u is actually in $C^\infty(U; H^\infty)$.

In order to do that, we need to clarify every concept in the set above which we will work with in this paper.

We begin the work introducing, in a precise way, the complex of differential operators which we want to study and talking about its local hypoellipticity in the "elliptic region" and after that its hypoellipticity out of it.

2. The Complex in Study

Let $A : D(A) \subset H \to H$ be a self-adjoint linear operator, positive with $0 \in \rho(A)$, in a Hilbert space H with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. Therefore, A is a sectorial operator with $\mathrm{Re}\,\sigma(A) > 0$ (see [6], for a definition) and, for each real s, let H^s be its fractional power space associated, that is, for $s \geq 0$, $H^s := \{A^{-s} f : f \in H\}$ with inner product $(u, v)_s := (A^s u, A^s v)_H$, for $u, v \in H^s$, where the operator A^{-s} is given by

$$A^{-s} := \frac{1}{\Gamma(s)} \int_0^\infty \theta^{s-1} e^{-A\theta} d\theta, \tag{6}$$

the one which is injective whose inverse is denoted by $A^s : H^s \to H$, $\{e^{-A\theta} : \theta \geq 0\}$ being the analytic semigroup generated by $-A$, and, for $s < 0$, H^s is the topological dual space of H^{-s}; that is, $H^s := (H^{-s})^*$.

That way, as the spaces H^s are Hilbert spaces, we obtain that, for each real s, H^{-s} is the topological dual of H^s.

Now, we put $H^\infty := \bigcap_{s\in\mathbb{R}} H^s$; with the topology projective limit, we mean the topology generated by the family of norms $(\|\cdot\|_s)_{s>0}$, and $H^{-\infty} := \bigcup_{s\in\mathbb{R}} H^s$, with the topology weak star, namely, the one such that "a net $(x_\lambda)_{\lambda\in\Lambda}$ in $H^{-\infty}$ converges to $x \in H^{-\infty}$ if, and only if, the net $(\langle x_\lambda - x, u\rangle)_{\lambda\in\Lambda}$ converges to zero, in $\mathbb{C}$, when λ runs in directed set Λ, for every $u \in H^\infty$." That is, $H^{-\infty}$ is the topological dual space of H^∞.

When we have $A = 1 - \Delta : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, A fulfill the properties above, the fractional power spaces are the usual Sobolev spaces in $\mathbb{R}^N$, and, as we well know, in this case holds

$$\begin{gathered} H^\infty \subset C^\infty\left(\mathbb{R}^N\right) \cap L^2\left(\mathbb{R}^N\right), \\ H^{-\infty} \subset \mathscr{D}'_{(F)}\left(\mathbb{R}^N\right) \cap \mathscr{S}'\left(\mathbb{R}^N\right), \end{gathered} \tag{7}$$

where $\mathscr{D}'_{(F)}(\mathbb{R}^N)$ stands for the finite order distribution on $\mathbb{R}^N$ and $\mathscr{S}'(\mathbb{R}^N)$ for the tempered distribution on $\mathbb{R}^N$ (go to [7, 8] for a proof).

On the other hand, let

$$\phi(t, A) = \sum_{k=0}^{\infty} \phi_k(t) A^{-k}, \tag{8}$$

with convergence in $\mathscr{L}(H)$, uniform in compacts of Ω, where Ω is an open set of $\mathbb{R}^n$, and $\phi_k \in C^\infty(\Omega)$ for every $k \in \mathbb{N} \cup \{0\}$.

We define, for $j = 1, 2, \ldots, n$, the differential operators $L_j : C^\infty(\Omega; H^\infty) \to C^\infty(\Omega; H^\infty)$, by

$$L_j u := \frac{\partial u}{\partial t_j} + \frac{\partial \phi}{\partial t_j}(t, A) Au. \tag{9}$$

Taking the leading coefficient of $\phi(t, A)$, that is, $\phi_0 \in C^\infty(\Omega)$, we also define, for each $j = 1, 2, \ldots, n$, the differential operator $L_{j,0} : C^\infty(\Omega; H^\infty) \to C^\infty(\Omega; H^\infty)$, by

$$L_{j,0} u := \frac{\partial u}{\partial t_j} + \frac{\partial \mathrm{Re}\phi_0}{\partial t_j}(t) Au. \tag{10}$$

It is easy to see that, for each $j = 1, 2, \ldots, n$, the operator given by

$$L^*_{j,0} u := -\frac{\partial u}{\partial t_j} + \frac{\partial \mathrm{Re}\phi_0}{\partial t_j}(t) Au \tag{11}$$

is the adjoint of $L_{j,0}$.

Indeed, if $\varphi, \psi \in C_c^\infty(\Omega; H^\infty)$, by the fact that A is self-adjoint, integrating by parts, we see

$$\begin{aligned} &\left\langle L_{j,0}\varphi, \psi\right\rangle \\ &= \int_\Omega \left(\frac{\partial \varphi(t)}{\partial t_j} + \frac{\partial \mathrm{Re}\phi_0}{\partial t_j}(t) A\varphi(t), \psi(t) \right)_H dt \\ &= -\int_\Omega \left(\varphi(t), \frac{\partial \psi(t)}{\partial t_j} \right)_H dt \\ &\quad + \int_\Omega \left(\varphi(t), \frac{\partial \mathrm{Re}\phi_0}{\partial t_j}(t) A\psi(t) \right)_H dt \\ &= \left\langle \varphi, L^*_{j,0}\psi \right\rangle. \end{aligned} \tag{12}$$

Observe that $\mathrm{supp}(L_{j,0}u) \subset \mathrm{supp}(u)$, for every $u \in C^\infty(\Omega)$.

In the same way we can see that, for each $j = 1, 2, \ldots, n$, the operator

$$L^*_j = -\frac{\partial}{\partial t_j} + \frac{\partial \overline{\phi}}{\partial t_j}(t, A) A \tag{13}$$

is the adjoint of L_j, where $\overline{\phi}(t, A)$ is the series $\sum_{k=0}^\infty \overline{\phi_k}(t) A^{-k}$, whose coefficients are the complex conjugated of the ones from $\phi(t, A)$.

