Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2015, Article ID 919608, 15 pages
http://dx.doi.org/10.1155/2015/919608

Research Article

Entropy Solution for Doubly Nonlinear Elliptic Anisotropic Problems with Robin Boundary Conditions

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Received 4 July 2015; Accepted 21 October 2015

Academic Editor: M. El-Gebeily

We study in this paper nonlinear anisotropic problems with Robin boundary conditions. We prove, by using the technic of monotone operators in Banach spaces, the existence of a sequence of weak solutions of approximation problems associated with the anisotropic Robin boundary value problem. For the existence and uniqueness of entropy solutions, we prove that the sequence of weak solutions converges to a measurable function which is the entropy solution of the anisotropic Robin boundary value problem.

1. Introduction

The aim of this paper is to study the following nonlinear anisotropic elliptic Robin boundary value problem:

$$
\begin{gathered}
-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) + b(u) = f \quad \text{in } \Omega, \\
\sum_{i=1}^{N} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \eta_i = -\gamma(u) \quad \text{on } \partial\Omega,
\end{gathered} \tag{1}
$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary and $\mathrm{meas}(\Omega) > 0$, $f \in L^1(\Omega)$, $\eta = (\eta_1, \dots, \eta_N)$ is the unit outward normal on $\partial\Omega$, and $\gamma(u) = |u|^{r(x)-2}u$.

All papers on problems like (1) considered particular cases of function b. Indeed, in [1], Bonzi et al. studied the following problems:

$$
\begin{gathered}
-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) + |u|^{p_M(x)-2} u = f \quad \text{in } \Omega, \\
\sum_{i=1}^{N} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \eta_i = -|u|^{r(x)-2} u \\
\text{on } \partial\Omega,
\end{gathered} \tag{2}
$$

where $f \in L^1(\Omega)$. The authors use minimization technics used in [2] or [3] (see also [4, 5]) to prove the existence and uniqueness of entropy solution.

The Robin type boundary conditions in the variable exponents setting are new and interesting problems and were for the first time studied by Boureanu and Radulescu in [3]. The main difficulty for the study of problem in [3] was the definition of an admissible space of solutions. The authors defined the appropriate space and obtained its properties which permit them to use minimization method to prove the existence of weak solutions to the following problem:

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\left(x,\frac{\partial u}{\partial x_i}\right)+b(x)|u|^{p_M(x)-2}u=f(x,u) \quad \text{in } \Omega,$$
$$u\geq 0 \quad \text{in } \Omega, \tag{3}$$
$$\sum_{i=1}^{N}a_i\left(x,\frac{\partial u}{\partial x_i}\right)\eta_i=g(x,u) \quad \text{on } \partial\Omega.$$

Since we consider L^1-data f instead of the function $f(x,u)$ considered in [3], the suitable notion of solution is the entropy solution introduced by Bénilan et al. in [6] (see also [7]). With the Robin type boundary conditions, the values of the solutions at the boundary must be precise and the notion of solutions considered must include the boundary condition. In this paper, as the function b is more general, it is not possible to use minimization technic to get the existence of solution. Therefore, we used the technic of monotone operators in Banach spaces (see [8]) to get the existence of entropy solutions of (1).

For presenting our main result, we first have to describe the data involved in our problem. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N\geq 3)$ with smooth boundary domain $\partial\Omega$ and $\vec{p}(\cdot)=(p_1(\cdot),\ldots,p_N(\cdot))$ such that, for any $i=1,\ldots,N$, $p_i(\cdot):\overline{\Omega}\to[2;N)$ is a continuous function with

$$1<p_i^{-}\operatorname*{ess\,inf}_{x\in\Omega}p_i(x)\leq\operatorname*{ess\,sup}_{x\in\Omega}p_i(x)\,p_i^{+}<+\infty. \tag{4}$$

For any $i=1,\ldots,N$, let $a_i:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying the following:

(i) There exists a positive constant C_1 such that

$$|a_i(x,\xi)|\leq C_1\left(j_i(x)+|\xi|^{p_i(x)-1}\right), \tag{5}$$

for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}$, where j_i is a nonnegative function lying in $L^{p_i'(\cdot)}(\Omega)$, with $(1/p_i(x))+(1/p_i'(x))=1$.

(ii) For $\xi,\eta\in\mathbb{R}$ with $\xi\neq\eta$ and for almost every $x\in\Omega$, there exists a positive constant C_2 such that

$$(a_i(x,\xi)-a_i(x,\eta))(\xi-\eta)\geq\begin{cases}C_2|\xi-\eta|^{p_i(x)} & \text{if } |\xi-\eta|\geq 1,\\ C_2|\xi-\eta|^{p_i^-} & \text{if } |\xi-\eta|<1.\end{cases} \tag{6}$$

(iii) There exists a positive constant C_3 such that

$$a_i(x,\xi)\cdot\xi\geq C_3|\xi|^{p_i(x)}, \tag{7}$$

for $\xi\in\mathbb{R}$ and for almost every $x\in\Omega$.

The hypotheses on a_i are classical in the study of nonlinear problems (see [1, 3]).

