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Research Article

A Formula for the Energy of Circulant Graphs with Two Generators

Justine Louis

Section de Mathématiques, Université de Genève, 1211 Geneva, Switzerland

Correspondence should be addressed to Justine Louis; justine.louis@unige.ch

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We derive closed formulas for the energy of circulant graphs generated by 1 and γ, where $\gamma \geqslant 2$ is an integer. We also find a formula for the energy of the complete graph without a Hamilton cycle.

Let $1 \leqslant \gamma_1 \leqslant \cdots \leqslant \gamma_d$ be integers. The circulant graph $C_n^{\gamma_1,\ldots,\gamma_d}$ generated by $\gamma_1,\ldots,\gamma_d$ on n vertices labelled $0,1,\ldots,n-1$ is the 2D-regular graph such that, for all $v \in \mathbb{Z}/n\mathbb{Z}$, v is connected to $v+\gamma_i \bmod n$ and to $v-\gamma_i \bmod n$, for all $i = 1,\ldots,d$. The adjacency matrix $A = (A_{ij})$ of a graph on n vertices is the $n\times n$ matrix with rows and columns indexed by the vertices such that A_{ij} is the number of edges connecting vertices i and j. Let λ_k, $k = 1,\ldots,n$, denote the eigenvalues of the adjacency matrix. The energy of a graph G on n vertices is defined by the sum of the absolute values of the eigenvalues of A; that is,

$$E(G) = \sum_{k=1}^{n} \left|\lambda_k\right|. \tag{1}$$

The energy of circulant graphs and integral circulant graphs is widely studied; see, for example, [1–4]. It has interesting applications in theoretical chemistry; namely, it is related to the π-electron energy of a conjugated carbon molecule; see [5]. In the following theorem, we give a formula for the energy of circulant graphs with two generators, 1 and γ, $\gamma \geqslant 2$. The formula is interesting as n is larger than γ.

Theorem 1. *Let $D_n(x)$ denote the Dirichlet kernel. The energy of the circulant graph $C_n^{1,2}$ is given by*

$$E\left(C_n^{1,2}\right) = 4\left(D_{\lfloor n/6\rfloor}\left(\frac{2\pi}{n}\right) + D_{\lfloor n/6\rfloor}\left(\frac{4\pi}{n}\right)\right). \tag{2}$$

For $\gamma \geqslant 3$, the energy of the circulant graph $C_n^{1,\gamma}$ is given by

$$E\left(C_n^{1,\gamma}\right) = 4\sum_{m\in\{1,\gamma\}}\left(\sum_{l=0}^{\lceil\gamma/2\rceil-1} D_{\lfloor(2l+1)n/(2(\gamma+1))\rfloor}\left(\frac{2\pi m}{n}\right) - \sum_{l=0}^{\lceil\gamma/2\rceil-2} D_{\lfloor(2l+1)n/(2(\gamma-1))\rfloor}\left(\frac{2\pi m}{n}\right)\right), \tag{3}$$

where $\lfloor x\rfloor$ denotes the greatest integer smaller than or equal to x and $\lceil x\rceil$ denotes the smallest integer greater than or equal to x.

Proof. The adjacency matrix of a circulant graph is circulant; it follows that the eigenvalues of $C_n^{1,\gamma}$ are given by $\lambda_k = 2\cos(2\pi k/n) + 2\cos(2\pi\gamma k/n)$, $k = 0,\ldots,n-1$ (see [6]). The energy of $C_n^{1,\gamma}$ is then given by

$$E\left(C_n^{1,\gamma}\right) = 2\sum_{k=0}^{n-1}\left|\cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{2\pi\gamma k}{n}\right)\right|. \tag{4}$$

Let $\gamma = 2$. The two roots of the equation $\cos x + \cos(2x) = 0$ for $x \in [0,\pi]$ are $\pi/3$ and π. We write the energy as

$$E\left(C_n^{1,2}\right) = 4 + 4\sum_{k=1}^{\lceil n/2\rceil-1}\left|\cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{4\pi k}{n}\right)\right|$$

$$= 4 + 4 \sum_{k=1}^{\lfloor n/6 \rfloor} \left(\cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{4\pi k}{n}\right) \right) - 4 \sum_{k=\lfloor n/6 \rfloor + 1}^{\lceil n/2 \rceil - 1} \left(\cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{4\pi k}{n}\right) \right). \tag{5}$$