That observation allows us to define L_j and $L_{j,0}$ on distributions, $L_j : \mathscr{D}'(\Omega; H^{-\infty}) \to \mathscr{D}'(\Omega; H^{-\infty})$ and $L_{j,0} : \mathscr{D}'(\Omega; H^{-\infty}) \to \mathscr{D}'(\Omega; H^{-\infty})$, putting

$$\begin{aligned} &\left\langle L_j u, \varphi \right\rangle := \left\langle u, L^*_j \varphi \right\rangle, \\ &\qquad \text{for } u \in \mathscr{D}'\left(\Omega; H^{-\infty}\right), \ \varphi \in C_c^\infty\left(\Omega; H^\infty\right), \\ &\left\langle L_{j,0} u, \varphi \right\rangle := \left\langle u, L^*_{j,0} \varphi \right\rangle, \\ &\qquad \text{for } u \in \mathscr{D}'\left(\Omega; H^{-\infty}\right), \ \varphi \in C_c^\infty\left(\Omega; H^\infty\right), \end{aligned} \tag{14}$$

just recalling that $\mathscr{D}'(\Omega; H^{-\infty})$ is the topological dual space of $C_c^\infty(\Omega; H^\infty)$, where the last one is equipped with the inductive limit.

The operators L_j and $L_{j,0}$, defined above, can be used to define complexes of differential operator,

$$\begin{aligned}\mathbb{L} &: \Lambda^p C^\infty(\Omega; H^\infty) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; H^\infty),\\ \mathbb{L} &: \Lambda^p \mathscr{D}'(\Omega; H^{-\infty}) \longrightarrow \Lambda^{p+1} \mathscr{D}'(\Omega; H^{-\infty}),\end{aligned} \tag{15}$$

$0 \le p \le n$, and

$$\begin{aligned}\mathbb{L}_0 &: \Lambda^p C^\infty(\Omega; H^\infty) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; H^\infty),\\ \mathbb{L}_0 &: \Lambda^p \mathscr{D}'(\Omega; H^{-\infty}) \longrightarrow \Lambda^{p+1} \mathscr{D}'(\Omega; H^{-\infty}),\end{aligned} \tag{16}$$

also with $0 \le p \le n$, by

$$\begin{aligned}\mathbb{L}u &:= \sum_{|J|=p} \sum_{j=1}^{n} L_j u_J dt_j \wedge dt_J, \quad \text{for } u = \sum_{|J|=p} u_J dt_J,\\ \mathbb{L}_0 u &:= \sum_{|J|=p} \sum_{j=1}^{n} L_{j,0} u_J dt_j \wedge dt_J, \quad \text{for } u = \sum_{|J|=p} u_J dt_J,\end{aligned} \tag{17}$$

where $dt_J = dt_{j_1} \wedge \cdots \wedge dt_{j_p}$, $J = \{j_1 < \cdots < j_p\} \subset I_n = \{1, 2, \ldots, n\}$, are the basic elements from the canonical basis of the $C^\infty(\Omega)$-module $\Lambda^p C^\infty(\Omega)$.

Thus, we get the global form of these complexes

$$\begin{aligned}\mathbb{L}u &:= d_t u + \omega(t, A) \wedge Au,\\ \mathbb{L}_0 u &:= d_t u + \operatorname{Re}\omega_0(t) \wedge Au,\end{aligned} \tag{18}$$

with

$$\omega(t, A) := \sum_{k=0}^{\infty} \omega_k(t) A^{-k} \in \Lambda^1 C^\infty(\Omega; \mathscr{L}(H)), \tag{19}$$

where

$$\omega_k(t) := \sum_{j=1}^{n} \frac{\partial \phi_k}{\partial t_j}(t)\, dt_j, \tag{20}$$

where, for every nonnegative integer k, d_t stands for the exterior derivative in the t variable in Ω, being $u \in \Lambda^p C^\infty(\Omega; H^\infty)$ or $u \in \Lambda^p \mathscr{D}'(\Omega; H^{-\infty})$ and $Au := \sum_{|J|=p} Au_J dt_J$.

Consequently, $\mathbb{L} \circ \mathbb{L} = 0$ and $\mathbb{L}_0 \circ \mathbb{L}_0 = 0$, condition which defines the concept of a differential complex.

Of course, just by restriction, we see that $\mathbb{L}$ and $\mathbb{L}_0$ define complexes on currents with coefficients in $C^\infty(\Omega; H^{-\infty})$ (see [2]); that is, we can look at

$$\begin{aligned}\mathbb{L} &: \Lambda^p C^\infty(\Omega; H^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; H^{-\infty}), \quad \text{for } 0 \le p \le n,\\ \mathbb{L}_0 &: \Lambda^p C^\infty(\Omega; H^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; H^{-\infty}), \quad \text{for } 0 \le p \le n.\end{aligned} \tag{21}$$

In these conditions, we can introduce the kind of hypoellipticity that we are going to work with.

Definition 1. Let Ω be an open set of $\mathbb{R}^n$. Given U, an open set of Ω, one says that an operator

$$\mathbb{M} : C^\infty(\Omega; H^{-\infty}) \longrightarrow \Lambda^1 C^\infty(\Omega; H^{-\infty}) \tag{22}$$

is hypoelliptic in U, in the first degree, when, for every distribution $u \in C^\infty(U; H^{-\infty})$ such that $\mathbb{M}u \in \Lambda^1 C^\infty(U; H^\infty)$, one actually has $u \in C^\infty(U; H^\infty)$.

When $\mathbb{M}$ is hypoelliptic in U, where $U = \Omega$, one says that $\mathbb{M} : C^\infty(\Omega; H^{-\infty}) \to \Lambda^1 C^\infty(\Omega; H^{-\infty})$ is globally hypoelliptic (in Ω) and when $\mathbb{M} : C^\infty(\Omega; H^{-\infty}) \to \Lambda^1 C^\infty(\Omega; H^{-\infty})$ is hypoelliptic in U, for every open set $U \subset \Omega$, one says that $\mathbb{M}$ is locally hypoelliptic in Ω.