The function b is such that

$$b:\mathbb{R}\longrightarrow\mathbb{R} \text{ is continuous, surjective, nondecreasing with } b(0)=0. \tag{8}$$

Throughout this paper, for any $i=1,\ldots,N$, we assume that

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)}<p_i^-<\frac{\overline{p}(N-1)}{N-\overline{p}},$$
$$\frac{p_i^+-p_i^--1}{p_i^-}<\frac{\overline{p}-N}{\overline{p}(N-1)}, \tag{9}$$
$$\sum_{i=1}^{N}\frac{1}{p_i^-}>1,$$

where $N/\overline{p}=\sum_{i=1}^{N}(1/p_i^-)$.

We put for all $x\in\Omega$,

$$p_M(x)\max\{p_1(x),\ldots,p_N(x)\},$$
$$p_m(x)\min\{p_1(x),\ldots,p_N(x)\} \tag{10}$$

and for all $x\in\partial\Omega$,

$$p^{\partial}(x)=\begin{cases}\dfrac{(N-1)p(x)}{N-p(x)} & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\geq N.\end{cases} \tag{11}$$

We make the following assumption:

$$r\in C(\overline{\Omega}) \text{ with } 1<r^-\leq r^+<\min_{x\in\partial\Omega}\left\{p_1^{\partial}(x),\ldots,p_N^{\partial}(x)\right\}. \tag{12}$$

Note that the function γ is continuous, defined on $\mathbb{R}$ with $\gamma(t)t\geq 0$ for all t in $\mathbb{R}$ and $\gamma(0)=0$.

A prototype example that is covered by our assumptions is the following anisotropic $\vec{p}(\cdot)$-harmonic system:

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)=f, \tag{13}$$

which, in the particular case when $p_i=p$ for any $i=1,\ldots,N$, is the p-Laplace equation.

The rest of the paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we study the existence and uniqueness of entropy solution.

2. Preliminaries

We recall in this section some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces with variable exponent. Set

$$C_+(\overline{\Omega})=\left\{p\in C(\overline{\Omega}) \text{ such that } \min_{x\in\overline{\Omega}}p(x)>1\right\}. \tag{14}$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega)\ \Big\{u : \Omega \longrightarrow \mathbb{R},\ \text{measurable such that } \int_\Omega |u|^{p(x)}\,dx < \infty\Big\}, \tag{15}$$

endowed with the so-called Luxemburg norm

$$|u|_{p(\cdot)}\ \inf\left\{\lambda > 0 \text{ such that } \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}. \tag{16}$$

The $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u)\ \int_\Omega |u|^{p(x)}\,dx. \tag{17}$$

For any $u \in L^{p(\cdot)}(\Omega)$, the following inequality (see [9, 10]) will be used later:

$$\min\left\{|u|_{p(\cdot)}^{p^-}; |u|_{p(\cdot)}^{p^+}\right\} \le \rho_{p(\cdot)}(u) \le \max\left\{|u|_{p(\cdot)}^{p^-}; |u|_{p(\cdot)}^{p^+}\right\}. \tag{18}$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$ with $(1/p(x)) + (1/q(x)) = 1$ in Ω, we have the following Hölder type inequality:

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(\cdot)}|v|_{q(\cdot)}. \tag{19}$$

If Ω is bounded and $p, q \in C_+(\overline{\Omega})$ such that $p(x) \le q(x)$ for any $x \in \Omega$, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous (see [11, Theorem 2.8]).

Herein, we need the following anisotropic Sobolev space with variable exponent:

$$W^{1,\vec{p}(\cdot)}(\Omega)\ \left\{u \in L^{p_M(\cdot)}(\Omega) \text{ such that } \frac{\partial u}{\partial x_i} \in L^{p_i(\cdot)}(\Omega),\ i = 1, \dots, N\right\}. \tag{20}$$

$W^{1,\vec{p}(\cdot)}(\Omega)$ is a separable and reflexive Banach space (see [2]) under the norm

$$\|u\|_{\vec{p}(\cdot)} = |u|_{p_M(\cdot)} + \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}. \tag{21}$$

We need the following embedding and trace results.

Theorem 1 (see [9, Corollary 2.1]). *Let $\Omega \subset \mathbb{R}^N$ ($N \ge 3$) be a bounded open set and for all $i = 1, \dots, N$, $p_i \in L^\infty(\Omega)$, $p_i(x) \ge 1$ a.e. in Ω. Then, for any $q \in L^\infty(\Omega)$ with $q(x) \ge 1$ a.e. in Ω such that*

$$\operatorname*{ess\,inf}_{x\in\Omega}\left(p_M(x) - q(x)\right) > 0, \tag{22}$$

one has the compact embedding

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega). \tag{23}$$

Theorem 2 (see [3, Theorem 6]). *Let $\Omega \subset \mathbb{R}^N$ ($N \ge 2$) be a bounded open set with smooth boundary and let $\vec{p}(\cdot) \in (C_+(\overline{\Omega}))^N$, $r \in C(\overline{\Omega})$ satisfy the condition*

$$1 \le r(x) < \min\left\{p_1^\partial(x), \dots, p_N^\partial(x)\right\}, \quad \forall x \in \partial\Omega. \tag{24}$$

Then, there is a compact boundary trace embedding

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega). \tag{25}$$

We introduce the numbers

$$q = \frac{N(\overline{p} - 1)}{N - 1}, \qquad q^* = \frac{N(\overline{p} - 1)}{N - \overline{p}} = \frac{Nq}{N - q}. \tag{26}$$

The following result is due to Troisi (see [12]).