The sum of $\cos(kx)$ over consecutive k's can be expressed in terms of the Dirichlet kernel; namely,

$$D_n(x) = 1 + 2\sum_{k=1}^{n} \cos(kx) = \frac{\sin((n+1/2)x)}{\sin(x/2)}. \tag{6}$$

As a consequence,

$$2 \sum_{k=n+1}^{m} \cos(kx) = D_m(x) - D_n(x). \tag{7}$$

The energy of $C_n^{1,2}$ is thus given by

$$E\left(C_n^{1,2}\right) = 4D_{\lfloor n/6 \rfloor}\left(\frac{2\pi}{n}\right) + 4D_{\lfloor n/6 \rfloor}\left(\frac{4\pi}{n}\right) - 2D_{\lceil n/2 \rceil - 1}\left(\frac{2\pi}{n}\right) - 2D_{\lceil n/2 \rceil - 1}\left(\frac{4\pi}{n}\right). \tag{8}$$

The formula then follows from the fact that, for odd n, $D_{(n-1)/2}(2\pi m/n) = 0$ for $m = 1, 2$, and, for even n, $D_{n/2-1}(2\pi/n) = 1$ and $D_{n/2-1}(4\pi/n) = -1$.

Let $\gamma \geqslant 3$. For odd γ, the γ solutions of the equation $\cos x + \cos \gamma x = 0$ for $x \in [0, \pi]$ are given in the increasing order by $\pi/(\gamma+1), \pi/(\gamma-1), 3\pi/(\gamma+1), 3\pi/(\gamma-1), \ldots, (\gamma-2)\pi/(\gamma-1), \gamma\pi/(\gamma+1)$. For even γ, they are given by $\pi/(\gamma+1), \pi/(\gamma-1), 3\pi/(\gamma+1), 3\pi/(\gamma-1), \ldots, (\gamma-3)\pi/(\gamma-1), (\gamma-1)\pi/(\gamma+1), \pi$. Let n be odd. We split the sum over k of cosines to group the positive terms together and the negative terms together. The energy is given by

$$\begin{aligned} E\left(C_n^{1,\gamma}\right) &= 4 + 4 \sum_{k=1}^{(n-1)/2} \left| \cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{2\pi \gamma k}{n}\right) \right| \\ &= 4 + 4 \sum_{k=1}^{\lfloor n/(2(\gamma+1)) \rfloor} \left(\cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{2\pi \gamma k}{n}\right) \right) \\ &+ 4 \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} \sum_{k=\lfloor (2l+1)n/(2(\gamma-1)) \rfloor + 1}^{\lfloor (2l+3)n/(2(\gamma+1)) \rfloor} \left(\cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{2\pi \gamma k}{n}\right) \right) \\ &- 4 \sum_{l=0}^{\lceil \gamma/2 \rceil - 1} \sum_{k=\lfloor (2l+1)n/(2(\gamma+1)) \rfloor + 1}^{\lfloor (2l+1)n/(2(\gamma-1)) \rfloor} \left(\cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{2\pi \gamma k}{n}\right) \right). \end{aligned} \tag{9}$$

Writing the above relation in terms of Dirichlet kernels, we have

$$\begin{aligned} E\left(C_n^{1,\gamma}\right) = 2 \sum_{m \in \{1,\gamma\}} \Bigg(& D_{\lfloor n/(2(\gamma+1)) \rfloor}\left(\frac{2\pi m}{n}\right) + \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} \left(D_{\lfloor (2l+3)n/(2(\gamma+1)) \rfloor}\left(\frac{2\pi m}{n}\right) - D_{\lfloor (2l+1)n/(2(\gamma-1)) \rfloor}\left(\frac{2\pi m}{n}\right) \right) \\ & - \sum_{l=0}^{\lceil \gamma/2 \rceil - 1} \left(D_{\lfloor (2l+1)n/(2(\gamma-1)) \rfloor}\left(\frac{2\pi m}{n}\right) - D_{\lfloor (2l+1)n/(2(\gamma+1)) \rfloor}\left(\frac{2\pi m}{n}\right) \right) \Bigg). \end{aligned} \tag{10}$$