We should say that, in this work, our concern is the regularity of the distributions $u \in C^\infty(\Omega; H^{-\infty})$ in the "x variable," by which we mean the regularity relatively to the scale of spaces H^s, where the distributions have their image.

To be more precise, in this work, we are not able, yet, to show in the more general framework that $\mathbb{L} : C^\infty(\Omega; H^{-\infty}) \to \Lambda^1 C^\infty(\Omega; H^{-\infty})$ is locally hypoelliptic in the whole Ω. What we actually are going to do is to show that $\mathbb{L}$ is locally hypoelliptic in $\Omega_0 := \Omega \setminus \mathscr{E}$, where $\mathscr{E} := \{t^* \in \Omega : \nabla \operatorname{Re}\phi_0(t^*) = 0\}$, set we will call the *elliptic region* of $\mathbb{L}$ and $\mathbb{L}_0$, and after that, using the techniques we have learned from [5], we will consider $A := 1 - \Delta$ and get the local hypoellipticity for $\mathbb{L}$ associated.

In other words, in the general case, we do not have the total information about $\mathbb{L}$ which allows us to obtain its local hypoellipticity in Ω, but our knowledge of the dynamics properties of the solution of the Cauchy problem

$$\begin{aligned}t' &= -\nabla \operatorname{Re}\phi_0(t), \quad s \ge 0,\\ t(0) &= t_0 \in \Omega,\end{aligned} \tag{23}$$

will give us the local hypoellipticity in Ω_0 and the nature, or noble structure, of the operator $1 - \Delta$ will be used to solve the problem out of Ω_0, that is, in some neighborhood of $\mathscr{E}$.

The analysis we will do below in Ω_0 will be strongly inspired by the study made in [9], where the author considers the same kind of problem as us, but only in one dimension, getting complete characterization of the global hypoellipticity, in the abstract framework, by the conditions (ψ) and (τ). Such conditions, however, we will not assume, explicitly, here.

Before we start to study the hypoellipticity of the operator $\mathbb{L}$ let us point out that as was done in [1–3] we can isolate the "principal part" of $\mathbb{L}$ and conclude that to study its hypoellipticity is equivalent to study the hypoellipticity of the simpler operator $\mathbb{L}_0$.

Lemma 2. *For each $0 \le p \le n$ and each open set $U \subset \Omega$,*

$$\mathbb{L} : \Lambda^p C^\infty(U; H^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(U; H^{-\infty}) \tag{24}$$

is hypoelliptic in U if and only if

$$\mathbb{L}_0 : \Lambda^p C^\infty(U; H^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(U; H^{-\infty}) \tag{25}$$

is hypoelliptic in U.

Proof. We just have to define, for each $t \in \Omega$, the operator

$$\begin{aligned}\alpha(t, A) &:= \mathrm{Re}\phi_0(t) - \phi(t, A) \\ &= [\phi_0(t) - \phi(t, A)] - i\mathrm{Im}\phi_0(t)\end{aligned} \tag{26}$$

and to observe that the composition $\alpha(t, A)A$ is the sum of an operator of type Schrödinger (hence, infinitesimal generator of a group of linear operators; see [10]) and a bound.

Therefore, we can define the operator $U(t) := e^{\alpha(t,A)A}$, $t \in \Omega$.

Thus, this one can be used to generate an automorphism of $\Lambda^p C^\infty(U; H^\infty)$ and $\Lambda^p C^\infty(U; H^{-\infty})$, for each $0 \le p \le n$, putting

$$(\mathscr{U}u)(t) := U(t)u(t) = e^{\alpha(t,A)A}u(t), \quad \text{for } u \in C^\infty(U; H^\infty),\ t \in U. \tag{27}$$

It is not hard to see that $\mathscr{U} : C^\infty(U; H^\infty) \to C^\infty(U; H^\infty)$ defines an automorphism, because $e^{\alpha(t,A)A}$ is invertible for every $t \in \Omega$, which extends to another $\mathscr{U} : C^\infty(U; H^{-\infty}) \to C^\infty(U; H^{-\infty})$, just by taking its adjoint.

From the definition of $\mathscr{U}$ it is just a calculation to get, for $j = 1, 2, \ldots, n$, the equality

$$[L_j(\mathscr{U}u)](t) = [\mathscr{U}(L_{j,0}u)](t), \quad \text{for } u \in C^\infty(U; H^\infty),\ t \in U. \tag{28}$$

If we define, for $u = \sum_{|J|=p} u_J dt_J$,

$$\mathscr{U}u := \sum_{|J|=p} (\mathscr{U}u_J)\, dt_J \tag{29}$$

equality (28) tells us that

$$\mathbb{L}(\mathscr{U}u) = (\mathscr{U}\mathbb{L}_0)u, \quad \text{for } u \in C^\infty(U; H^\infty). \tag{30}$$

As the same equality above it is true for $u \in C^\infty(U; H^{-\infty})$; our claim holds. □

3. The Main Theorems

We begin our contribution introducing a very simple result, from the ordinary differential equations theory, whose proof will be left to the reader.

Lemma 3. *Let $\phi_0 \in C^\infty(\Omega)$, consider the Cauchy problem*

$$\begin{aligned}t'(s) &= -\nabla\mathrm{Re}\phi_0(t(s)), \quad s \ge 0, \\ t(0) &= t_0 \in \Omega,\end{aligned} \tag{31}$$

and let $\mathscr{E} := \{t^* \in \Omega : \nabla\mathrm{Re}\phi_0(t^*) = 0\}$ be the set of all equilibrium points of it.