Theorem 3. *Let $p_1, \dots, p_N \in [1, +\infty)$; $g \in W^{1,(p_1,\dots,p_N)}(\Omega)$ and*

$$q = (\overline{p})^* \quad \text{if } (\overline{p})^* < N, \qquad q \in [1, +\infty) \quad \text{if } (\overline{p})^* \ge N. \tag{27}$$

Then, there exists a constant $C_4 > 0$ depending on $N, p_1, \dots, p_N$ if $\overline{p} < N$ and also on q and $\operatorname{meas}(\Omega)$ *if $\overline{p} \ge N$ such that*

$$\|g\|_{L^q(\Omega)} \le C_4 \prod_{i=1}^N \left[\|g\|_{L^{p_M}(\Omega)} + \left\|\frac{\partial g}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\right]^{1/N}. \tag{28}$$

In this paper, we will use the Marcinkiewicz space $\mathscr{M}^q(\Omega)$ ($1 < q < +\infty$) as the set of measurable functions $g : \Omega \to \mathbb{R}$ for which the distribution function

$$\lambda_g(k) = \operatorname{meas}\left(\{x \in \Omega : |g(x)| > k\}\right), \quad k \ge 0 \tag{29}$$

satisfies an estimate of the form

$$\lambda_g(k) \le Ck^{-q}, \quad \text{for some finite constant } C > 0. \tag{30}$$

We will use the following pseudonorm in $\mathscr{M}^q(\Omega)$:

$$\|g\|_{\mathscr{M}^q(\Omega)}\ \inf\left\{C > 0 : \lambda_g(k) \le Ck^{-q},\ \forall k > 0\right\}. \tag{31}$$

For any $k > 0$, the truncation function T_k is defined by

$$T_k(s) = \max\{-k; \min\{k; s\}\}. \tag{32}$$

It is clear that $\lim_{k\to+\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.

In order to simplify the notation, for any $v \in W^{1,\vec{p}(\cdot)}(\Omega)$, we use v instead of $v_{|\partial\Omega}$ for the trace of v on $\partial\Omega$.

Set $\mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$ as the set of the measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that, for any $k > 0$, $T_k(u) \in W^{1,\vec{p}(\cdot)}(\Omega)$. We define the space $\mathcal{T}_{\mathrm{tr}}^{1,\vec{p}(\cdot)}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$ such that there exists a sequence $(u_n)_n \subset W^{1,\vec{p}(\cdot)}(\Omega)$ satisfying

$$\begin{aligned} & u_n \longrightarrow u \quad \text{a.e. in } \Omega, \\ & \frac{\partial T_k(u_n)}{\partial x_i} \longrightarrow \frac{\partial T_k(u)}{\partial x_i} \quad \text{in } L^1(\Omega),\ \forall k > 0, \\ & \text{there exists a measurable function } v \text{ on } \partial\Omega \text{ such that } u_n \longrightarrow v \quad \text{a.e. on } \partial\Omega. \end{aligned} \tag{33}$$

We need the following lemma proved in [13].

Lemma 4. *Let g be a nonnegative function in $W^{1,\vec{p}(\cdot)}(\Omega)$. Assume $\overline{p} < N$ and there exists a constant $C > 0$ such that*

$$\int_\Omega |T_k(g)|^{p_M^-}\,dx + \sum_{i=1}^N \int_{\{|g|\le k\}} \left|\frac{\partial g}{\partial x_i}\right|^{p_i^-} dx \le C(1+k), \quad \forall k > 0. \tag{34}$$

Then, there exists a constant D, depending on C, such that

$$\|g\|_{\mathcal{M}^{q^*}(\Omega)} \le D, \tag{35}$$

where $q^* = N(\overline{p}-1)/(N-\overline{p})$.

3. Entropy Solutions

The notion of entropy solutions to problem (1) where the data f belongs to $L^1(\Omega)$ is the following.

Definition 5. A measurable function $u \in \mathcal{T}_{\mathrm{tr}}^{1,\vec{p}(\cdot)}(\Omega)$ is an entropy solution of problem (1) if $b(u) \in L^1(\Omega)$, $\gamma(u) \in L^1(\partial\Omega)$, and, for every $k > 0$,

$$\begin{aligned} & \sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u-\varphi)\,dx \\ & \quad + \int_\Omega b(u)\,T_k(u-\varphi)\,dx \\ & \quad + \int_{\partial\Omega} \gamma(u)\,T_k(u-\varphi)\,d\sigma \le \int_\Omega f T_k(u-\varphi)\,dx, \end{aligned} \tag{36}$$

for every $\varphi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

The existence result is the following theorem.