Hence,

$$\begin{aligned} E\left(C_n^{1,\gamma}\right) = \sum_{m \in \{1,\gamma\}} \Bigg(& 4 \sum_{l=0}^{\lceil \gamma/2 \rceil - 1} D_{\lfloor (2l+1)n/(2(\gamma+1)) \rfloor}\left(\frac{2\pi m}{n}\right) \\ & - 4 \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} D_{\lfloor (2l+1)n/(2(\gamma-1)) \rfloor}\left(\frac{2\pi m}{n}\right) \\ & - 2D_{\lfloor n/2 \rfloor}\left(\frac{2\pi m}{n}\right) \Bigg). \end{aligned} \tag{11}$$

The formula follows from the fact that $D_{\lfloor n/2 \rfloor}(2\pi m/n) = 0$ for $m = 1, \gamma$.

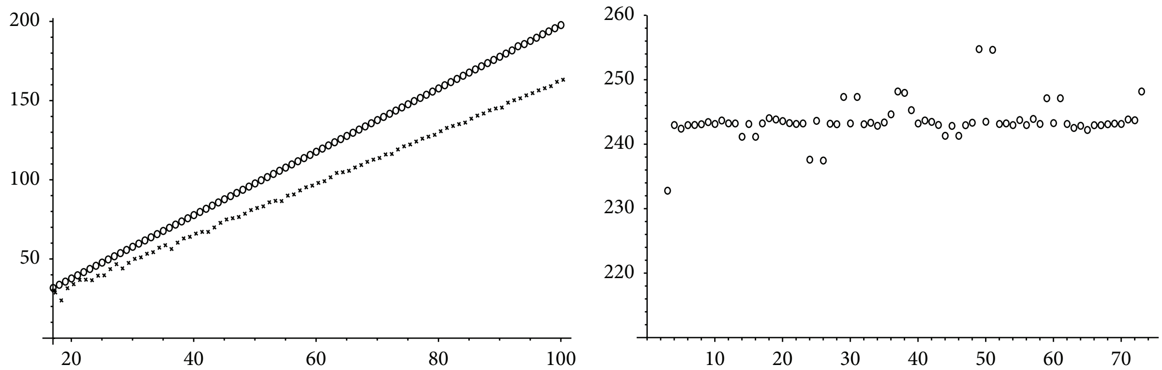
Let n be even. As for the case when n is odd, we write the energy as follows:

$$E\left(C_n^{1,\gamma}\right) = 4\left(1 + \delta_{\gamma\,\mathrm{odd}}\right) + 4 \sum_{k=1}^{n/2-1} \left| \cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{2\pi \gamma k}{n}\right) \right|, \tag{12}$$

where $\delta_{\gamma\,\mathrm{odd}} = 1$ if γ is odd and 0 otherwise.

For even γ, relations (9), (10), and (11) also hold. The theorem then follows from the fact that $D_{n/2}(2\pi/n) = -1$ and $D_{n/2}(2\pi\gamma/n) = 1$. For odd γ, we have

$$E\left(C_n^{1,\gamma}\right) = 8 + 4 \sum_{k=1}^{\lfloor n/(2(\gamma+1)) \rfloor} \left(\cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{2\pi \gamma k}{n}\right) \right)$$



(a) Energy of $C_n^{1,8}$ (crosses) and of K_n (circles) with respect to n

(b) Energy of $C_{150}^{1,\gamma}$ with respect to γ

Figure 1: Energy of circulant graphs.