If, for each $t_0 \in \Omega$, $\omega(t_0) > 0$ indicates the maximal time of existence of the solution $T(s)t_0$, $s > 0$, of this problem, then, for each $t_0 \in \Omega_0 := \Omega \setminus \mathscr{E}$ and $\delta > 0$ with $d(t_0, \mathscr{E}) > 2\delta$, there exist an open set $U \subset \Omega$ with $t_0 \in U$ and $\tau > 0$, such that

(i) *$\omega(t) \ge \tau$ for every $t \in U$,*

(ii) *$T(s)U \subset \mathscr{O}_\delta(\mathscr{E} \cup \partial\Omega)$ whenever $s \ge \tau$ (when $X \subset \Omega$, the symbol $\mathscr{O}_\delta(X)$ stands for the union of all open balls with radius $\delta > 0$ and center in some point of X),*

(iii) *$T(s)U \subset \Omega_0$ when $0 \le s \le \tau$,*

(iv) *$U \cap \mathscr{O}_\delta(\mathscr{E} \cup \partial\Omega) = \varnothing$.*

As we have seen in Lemma 2, we just need to study the complex generated by $\mathbb{L}_0$. That fact will be implicit in the results we establish below.

Theorem 4. *In the conditions above, given $t_0 \in \Omega \setminus \mathscr{E}$, there exists an open set $U \subset \Omega \setminus \mathscr{E}$, with $t_0 \in U$, such that $\mathbb{L}$ is hypoelliptic in U.*

Proof. Indeed, given $t_0 \in \Omega_0 = \Omega \setminus \mathscr{E}$ and $\delta > 0$ with $d(t_0, \mathscr{E}) > 2\delta$, let U and $\tau > 0$ be the ones given by the lemma above.

Also, let $\{e^{-sA} : s \ge 0\}$ be the analytic semigroup generated by the minus sectorial operator $-A$. As we well know, $e^{-As}u \in H^\infty$ for every $u \in H^{-\infty}$ whenever $s > 0$ (see [6]).

Now, for $\omega \in \Lambda^1 C^\infty(U; H^\infty)$ (or $\omega \in \Lambda^1 C^\infty(U; H^{-\infty})$) and for $t \in U$, inspired in work [9], we define the linear operator

$$(K\omega)(t) := -\int_{\gamma_t} e^{\mathrm{Re}(\phi_0(z)-\phi_0(t))A}\omega(z)\, dz, \tag{32}$$

where the integration path is $\gamma_t(s) := T(s)t$, $s \in [0, \tau]$.

In the same way, we can define K in each open subset W of U.

We have to say that the value $(K\omega)(t)$ is well defined because the function $\mathrm{Re}\phi_0$ is a Lyapunov function for Cauchy problem (31), so $\mathrm{Re}\phi_0(T(s)t) \le \mathrm{Re}\phi_0(t)$ for every $s \in [0, \tau]$ and $t \in U$; hence we may apply the semigroup $\{e^{-As} : s \ge 0\}$ in $s = -\mathrm{Re}(\phi_0(T(s)t) - \phi_0(t)) \ge 0$ and, for the case when $\omega \in \Lambda^1 C^\infty(U; H^{-\infty})$, $H^{-\infty}$, endowed with the weak star topology, is complete.

Besides, it is not hard to see that K maps $\Lambda^1 C^\infty(U'; H^\infty)$ into $C^\infty(U'; H^\infty)$ and $\Lambda^1 C^\infty(U'; H^{-\infty})$ into $C^\infty(U'; H^{-\infty})$, for every open subset $U' \subset U$.

On the other hand, let $g \in C_c^\infty(U; H^{-\infty})$, consider $\mathbb{L}_0 g \in \Lambda^1 C^\infty(U; H^{-\infty})$, and define $K(\mathbb{L}_0 g)$.

From this, for every $t \in U$ we have, by Lemma 3, that $T(\tau)t \notin U$; hence $T(\tau)t \notin \mathrm{supp}(g)$, so, integrating by parts and using the fact that $T(s)t$ is the solution of (31), we see that for $t \in U$

$$\begin{aligned}[K(\mathbb{L}_0 g)](t) &= -\int_{\gamma_t} e^{\mathrm{Re}(\phi_0(z)-\phi_0(t))A}(\mathbb{L}_0 g)(z)\, dz \\ &= -\int_{\gamma_t} e^{\mathrm{Re}(\phi_0(z)-\phi_0(t))A}(d_t g)(z)\, dz \\ &\quad - \int_{\gamma_t} e^{\mathrm{Re}(\phi_0(z)-\phi_0(t))A}\mathrm{Re}\omega_0(z) \wedge Ag(z)\, dz\end{aligned}$$

$$\begin{aligned}
&= -\left[e^{\mathrm{Re}(\phi_0(T(s)t)-\phi_0(t))A} g(z)\right]\Big|_{s=0}^{\tau} \\
&+ \int_{\gamma_t} e^{\mathrm{Re}(\phi_0(z)-\phi_0(t))A} \mathrm{Re}\omega_0(z) \wedge Ag(z)\,dz \\
&- \int_{\gamma_t} e^{\mathrm{Re}(\phi_0(z)-\phi_0(t))A} \mathrm{Re}\omega_0(z) \wedge Ag(z)\,dz \\
&= -\left[e^{\mathrm{Re}(\phi_0(T(\tau)t)-\phi_0(t))A} g(T(\tau)t)\right. \\
&\left. - e^{\mathrm{Re}(\phi_0(t)-\phi_0(t))A} g(t)\right] = g(t).
\end{aligned} \tag{33}$$

In resume

$$[K(\mathbb{L}_0 g)](t) = g(t), \quad \text{for every } t \in U. \tag{34}$$

Thus, if $u \in C^\infty(U; H^{-\infty})$ has $\mathbb{L}_0 u = f \in \Lambda^1 C^\infty(U; H^\infty)$, for each $t' \in U$ we may choose $\varphi \in C_c^\infty(U; \mathbb{R})$, with $\varphi = 1$ in some neighborhood of U' of t'. Then, $g := \varphi u \in C_c^\infty(U; H^{-\infty})$ and we have

$$\begin{aligned}
\mathbb{L}_0(\varphi u) &= \varphi \mathbb{L}_0 u + \sum_{j=1}^{n} \frac{\partial \varphi}{\partial t_j}(t)\, u(t)\, dt_j \\
&= \varphi f + \sum_{j=1}^{n} \frac{\partial \varphi}{\partial t_j}(t)\, u(t)\, dt_j.
\end{aligned} \tag{35}$$

So, by (34), we have

$$\begin{aligned}
&[K(\varphi f)](t) + \left[K\left(\sum_{j=1}^{n} \frac{\partial \varphi}{\partial t_j} u\, dt_j\right)\right](t) \\
&= [K \mathbb{L}_0(\varphi u)](t) = (\varphi u)(t), \quad \forall t \in U.
\end{aligned} \tag{36}$$

Since $\varphi f \in \Lambda^1 C^\infty(U; H^\infty)$, we have $K(\varphi f) \in C^\infty(U; H^\infty)$. So if we show that

$$K\left(\sum_{j=1}^{n} \frac{\partial \varphi}{\partial t_j} u\, dt_j\right) \tag{37}$$

is in $C^\infty(U'; H^\infty)$, then the theorem follows, once U' was arbitrary.