Theorem 6. *Assume that (4)–(12) hold. Then, problem (1) admits at least one entropy solution.*

Proof. The proof is done in three steps.

Step 1 (the approximate problem). We define the reflexive space

$$E = W^{1,\vec{p}(\cdot)}(\Omega) \times L^{p_M(\cdot)}(\partial\Omega). \tag{37}$$

Let X_0 be the subspace of E defined by

$$X_0 = \{(u,v) \in E : v = \tau(u)\}, \tag{38}$$

where $\tau(u)$ is the trace of $u \in \mathcal{T}_{\mathrm{tr}}^{1,p(\cdot)}(\Omega)$ in the usual sense, since $u \in W^{1,\vec{p}(\cdot)}(\Omega)$. In the sequel, we will identify an element $(u,v) \in X_0$ with its representative $u \in W^{1,\vec{p}(\cdot)}(\Omega)$.

For any $n \in \mathbb{N}$ and $\varepsilon > 0$, we consider the sequence of approximate problems:

$$\begin{aligned} & \sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \frac{\partial v}{\partial x_i} dx + \varepsilon \int_\Omega |u_n|^{p_M(x)-2} u_n v\,dx \\ & \quad + \int_\Omega T_n(b(u_n))\,v\,dx + \int_{\partial\Omega} T_n(\gamma(u_n))\,v\,d\sigma \\ & = \int_\Omega f_n v\,dx, \end{aligned} \tag{39}$$

where $f_n = T_n(f)$, and we define an operator A_n by

$$\begin{aligned} \langle A_n(u), v\rangle &= \langle A(u), v\rangle + \int_\Omega T_n(b(u))\,v\,dx \\ & \quad + \int_{\partial\Omega} T_n(\gamma(u))\,v\,d\sigma \quad \forall u, v \in X_0, \end{aligned} \tag{40}$$

where

$$\begin{aligned} \langle A(u), v\rangle &= \sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial v}{\partial x_i} dx \\ & \quad + \varepsilon \int_\Omega |u|^{p_M(x)-2} u v\,dx. \end{aligned} \tag{41}$$

Note that

$$\begin{aligned} & \|f_n\|_\infty \le \frac{\|f\|_1}{\operatorname{meas}(\Omega)}, \\ & \|f_n\|_1 = \int_\Omega |f_n|\,dx \le \int_\Omega |f|\,dx = \|f\|_1, \\ & f_n \xrightarrow[n \to +\infty]{} f \text{ in } L^1(\Omega), \text{ a.e. in } \Omega. \end{aligned} \tag{42}$$

Assertion 1 (the operator A is of type M). (i) The operator A is monotone. Indeed, for $u, v \in W^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned}\langle A(u)-A(v), u-v\rangle &= \langle A(u), u-v\rangle + \langle A(v), v \\ &- u\rangle = \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial(u-v)}{\partial x_i}dx \\ &+ \varepsilon\int_\Omega |u|^{p_M(x)-2}u(u-v)\,dx \\ &+ \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial v}{\partial x_i}\right)\frac{\partial(v-u)}{\partial x_i}dx + \varepsilon\int_\Omega |v|^{p_M(x)-2} \\ &\cdot v(v-u)\,dx \\ &= \int_\Omega \sum_{i=1}^N \left[a_i\left(x, \frac{\partial u}{\partial x_i}\right) - a_i\left(x, \frac{\partial v}{\partial x_i}\right)\right] \\ &\cdot \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right)dx + \varepsilon\int_\Omega \left(|u|^{p_M(x)-2}u\right. \\ &\left. - |v|^{p_M(x)-2}v\right)(u-v)\,dx.\end{aligned} \tag{43}$$

Therefore,

$$\langle A(u)-A(v), u-v\rangle \geq 0, \tag{44}$$

since for $i = 1, \ldots, N$, for almost every $x \in \Omega$, $a_i(x,\cdot)$ and $t \longmapsto |t|^{p_M(x)-2}t$ are monotone.

(ii) The operator A is hemicontinuous. Indeed, for every u, v in $W^{1,\vec{p}(\cdot)}(\Omega)$, let

$$\varphi : t \in \mathbb{R} \longmapsto \varphi(t) = \langle A(u+tv), v\rangle \tag{45}$$

and let $t, t_0 \in \mathbb{R}$ be such that $t \to t_0$. We have $w = u + tv \to w_0 = u + t_0 v$ in $W^{1,\vec{p}(\cdot)}(\Omega)$.