$$+4\sum_{l=0}^{\lceil\gamma/2\rceil-2}\sum_{k=\lfloor(2l+1)n/(2(\gamma-1))\rfloor+1}^{\lfloor(2l+3)n/(2(\gamma+1))\rfloor}\left(\cos\left(\frac{2\pi k}{n}\right)+\cos\left(\frac{2\pi\gamma k}{n}\right)\right)$$
$$-4\sum_{l=0}^{\lceil\gamma/2\rceil-2}\sum_{k=\lfloor(2l+1)n/(2(\gamma+1))\rfloor+1}^{\lfloor(2l+1)n/(2(\gamma-1))\rfloor}\left(\cos\left(\frac{2\pi k}{n}\right)+\cos\left(\frac{2\pi\gamma k}{n}\right)\right)$$
$$-4\sum_{k=\lfloor(2\lceil\gamma/2\rceil-1)n/(2(\gamma+1))\rfloor+1}^{n/2-1}\left(\cos\left(\frac{2\pi k}{n}\right)+\cos\left(\frac{2\pi\gamma k}{n}\right)\right). \tag{13}$$

Expressing it in terms of Dirichlet kernels, we have

$$E\left(C_n^{1,\gamma}\right)=4+2\sum_{m\in\{1,\gamma\}}\left(D_{\lfloor n/(2(\gamma+1))\rfloor}\left(\frac{2\pi m}{n}\right)+\sum_{l=0}^{\lceil\gamma/2\rceil-2}\left(D_{\lfloor(2l+3)n/(2(\gamma+1))\rfloor}\left(\frac{2\pi m}{n}\right)-D_{\lfloor(2l+1)n/(2(\gamma-1))\rfloor}\left(\frac{2\pi m}{n}\right)\right)\right.$$
$$\left.-\sum_{l=0}^{\lceil\gamma/2\rceil-2}\left(D_{\lfloor(2l+1)n/(2(\gamma-1))\rfloor}\left(\frac{2\pi m}{n}\right)-D_{\lfloor(2l+1)n/(2(\gamma+1))\rfloor}\left(\frac{2\pi m}{n}\right)\right)-D_{n/2-1}\left(\frac{2\pi m}{n}\right)+D_{\lfloor(2\lceil\gamma/2\rceil-1)n/(2(\gamma+1))\rfloor}\left(\frac{2\pi m}{n}\right)\right). \tag{14}$$

The theorem follows from the fact that $D_{n/2-1}(2\pi m/n) = 1$ for $m = 1, \gamma$. □

A graph is called hyperenergetic if its energy is greater than the one of the complete graph K_n. The eigenvalues of the adjacency matrix of K_n are given by $n-1$ and -1 with multiplicity $n-1$, so that its energy is given by $E(K_n) = 2(n-1)$.

Figure 1(a) shows how the energy of $C_n^{1,\gamma}$ grows with respect to n for $\gamma = 8$. We see that it is not hyperenergetic and that the energy grows more or less linearly with respect to n. Figure 1(b) shows the energy of $C_n^{1,\gamma}$ with fixed n as γ varies. We observe that the energy stays more or less constant independently of γ.

As a consequence of the theorem, we can carry out the sum of the Dirichlet kernels when the number of vertices is proportional to $2(\gamma-1)(\gamma+1)$.

Corollary 2. *Given integers $\gamma \geqslant 3$ and $\alpha \geqslant 1$, the energy of the circulant graph $C_{2\alpha(\gamma-1)(\gamma+1)}^{1,\gamma}$ is given by*

$$E\left(C_{2\alpha(\gamma-1)(\gamma+1)}^{1,\gamma}\right)=4\sum_{m\in\{1,\gamma\}}\left(\frac{\sin\left(\pi m\left(\lceil\gamma/2\rceil+1/\left(2\alpha\left(\gamma-1\right)\right)\right)/\left(\gamma+1\right)\right)\sin\left(\lceil\gamma/2\rceil\pi m/\left(\gamma+1\right)\right)}{\sin\left(\pi m/\left(2\alpha\left(\gamma-1\right)\left(\gamma+1\right)\right)\right)\sin\left(\pi m/\left(\gamma+1\right)\right)}\right.$$
$$\left.-\frac{\sin\left(\pi m\left(\lceil\gamma/2\rceil-1+1/\left(2\alpha\left(\gamma+1\right)\right)\right)/\left(\gamma-1\right)\right)\sin\left(\left(\lceil\gamma/2\rceil-1\right)\pi m/\left(\gamma-1\right)\right)}{\sin\left(\pi m/\left(2\alpha\left(\gamma-1\right)\left(\gamma+1\right)\right)\right)\sin\left(\pi m/\left(\gamma-1\right)\right)}\right). \tag{15}$$