Indeed, on one hand, since φ is constant in U' we have $\sum_{j=1}^{n} (\partial \varphi / \partial t_j)(r) u(r) dt_j = 0$ as long as $r \in U'$.

On the other, for each $t' \in U'$ there exist a neighborhood V' in U', for it, and $\tau_1 > 0$ such that $T(s)t \in U'$ whenever $s \in [0, \tau_1]$ and $t \in V'$.

So, for $t \in V'$

$$\begin{aligned}
&K\left(\sum_{j=1}^{n} \frac{\partial \varphi}{\partial t_j} u\, dt_j\right)(t) = -\int_{\tau_1}^{\tau} e^{\mathrm{Re}(\phi_0(T(s)t)-\phi_0(t)+\eta)A} \\
&\cdot \left[e^{-\eta A} \left(\sum_{j=1}^{n} \frac{\partial \varphi}{\partial t_j}(T(s)t)\right.\right. \\
&\left.\left. \cdot\, u(T(s)t)\, \frac{d(T(s)t)_j}{ds}\right)\right] ds,
\end{aligned} \tag{38}$$

where $\eta := \mathrm{Re}(\phi_0(t) - \phi_0(T(s)\tau_1)) > 0$ and $d(T(s)t)_j/ds$ stands for the components, $j = 1, 2, \ldots, n$, of the vector $dT(s)t/ds \in \mathbb{R}^n$.

Observe that $\eta > 0$, because, for $t \in U$ fixed, we only have $\mathrm{Re}\phi_0(T(s)t) = \mathrm{Re}\phi_0(t)$ to a finite number of s in $[0, \tau]$. Otherwise, there exists a sequence $(s_j)_{j \in \mathbb{N}}$ in $[0, \tau]$ with $s_j \to s_0 \in [0, \tau]$, so $\nabla \mathrm{Re}\phi_0(T(s_0)t) = 0$; that is, $T(s_0)t \in \mathscr{E}$, but it cannot be true, because $T(s)U \subset \Omega_0$ when $0 \le s \le \tau$.

Finally, it is not hard to see that if $\alpha \in \mathbb{R}$ is fixed, for every $h \in C^\infty([\tau_1, \tau]; H^{-\infty})$ we have that $e^{-\eta A} h \in C^\infty([\tau_1, \tau]; H^\alpha)$ and, by that,

$$e^{\mathrm{Re}(\phi_0(T(s)t)-\phi_0(t)+\eta)A} \left[e^{-\eta A} h\right] \in C^\infty([\tau_1, \tau]; H^\infty). \tag{39}$$

Putting all these results together we get that for every $t \in U'$ holds

$$\begin{aligned}
(\varphi u)(t) &= K(\varphi f)(t) - \int_{\tau_1}^{\tau} e^{\mathrm{Re}(\phi_0(T(s)t)-\phi_0(t)+\eta)A} e^{-\eta A} \\
&\cdot \left[\sum_{j=1}^{n} \frac{\partial \varphi}{\partial t_j}(T(s)t)\, u(T(s)t)\, \frac{d(T(s)t)_j}{ds}\right] ds,
\end{aligned} \tag{40}$$

so the second term in the sum above defines also an element of $C^\infty(U'; H^\infty)$; therefore $\varphi u \in C^\infty(U'; H^\infty)$. But $\varphi u = u$ in U' and the proof is complete. □

As we saw in the theorem above, we did not give the answer to our problem for points in the set $\mathscr{E}$, yet. However, the next result shows us that there might exist points in $\mathscr{E}$, where we can not obtain the hypoellipticity.

Proposition 5. *If $t^* \in \mathscr{E}$ is a local minimal point for $\mathrm{Re}\phi_0$, then t^* has a neighborhood V in Ω, where $\mathbb{L}$ is not hypoelliptic.*

Proof. Indeed, let V be an open set of Ω, where $\mathrm{Re}\phi_0(t^*) \le \mathrm{Re}\phi_0(t)$ for all $t \in V$.

Take $u_0 \in H \setminus H^\infty$ and define $u : V \to H^{-\infty}$ by

$$u(t) := e^{\mathrm{Re}(\phi_0(t^*)-\phi_0(t))A} u_0, \quad t \in V. \tag{41}$$

It follows that u is well defined and $u \in C^\infty(V; H^{-\infty})$.

Now, it is pretty easy to see that $\mathbb{L}_0 u = 0$ in V, so $\mathbb{L}_0 u \in \Lambda^1 C^\infty(V; H^\infty)$. However, since $u(t^*) = u_0 \notin H^\infty$, we do not have $u \in C^\infty(V; H^\infty)$, and the claim is true. □

Remark 6. It is easy to see that when $t^* \in \mathscr{E}$ is an isolated saddle point, then $\mathrm{Re}\phi_0$ is an open map in the same neighborhood of t^*.

We finish this section restricting us to the case where the operator $A : \mathscr{D}(A) \subset H \to H$ and the Hilbert space H are $A = 1 - \Delta$, $\mathscr{D}(A) = H^2(\mathbb{R}^N)$, and $H = L^2(\mathbb{R}^N)$, the ones which have the properties we consider in the abstract framework above.