Using the Hölder type inequality, there exists $i_0 \in \{1, \ldots, N\}$ such that

$$\begin{aligned}|\varphi(t)-\varphi(t_0)| &= \left|\langle A(u+tv), v\rangle - \langle A(u+t_0v), v\rangle\right| \\ &\leq \sum_{i=1}^N \int_\Omega \left|a_i\left(x, \frac{\partial w}{\partial x_i}\right) - a_i\left(x, \frac{\partial w_0}{\partial x_i}\right)\right|\left|\frac{\partial v}{\partial x_i}\right|dx \\ &+ \varepsilon\int_\Omega \left||w|^{p_M(x)-2}w - |w_0|^{p_M(x)-2}w_0\right||v|\,dx \\ &\leq N\left(\frac{1}{p_{i_0}^-} + \frac{1}{(p'_{i_0})^-}\right) \\ &\cdot \left|a_{i_0}\left(x, \frac{\partial w}{\partial x_{i_0}}\right) - a_{i_0}\left(x, \frac{\partial w_0}{\partial x_{i_0}}\right)\right|_{p'_{i_0}(\cdot)}\left|\frac{\partial v}{\partial x_{i_0}}\right|_{p_{i_0}(\cdot)} \\ &+ \varepsilon\left(\frac{1}{p_M^-} + \frac{1}{(p'_M)^-}\right) \\ &\cdot \left||w|^{p_M(x)-2}w - |w_0|^{p_M(x)-2}w_0\right|_{p'_M(\cdot)}|v|_{p_M(\cdot)}.\end{aligned} \tag{46}$$

Let us denote $\psi_{i_0}(x,w) = a_{i_0}(x, \partial w/\partial x_{i_0})$.

Using assumption (5) and [11, Theorems 4.1 and 4.2] we have $\psi_{i_0}(x,w) \to \psi_{i_0}(x,w_0)$ in $L^{p'_{i_0}(\cdot)}(\Omega)$. Then, we deduce that φ is continuous; namely, the operator A is hemicontinuous.

Since the operator A is *monotone* and *hemicontinuous*, then according to Lemma 2.1 in [8], A is of type M.

Assertion 2 (the operator A_n is of type M). Indeed, let $(u_k)_{k\in\mathbb{N}}$ be a sequence in X_0 such that

$$\begin{aligned}u_k &\rightharpoonup u \quad \text{weakly in } X_0, \\ A_n u_k &\rightharpoonup \chi \quad \text{weakly in } X'_0, \\ \limsup_{k\to+\infty}\langle A_n(u_k), u_k\rangle &\leq \langle \chi, u\rangle.\end{aligned} \tag{47}$$

Since

$$\begin{aligned}T_n(b(u))u &\geq 0, \\ T_n(\gamma(u))u &\geq 0,\end{aligned} \tag{48}$$

by Fatou's lemma, we obtain that

$$\begin{aligned}&\liminf_{k\to+\infty}\left(\int_\Omega T_n(b(u_k))u_k dx + \int_{\partial\Omega} T_n(\gamma(u_k))u_k d\sigma\right) \\ &\quad\geq \int_\Omega T_n(b(u))u\,dx + \int_{\partial\Omega} T_n(\gamma(u))u\,d\sigma\end{aligned} \tag{49}$$

and thanks to the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}&\lim_{k\to+\infty}\left(\int_\Omega T_n(b(u_k))v\,dx + \int_{\partial\Omega} T_n(\gamma(u_k))v\,d\sigma\right) \\ &\quad= \int_\Omega T_n(b(u))v\,dx + \int_{\partial\Omega} T_n(\gamma(u))v\,d\sigma,\end{aligned} \tag{50}$$

for all v in X_0. Consequently,

$$T_n(b(u_k)) + T_n(\gamma(u_k)) \rightharpoonup T_n(b(u)) + T_n(\gamma(u)) \quad \text{weakly in } X'_0. \tag{51}$$

Therefore, we deduce that

$$Au_k \rightharpoonup \chi - (T_n(b(u)) + T_n(\gamma(u))) \quad \text{weakly in } X'_0. \tag{52}$$

As in Assertion 1, we prove that the operator A is of type M, so we have

$$Au = \chi - (T_n(b(u)) + T_n(\gamma(u))). \tag{53}$$

Thus, it follows that

$$A_n u = \chi. \tag{54}$$

Hence, A_n is of type M.

Assertion 3 (the operator A_n is coercive). Indeed, since

$$T_n(b(u))u + T_n(\gamma(u))u \geq 0, \tag{55}$$

then

$$\langle A_n(u), u\rangle \geq \langle A(u), u\rangle. \tag{56}$$

According to (7), we have

$$\langle A(u), u\rangle \geq C_3 \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx + \varepsilon \int_\Omega |u|^{p_M(x)} dx. \tag{57}$$

Denote

$$\begin{aligned} \mathcal{I} &= \left\{ i \in \{1,\dots,N\} : \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)} \leq 1 \right\}, \\ \mathcal{J} &= \left\{ i \in \{1,\dots,N\} : \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)} > 1 \right\}. \end{aligned} \tag{58}$$

We have

$$\begin{aligned} \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx &= \sum_{i\in\mathcal{I}} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \\ &\quad + \sum_{i\in\mathcal{J}} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \\ &\geq \sum_{i\in\mathcal{I}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_i^+} + \sum_{i\in\mathcal{J}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_i^-} \\ &\geq \sum_{i\in\mathcal{J}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_i^-} \geq \sum_{i\in\mathcal{J}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_m^-} \\ &\geq \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_m^-} - \sum_{i\in\mathcal{I}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_m^-} \\ &\geq \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}^{p_m^-} - N. \end{aligned} \tag{59}$$

Using the convexity of the application $t \in \mathbb{R}^+ \mapsto t^{p_m^-}$, $p_m^- > 1$, we obtain

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \geq \frac{1}{N^{p_m^- -1}} \left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}\right)^{p_m^-} - N. \tag{60}$$

Then,

$$\begin{aligned} \langle A_n(u), u\rangle &\geq \frac{C_3}{N^{p_m^- -1}} \left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}\right)^{p_m^-} \\ &\quad + \varepsilon \int_\Omega |u|^{p_M(x)} dx - C_3 N. \end{aligned} \tag{61}$$

(i) Assume $|u|_{p_M(\cdot)} > 1$. Then, (18) gives $\int_\Omega |u|^{p_M(x)} dx \geq |u|_{p_M(\cdot)}^{p_m^-}$.