Proof. Let $a \geqslant 1$ and $K \geqslant 0$ be integers. The sum over k of Dirichlet kernels of index $(2k+1)a$ is given by

$$\sum_{k=0}^{K} D_{(2k+1)a}(x) = \sum_{k=0}^{K} \frac{\sin(((2k+1)a + 1/2)x)}{\sin(x/2)}. \quad (16)$$

By multiplying the summation by $\sin(ax)/\sin(ax)$ and using the trigonometric identity $2\sin\theta\sin\phi = \cos(\theta-\phi) - \cos(\theta+\phi)$, we have

$$\begin{aligned}\sum_{k=0}^{K} D_{(2k+1)a}(x) &= \frac{\cos(x/2) - \cos(((2K+2)a + 1/2)x)}{2\sin(x/2)\sin(ax)} \\ &= \frac{\sin(((2K+2)a+1)x/2)\sin((K+1)ax)}{\sin(x/2)\sin(ax)}.\end{aligned} \quad (17)$$

The corollary then follows by applying the above relation first with $a = \alpha(\gamma-1)$, $K = \lceil \gamma/2 \rceil - 1$ and second with $a = \alpha(\gamma+1)$, $K = \lceil \gamma/2 \rceil - 2$, and $x = 2\pi m/n$, $m \in \{1, \gamma\}$. □

In [7], the author considered the graphs $K_n - H$, where K_n is the complete graph on n vertices and H is a Hamilton cycle of K_n, and asked whether these graphs are hyperenergetic. In [4], the authors showed that the energy of $K_n - H$ is given by

$$E(K_n - H) = n - 3 + \sum_{k=1}^{n-1} \left| 1 + 2\cos\left(\frac{2\pi k}{n}\right) \right| \quad (18)$$

and that as n goes to infinity, it is hyperenergetic. In the following proposition, we give a formula for it for all $n \geqslant 3$.

Proposition 3. *For all $n \geqslant 3$, the energy of $K_n - H$ is given by*

$$\begin{aligned}E(K_n - H) &= 2\left(n - 3 - \left(\left\lfloor \frac{2n}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor\right)\right) + 2 \\ &\cdot \frac{\sin((\lfloor n/3 \rfloor + 1/2)2\pi/n) - \sin((\lfloor 2n/3 \rfloor + 1/2)2\pi/n)}{\sin(\pi/n)}.\end{aligned} \quad (19)$$

Proof. We have

$$\begin{aligned}\sum_{k=1}^{n-1} \left| 1 + 2\cos\left(\frac{2\pi k}{n}\right) \right| &= \sum_{k=1}^{\lfloor n/3 \rfloor} \left(1 + 2\cos\left(\frac{2\pi k}{n}\right)\right) \\ &\quad - \sum_{k=\lfloor n/3 \rfloor + 1}^{\lfloor 2n/3 \rfloor} \left(1 + 2\cos\left(\frac{2\pi k}{n}\right)\right) \\ &\quad + \sum_{k=\lfloor 2n/3 \rfloor + 1}^{n-1} \left(1 + 2\cos\left(\frac{2\pi k}{n}\right)\right) \\ &= n - 2 - 2\left(\left\lfloor \frac{2n}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor\right) + 2D_{\lfloor n/3 \rfloor}\left(\frac{2\pi}{n}\right) \\ &\quad - 2D_{\lfloor 2n/3 \rfloor}\left(\frac{2\pi}{n}\right) + D_{n-1}\left(\frac{2\pi}{n}\right).\end{aligned} \quad (20)$$

Since $D_{n-1}(2\pi/n) = -1$, the proposition follows. □

By elementary analysis, one can show that $E(K_n - H) - 2(n-1)$ is increasing in n. As a consequence, we find that $K_n - H$ are hyperenergetic for all $n \geqslant 10$. This has been previously found in [4].

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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