The reason that leads us to do this hypothesis is the fact that the nature of this operator in the L^2 situation allows us to use the Fourier transform to get the regularity of the solutions of the equation $\mathbb{L}u = f$ by studying its Fourier transform decay rate in infinity, the same way the authors do to lay down work [5].

Just for completeness of this paper, we write below the technical lemma shown in [5] which we are also going to need here, with a little alteration, which does not change its proof.

Lemma 7 (see Lemma 4.4 in [5]). *Suppose that* $\mathrm{Re}\phi_0$ *is an analytic function.*

Let $t^* \in \mathscr{E}$ *and let* B *be an open ball contained in* Ω *such that* $B \cap \mathscr{E}$ *is connected by piecewise smooth paths and take* $t_0 \in B \cap \mathscr{E}$*. Then there exist*

(a) *an open neighborhood* $B^* \subset B$ *of* t^**;*

(b) *a constant* $K > 0$ *and* $\varepsilon > 0$*;*

(c) *a family* $(\gamma_t)_{t\in B^*}$ *of piecewise smooth paths* $\gamma_t : [0,1] \to B$*, such that one has the following:*

(I) $\gamma_t(0) = t$*, for every* $t \in B^*$*;*

(II) $\mathrm{Re}\phi_0(\gamma_t(s)) \leq \mathrm{Re}\phi_0(t)$*, for all* $s \in [0,1]$ *and all* $t \in B^*$*;*

(III) *the length* $l(\gamma_t)$ *of* γ_t *is such that* $l(\gamma_t) \leq K$ *for all* $t \in B^*$*;*

(IV) *if* $t \in B^*$*, then one of the following properties holds:*

$(\mathrm{IV})_1$ $\gamma_t(1) = t_0$,

$(\mathrm{IV})_2$ $\mathrm{Re}\phi_0(\gamma_t(1)) \leq \mathrm{Re}\phi_0(t) - \varepsilon$.

The reader must observe that we have made a little alteration in the statement of Lemma 7; more precisely, we have made the hypothesis that "$B\cap\mathscr{E}$ is *connected by piecewise smooth paths*" instead of the one stating that "$B \cap \mathscr{E}$ is *connected,*" only, as the authors consider there. We made this because our data $\mathrm{Re}\phi_0$ need not be constantly equal to zero on $\mathscr{E}$, as they have there, but the fact that "$B \cap \mathscr{E}$ is connected by piecewise smooth paths" allows us to get that $\mathrm{Re}\phi_0$ is constant on $B \cap \mathscr{E}$, an alteration which does not change the proof that we have in [5].

Another thing, the hypothesis that "$B \cap \mathscr{E}$ is connected by piecewise smooth paths" is always satisfied when $\mathscr{E}$ is discrete, just taking B with radius as small as it needs to be $B \cap \mathscr{E}$ a singleton.

Finally, the proof of Lemma 7 lies on the Łojasiewicz-Simon inequality, which can be obtained without the hypothesis of analyticity of $\mathrm{Re}\phi_0$ if we suppose, for example, that the second derivative of $\mathrm{Re}\phi_0$ in $t^* \in \mathscr{E}$ is an isomorphism, as we can see in [11].

We are now in position to prove our final theorem.

Theorem 8. *Suppose that* $\mathrm{Re}\phi_0$ *is an analytic function.*

Let $A = 1 - \Delta : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $u \in C^\infty(\Omega; H^{-\infty})$ *with* $\mathbb{L}_0 u = f \in \Lambda^1 C^\infty(\Omega; H^\infty)$, $t^* \in \mathscr{E}$, *and suppose that one of the following properties holds:*

(i) $\mathrm{Re}\phi_0$ *is an open map at* t^**; that is,* $\mathrm{Re}\phi_0$ *transforms neighborhoods of* t^* *in neighborhoods of* $\mathrm{Re}\phi_0(t^*)$*.*

(ii) *There is* $t_0 \in B \cap \mathscr{E}$ *such that* $u(t_0,\cdot) \in H^\infty$*, where* B *is taken from Lemma 7.*

Then, $u \in C^\infty(B^* \times \mathbb{R}^N)$ *for some neighborhood* $B^* \subset B$ *of* t^**.*

Proof. Well, applying the Fourier transform in variable $x \in \mathbb{R}^N$ to the equality $\mathbb{L}_0 u = f$ we get

$$d_t\hat{u} + \mathrm{Re}\omega_0(t) \wedge a(\xi)\hat{u} = \hat{f}, \quad \text{for } t \in B, \tag{42}$$

where the "hat" stands for the Fourier transform in the variable x, $a(\xi) = 1 + 4\pi^2|\xi|^2$ is the symbol of the operator $1 - \Delta$, and B^* is the one obtained in the last lemma.

Multiplying equality (42) by $e^{a(\xi)\mathrm{Re}\phi_0(t)}$ and using the product rule we may write

$$d_t\left(e^{a(\xi)\mathrm{Re}\phi_0(t)}\hat{u}(t,\xi)\right) = e^{a(\xi)\mathrm{Re}\phi_0(t)}\hat{f}(t,\xi), \quad \forall t \in B,\ \xi \in \mathbb{R}^N. \tag{43}$$

Also by Lemma 7, considering the family of paths $(\gamma_t)_{t\in B^*}$ and integrating the equality above along γ_t, for $t \in B^*$ and $\xi \in \mathbb{R}^N$, we get

$$\begin{aligned} &e^{a(\xi)\mathrm{Re}\phi_0(\gamma_t(1))}\hat{u}(\gamma_t(1),\xi) - e^{a(\xi)\mathrm{Re}\phi_0(t)}\hat{u}(t,\xi) \\ &\quad= \int_{\gamma_t} d_t\left(e^{a(\xi)\mathrm{Re}\phi_0(z)}\hat{u}(z,\xi)\right) \\ &\quad= \int_{\gamma_t} e^{a(\xi)\mathrm{Re}\phi_0(z)}\hat{f}(z,\xi), \end{aligned} \tag{44}$$