So, combining (56) and (61) we get

$$\begin{aligned} \langle A_n(u), u\rangle &\geq C\left[\left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}\right)^{p_m^-} + |u|_{p_M(\cdot)}^{p_m^-}\right] \\ &\quad - C_3 N \geq \frac{C}{2^{p_m^- -1}} \|u\|_{\vec{p}(\cdot)}^{p_m^-} - C_3 N, \\ &\quad \text{where } C = \min\left\{\frac{C_3}{N^{p_m^- -1}}; \varepsilon\right\}. \end{aligned} \tag{62}$$

(ii) Assume $|u|_{p_M(\cdot)} \leq 1$. Then, combining (56) and (61) we get

$$\begin{aligned} \langle A_n(u), u\rangle &\geq C\left[\left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(\cdot)}\right)^{p_m^-} + |u|_{p_M(\cdot)}^{p_m^-}\right] - 1 \\ &\quad - C_3 N + \varepsilon \int_\Omega |u|^{p_M(x)} dx \\ &\geq \frac{C}{2^{p_m^- -1}} \|u\|_{\vec{p}(\cdot)}^{p_m^-} - 1 - C_3 N, \\ &\quad \text{where } C = \min\left\{\frac{C_3}{N^{p_m^- -1}}; 1\right\}. \end{aligned} \tag{63}$$

Consequently, since $p_m^- > 1$, the operator A_n is coercive.

Besides, the operator A_n is bounded and hemicontinuous.

Then, for any $F_n = T_n(f) \in E' \subset X_0'$, we can deduce the existence of a function $u_n \in X_0$ such that

$$\langle A_n(u_n), v\rangle = \langle F_n, v\rangle, \quad \forall v \in X_0. \tag{64}$$

Namely, u_n is a weak solution of problem (39).

We are now going to prove that these approximated solutions u_n tend, as n goes to infinity, to a measurable function u which is an entropy solution of problem (1). To start with, we establish some a priori estimates.

Step 2 (a priori estimates). Assume (4)–(12) and let u_n be a solution of problem (39). We have the following results.

Lemma 7. *There exists a constant $C_5 > 0$ such that*

$$\begin{aligned} &\int_\Omega |T_k(u_n)|^{p_M^-} dx + \sum_{i=1}^N \int_{\{|u_n|\leq k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx \\ &\quad \leq C_5(k+1). \end{aligned} \tag{65}$$

Proof. Let us take $T_k(u_n)$ as a test function in (39). Since

$$\begin{aligned} &\int_\Omega T_n(b(u_n)) T_k(u_n) dx + \varepsilon \int_\Omega |u_n|^{p_M(x)-2} u_n T_k(u_n) \\ &\quad + \int_{\partial\Omega} T_n(\gamma(u_n)) T_k(u_n) d\sigma \geq 0, \end{aligned} \tag{66}$$

using relation (7), we obtain

$$C_3 \sum_{i=1}^{N} \int_{\{|u_n|\le k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} dx \le k \|f\|_1 . \tag{67}$$

Then, we have

$$\begin{aligned}\sum_{i=1}^{N} \int_{\{|u_n|\le k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx &= \sum_{i=1}^{N} \int_{\{|u_n|\le k; |\partial u_n/\partial x_i|>1\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx \\ &\quad + \sum_{i=1}^{N} \int_{\{|u_n|\le k; |\partial u_n/\partial x_i|\le 1\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx \\ &\le \sum_{i=1}^{N} \int_{\{|u_n|\le k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} dx + N \cdot \operatorname{meas}(\Omega) \\ &\le \frac{k}{C_3} \|f\|_1 + N \cdot \operatorname{meas}(\Omega) .\end{aligned} \tag{68}$$

Moreover, we have

$$\begin{aligned}\int_\Omega |T_k(u_n)|^{p_M^-} dx &= \int_{\{|T_k(u_n)|\le 1\}} |T_k(u_n)|^{p_M^-} dx + \int_{\{|T_k(u_n)|>1\}} |T_k(u_n)|^{p_M^-} dx \\ &\le \operatorname{meas}(\Omega) + \int_{\{|T_k(u_n)|>1\}} k^{p_M^-} dx \\ &\le \operatorname{meas}(\Omega) \left(1 + k^{p_M^-}\right).\end{aligned} \tag{69}$$