so, for all $t \in B^*$ and $\xi \in \mathbb{R}^N$ holds

$$\begin{aligned} \hat{u}(t,\xi) &= e^{a(\xi)[\mathrm{Re}\phi_0(\gamma_t(1))-\mathrm{Re}\phi_0(t)]}\hat{u}(\gamma_t(1),\xi) \\ &\quad - \int_{\gamma_t} e^{a(\xi)[\mathrm{Re}\phi_0(z)-\mathrm{Re}\phi_0(t)]}\hat{f}(z,\xi)\,dz, \end{aligned} \tag{45}$$

and hence

$$\begin{aligned} |\hat{u}(t,\xi)| &\leq e^{a(\xi)[\mathrm{Re}\phi_0(\gamma_t(1))-\mathrm{Re}\phi_0(t)]}\left|\hat{u}(\gamma_t(1),\xi)\right| \\ &\quad + \left|\int_{\gamma_t} e^{a(\xi)[\mathrm{Re}\phi_0(z)-\mathrm{Re}\phi_0(t)]}\hat{f}(z,\xi)\,dz\right|. \end{aligned} \tag{46}$$

At this point, we divide the proof into two cases.

Case 1. The conclusion $(\mathrm{IV})_1$ of Lemma 7 holds.

In this case, we use Theorem 8 hypothesis (ii); therefore for every $s \in \mathbb{R}$ we have that

$$\left(1 + |\xi|^2\right)^{s/2}\hat{u}(t_0,\cdot) \in L^2\left(\mathbb{R}^N\right). \tag{47}$$

Thanks to the fact that $f \in \Lambda^1 C^\infty(\Omega; H^\infty)$, for every $s \in \mathbb{R}$ we also have

$$\left(1 + |\xi|^2\right)^{s/2}\hat{f}_j(t,\cdot) \in L^2\left(\mathbb{R}^N\right) \tag{48}$$

for all $t \in \Omega$ (in particular, for $t \in B^*$), and the map $\Omega \ni t \mapsto f_j(t,\cdot) \in H^\infty$ is C^∞, for all j, where we have written $f = \sum_{j=1}^n f_j dt_j$.

Thus, using these facts and conclusion (III) from Lemma 7 in inequality (46) we obtain, for each real s, all $\xi \in \mathbb{R}^N$ and $t \in B^*$

$$\begin{aligned}(1+|\xi|^2)^{s/2}|\widehat{u}(t,\xi)| &\le (1+|\xi|^2)^{s/2}|\widehat{u}(t_0,\xi)| \\ &\quad + \left|\int_{\gamma_t}(1+|\xi|^2)^{s/2}\widehat{f}(z,\xi)\,dz\right|.\end{aligned} \tag{49}$$

Now, observe that, by the Minköwski inequality for integrals, we have

$$\begin{aligned}&\left(\int_{\mathbb{R}^N}\left|\int_{\gamma_t}(1+|\xi|^2)^{s/2}\widehat{f}(z,\xi)\,dz\right|^2 d\xi\right)^{1/2} \\ &\le \int_{\gamma_t}\left(\int_{\mathbb{R}^N}(1+|\xi|^2)^{s}\left|\widehat{f}(z,\xi)\right|^2 d\xi\right)^{1/2}|dz| \\ &\le K\sup_{z\in B}\|f(z,\cdot)\|_{H^s}<\infty.\end{aligned} \tag{50}$$

This and (47) give us that $(1+|\xi|^2)^{s/2}|\widehat{u}(t,\cdot)| \in L^2(\mathbb{R}^N)$ for all real s.

Case 2. The conclusion $(\mathrm{IV})_2$ of Lemma 7 holds.

In this situation, by Lemma 7, we are actually using Theorem 8 hypothesis (i) so estimate (46) gives us, for each real s,

$$\begin{aligned}(1+|\xi|^2)^{s/2}|\widehat{u}(t,\xi)| &\le (1+|\xi|^2)^{s/2}e^{-\varepsilon a(\xi)}|\widehat{u}(\gamma_t(1),\xi)| \\ &\quad + \left|\int_{\gamma_t}(1+|\xi|^2)^{s/2}\widehat{f}(z,\xi)\,dz\right|.\end{aligned} \tag{51}$$

From that we see that, to take care of $|\int_{\gamma_t}(1+|\xi|^2)^{s/2}\widehat{f}(z,\xi)dz|$, we may use the same method we have used in Case 1 and since

$$(1+|\xi|^2)^{\alpha/2}|\widehat{u}(\gamma_t(1),\cdot)| \in L^2(\mathbb{R}^N) \tag{52}$$

for the same real α, the exponential decay of $e^{-\varepsilon a(\xi)}$ gives us that, for every real s,

$$(1+|\xi|^2)^{s/2}e^{-\varepsilon a(\xi)}|\widehat{u}(\gamma_t(1),\cdot)| \in L^2(\mathbb{R}^N), \tag{53}$$

and hence

$$\begin{aligned}\|u(t,\cdot)\|_{H^s} &\le \left\|(1+|\xi|^2)^{s/2}e^{-\varepsilon a(\xi)}\widehat{u}(\gamma_t(1),\cdot)\right\|_{L^2} \\ &\quad + K\sup_{z\in B}\|f(z,\cdot)\|_{H^s}<\infty\end{aligned} \tag{54}$$

for all $t \in B^*$ and $s \in \mathbb{R}$, completing the proof of this case.

From the cases we have studied above, we conclude that $u(t,\cdot) \in H^\infty \subset C^\infty(\mathbb{R}^N)$ for all $t \in B^*$.

Finally, differentiating with respect to t_k the equation $\mathbb{L}_0 u = f$ we get

$$\begin{aligned}&\frac{\partial}{\partial t_k}\left(\frac{\partial u}{\partial t_j}\right)(t) + \frac{\partial \mathrm{Re}\phi_0}{\partial t_k}(t)\,A\left(\frac{\partial u}{\partial t_j}(t)\right) \\ &= \frac{\partial f_j}{\partial t_k}(t) - \frac{\partial^2 \mathrm{Re}\phi_0}{\partial t_k \partial t_j}(t)\,Au(t),\end{aligned} \tag{55}$$

so we can repeat the procedure we have made above to conclude that

$$\frac{\partial u}{\partial t_j}(t,\cdot) \in H^\infty \subset C^\infty(\mathbb{R}^N) \tag{56}$$

for all $t \in B^*$; thus the induction will show us $u \in C^\infty(B^* \times \mathbb{R}^N)$, and the proof is done. □

4. Final Comments

We must make some comments to ensure the reader that the question we have treated here was not done in [2] because even though the kind of problem treated there is similar to that we study here, the structure of the operator we consider is different from that seen there.

For example, our operator A is an abstract one in the Hilbert space framework, abstract as well, whereas in [2] the author considers a different class of operators in the specific space $\mathcal{F}^2_{\mathrm{loc}}(\mathbb{R}^n)$, topological dual space of the space $\mathcal{F}^2_c(\mathbb{R}^n)$, the one which is an inductive limit of Hilbert spaces.

There, the author does a systematic study of the problem $d_t u + b(t,D_x)\wedge u = f$, for $u \in \mathcal{F}^2_{\mathrm{loc}}(\mathbb{R}^n)$, where $b(t,D_x) : \mathcal{F}^2_{\mathrm{loc}}(\mathbb{R}^n) \to \mathcal{F}^2_{\mathrm{loc}}(\mathbb{R}^n)$, $t \in \Omega$, is a pseudodifferential operator which has no need to be in the same class as our operator $\mathrm{Re}\phi_0(t)A : D(A) : H \to H$, $t \in \Omega$.

Another situation we must point out is that if the operator $A : D(A) \subset H \to H$ fulfills all the properties we have made above to prove Theorem 4 and, besides these, H is separable and A^{-1} is compact, as we well know, in this case, the operator A admits the spectral resolution

$$Au = \sum_{j=1}^{\infty}\lambda_j P_j u, \quad u \in D(A), \tag{57}$$

where λ_j's are the eigenvalues of A and $P_j : H \to E_j$ are the sequence of projections into the eigenspaces E_j corresponding and the semigroup analytic is written like this:

$$e^{-As}u = \sum_{j=1}^{\infty}e^{-\lambda_j s}P_j u, \quad u \in H. \tag{58}$$

In this situation, for $s \ge 0$, the spaces H^s admit the characterization

$$H^s = \left\{u \in H : \left(\lambda_j^s\|P_j u\|_H\right)_{j\in\mathbb{N}} \in l^2(\mathbb{N})\right\} \tag{59}$$

and are equipped with the norm

$$H^s \ni u \longmapsto \|u\|_s := \left(\sum_{j=1}^{\infty}\lambda_j^{2s}\|P_j u\|_H^2\right)^{1/2}. \tag{60}$$

Also, for $s < 0$ the space H^s is the topological dual space of H^{-s} or even the completion of the set H_s defined in the same way as (59) with respect to the norm $\|\cdot\|_s$ defined just as (60).

For each $j \in \mathbb{N}$, it is possible to extend the projection $P_j : H \rightarrow E_j$ to a new projection $\widetilde{P}_j : H^{-\infty} \rightarrow E_j$. Therefore, in these conditions, considering the differential operator $\mathbb{L}$ associated with the operator A, we see that to get the regularity of the solutions of the equation $\mathbb{L}u = f$ we just have to study the decay behavior of the sequences $(\lambda_j^s \|\widetilde{P}_j u(t)\|_H)_{j\in\mathbb{N}}$ in the same way as we have done in Theorem 8, that is, to prove that this sequence is in $l^2(\mathbb{N})$ for every real s. This way, the same proof we gave for Theorem 8 applies to this case and we can state the following.

Theorem 9. *Besides the hypothesis one has made for the operator $A : D(A) \subset H \rightarrow H$, suppose also that H is separable and A^{-1} is compact.*

If $u \in C^\infty(\Omega; H^{-\infty})$ verify $\mathbb{L}u = f$ with $f \in \Lambda^1 C^\infty(\Omega; H^\infty)$; for $t^* \in \mathscr{E}$ suppose that one of the following properties holds:

(i) Reϕ_0 *is an open map at t^*.*

(ii) *There is $t_0 \in B \cap \mathscr{E}$ such that $u(t_0, \cdot) \in H^\infty$.*

Then, $u \in C^\infty(\Omega; H^\infty)$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is partially supported by 2011/52052-9 and 2014/02899-3 from São Paulo Research Foundation (FAPESP), Brazil.

References

[1] F. Trèves, "Concatenations of second-order evolution equations applied to local solvability and hypoellipticity," *Communications on Pure and Applied Mathematics*, vol. 26, pp. 201–250, 1973.

[2] F. Treves, "Study of a model in the theory of complexes of pseudodifferential operators," *Annals of Mathematics*, vol. 104, no. 2, pp. 269–324, 1976.

[3] L. C. Yamaoka, *Resolubilidade Local de uma Classe de Sistemas Subdeterminados Abstratos*, IME-USP Tese de Doutorado, 2011.

[4] Z. Han, "Local solvability of analytic pseudodifferential complexes in top degree," *Duke Mathematical Journal*, vol. 87, no. 1, pp. 1–28, 1997.

[5] A. P. Bergamasco, P. D. Cordaro, and P. A. Malagutti, "Globally hypoelliptic systems of vector fields," *Journal of Functional Analysis*, vol. 114, no. 2, pp. 267–285, 1993.

[6] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, vol. 840 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1981.

[7] L. Hörmander, *Linear Partial Differential Operators*, Springer, New York, NY, USA, 1963.

[8] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, NY, USA, 1967.

[9] J. Hounie, "Globally hypoelliptic and globally solvable first order evolution equations," *Transactions of the American Mathematical Society*, vol. 252, pp. 233–248, 1979.

[10] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, NY, USA, 1963.

[11] E. R. Aragao-Costa, A. N. Carvalho, P. Marín-Rubio, and G. Planas, "Gradient-like nonlinear semigroups with infinitely many equilibria and applications to cascade systems," *Topological Methods in Nonlinear Analysis*, vol. 42, no. 2, pp. 345–376, 2013.