Therefore, we get

$$\begin{aligned}\int_\Omega |T_k(u_n)|^{p_M^-} dx + \sum_{i=1}^{N} \int_{\{|u_n|\le k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx &\le \operatorname{meas}(\Omega) \left(1 + N + k^{p_M^-}\right) + \frac{k}{C_3} \|f\|_1 \\ &\le C_5 (1 + k),\end{aligned} \tag{70}$$

where $C_5 = \max\{\operatorname{meas}(\Omega)(1 + N + k^{p_M^-}); (1/C_3)\|f\|_1\}$. □

Lemma 8. *For any $k > 0$, there exist two constants $C_7 > 0$ and $C_8 > 0$ such that*

(i) $\|u_n\|_{\mathscr{M}^{q^*}(\Omega)} \le C_7$,

(ii) $\|\partial u_n/\partial x_i\|_{\mathscr{M}^{p_i^- q/\overline{p}}(\Omega)} \le C_8$, $\forall i = 1, \dots, N$.

Proof. (i) This is a consequence of Lemmas 4 and 7.
(ii)

(a) Let $\alpha \ge 1$. For any $k \ge 1$, we have

$$\begin{aligned}\lambda_{\partial u_n/\partial x_i}(\alpha) &= \operatorname{meas}\left(\left\{x \in \Omega : \left|\frac{\partial u_n}{\partial x_i}\right| > \alpha\right\}\right) \\ &= \operatorname{meas}\left(\left\{\left|\frac{\partial u_n}{\partial x_i}\right| > \alpha; |u_n| \le k\right\}\right) \\ &\quad + \operatorname{meas}\left(\left\{\left|\frac{\partial u_n}{\partial x_i}\right| > \alpha; |u_n| > k\right\}\right) \\ &\le \int_{\{|\partial u_n/\partial x_i|>\alpha; |u_n|\le k\}} dx + \lambda_{u_n}(k) \\ &\le \int_{\{|u_n|\le k\}} \left(\frac{1}{\alpha}\left|\frac{\partial u_n}{\partial x_i}\right|\right)^{p_i^-} dx + \lambda_{u_n}(k) \\ &\le \alpha^{-p_i^-} C' k + C k^{-q^*}.\end{aligned} \tag{71}$$

Then, there exists a positive constant C_6 such that

$$\lambda_{\partial u_n/\partial x_i}(\alpha) \le C_6 \left(k \alpha^{-p_i^-} + k^{-q^*}\right). \tag{72}$$

Let us consider the function

$$\begin{aligned}g &: [1, +\infty) \longrightarrow \mathbb{R}, \\ t &\longmapsto g(t) = \frac{t}{\alpha^{p_i^-}} + t^{-q^*}.\end{aligned} \tag{73}$$

We have $g'(t) = 0$ for $t = (q^* \alpha^{p_i^-})^{1/(q^*+1)}$. Thus, if we take $k = (q^* \alpha^{p_i^-})^{1/(q^*+1)} \ge 1$ in (72) we get

$$\begin{aligned}\lambda_{\partial u_n/\partial x_i}(\alpha) &\le C_6 k \left(\frac{q^*+1}{q^*} \frac{1}{\alpha^{p_i^-}}\right) \le C_6' \alpha^{-(q^*/(q^*+1)) p_i^-} \\ &\le C_6' \alpha^{-p_i^- q/\overline{p}}, \\ &\quad \forall \alpha \ge 1, \text{ where } C_6' \text{ is a positive constant.}\end{aligned} \tag{74}$$

(b) If $0 \le \alpha < 1$, we have

$$\begin{aligned}\lambda_{\partial u_n/\partial x_i}(\alpha) &= \operatorname{meas}\left(\left\{\left|\frac{\partial u_n}{\partial x_i}\right| > \alpha\right\}\right) \le \operatorname{meas}(\Omega) \\ &\le \operatorname{meas}(\Omega)\, \alpha^{-p_i^- q/\overline{p}}.\end{aligned} \tag{75}$$

Then

$$\lambda_{\partial u_n/\partial x_i}(\alpha) \le \left(C_6' + \operatorname{meas}(\Omega)\right) \alpha^{-p_i^- q/\overline{p}}, \quad \forall \alpha \ge 0. \tag{76}$$

Therefore, we deduce that there exists a positive constant C_8 such that

$$\left\|\frac{\partial u_n}{\partial x_i}\right\|_{\mathscr{M}^{p_i^- q/\overline{p}}(\Omega)} \le C_8, \quad \forall i = 1, \dots, N. \tag{77}$$

□

Step 3 (existence of entropy solution). Using Lemma 8, we have the following useful lemma (see [13]).

Lemma 9. *For $i = 1, \dots, N$, as $n \to +\infty$, one has*

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \quad \text{in } L^1(\Omega) \text{ a.e. } x \in \Omega. \tag{78}$$

In order to pass to the limit in relation (39), we also need the following convergence results which can be proved as in [7] (see also [1, 13]).

Proposition 10. *Assume (4)–(12). If $u_n \in W^{1,\vec{p}(\cdot)}(\Omega)$ is a weak solution of (P_n) then the sequence $(u_n)_{n\in\mathbb{N}^*}$ is Cauchy in measure. In particular, there exist a measurable function u and a subsequence still denoted by u_n such that $u_n \to u$ in measure.*

Proposition 11. *Assume (4)–(12). If $u_n \in W^{1,\vec{p}(\cdot)}(\Omega)$ is a weak solution of (P_n) then*

(i) *there exists $s > 1$ such that $u_n \to u$ a.e. in Ω and moreover $u_n \rightharpoonup u$ in $W^{1,s}(\Omega)$,*

(ii) *for all $i = 1, \dots, N$, $\partial u_n/\partial x_i$ converges strongly in $L^1(\Omega)$. Moreover $a_i(x, \partial u_n/\partial x_i)$ converges to $a_i(x, \partial u/\partial x_i)$ in $L^1(\Omega)$ strongly and in $L^{p_i'(\cdot)}(\Omega)$ weakly for all $i = 1, \dots, N$,*

(iii) *u_n converges to some measurable function v a.e. in $\partial\Omega$.*

We can now pass to the limit in relation (39).

Let $\varphi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and choosing $T_k(u_n - \varphi)$ as a test function in (39), we get

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u_n - \varphi)\, dx \\ &\quad + \int_\Omega T_n(b(u_n)) T_k(u_n - \varphi)\, dx \\ &\quad + \varepsilon \int_\Omega |u_n|^{p_M(x)-2} u_n T_k(u_n - \varphi)\, dx \\ &\quad + \int_{\partial\Omega} T_n(\gamma(u_n)) T_k(u_n - \varphi)\, d\sigma \\ &= \int_\Omega f_n T_k(u_n - \varphi)\, dx. \end{aligned} \tag{79}$$

For the right-hand side of (79), we have

$$\int_\Omega f_n(x) T_k(u_n - \varphi)\, dx \longrightarrow \int_\Omega f(x) T_k(u - \varphi)\, dx, \tag{80}$$

since f_n converges strongly to f in $L^1(\Omega)$ and $T_k(u_n - \varphi)$ converges weakly-* to $T_k(u - \varphi)$ in $L^\infty(\Omega)$ and a.e. in Ω.

For the first term of (79), we have (see [13])

$$\begin{aligned} &\liminf_{n\to+\infty} \sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u_n - \varphi)\, dx \\ &\quad \geq \sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u - \varphi)\, dx. \end{aligned} \tag{81}$$

We now focus our attention on the second term of (79). We have

$$\begin{aligned} &T_n(b(u_n)) T_k(u_n - \varphi) \longrightarrow b(u) T_k(u - \varphi) \\ &\qquad \text{a.e. } x \in \Omega, \\ &|T_n(b(u_n)) T_k(u_n - \varphi)| \leq k |b(u_n)|. \end{aligned} \tag{82}$$

Now we show that $|b(u_n)| \leq \|f\|_1/\operatorname{meas}(\Omega)$. Indeed, let us denote by

$$\begin{aligned} H_\delta(s) &= \min\left(\frac{s^+}{\delta}; 1\right), \\ \operatorname{sign}_0^+(s) &= \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0. \end{cases} \end{aligned} \tag{83}$$

If θ is a maximal monotone operator defined on $\mathbb{R}$, we denote by θ_0 the main section of θ; that is,

$$\begin{aligned} &\theta_0(s) \\ &= \begin{cases} \text{minimal absolute value of } \theta(s) & \text{if } \theta(s) \neq \emptyset, \\ +\infty & \text{if } [s, +\infty) \cap D(\theta) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap D(\theta) = \emptyset. \end{cases} \end{aligned} \tag{84}$$

Remark that as δ goes to 0, $H_\delta(s)$ goes to $\operatorname{sign}_0^+(s)$.

We take $\varphi = H_\delta(u_n - M)$ as a test function in (39) for the weak solution u_n and $M > 0$ (a constant to be chosen later) to get

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \frac{\partial}{\partial x_i} H_\delta(u_n - M)\, dx \\ &\quad + \varepsilon \int_\Omega |u_n|^{p_M(x)-2} u_n H_\delta(u_n - M)\, dx \\ &\quad + \int_\Omega T_n(b(u_n)) H_\delta(u_n - M)\, dx \\ &\quad + \int_{\partial\Omega} T_n(\gamma(u_n)) H_\delta(u_n - M)\, d\sigma \\ &= \int_\Omega f_n H_\delta(u_n - M)\, dx. \end{aligned} \tag{85}$$

We have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}H_\delta\left(u_n-M\right)dx\\
&\quad=\frac{1}{\delta}\sum_{i=1}^{N}\int_{\{(u_n-M)^+/\delta<1\}} a_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}\left(u_n-M\right)^+dx\\
&\quad=\frac{1}{\delta}\sum_{i=1}^{N}\int_{\{0<u_n-M<\delta\}} a_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}u_n dx\\
&\quad\geq 0 \text{ according to } (7),\\
&\int_{\Omega}\left|u_n\right|^{p_M(x)-2}u_nH_\delta\left(u_n-M\right)dx\\
&\quad=\int_{\{(u_n-M)^+/\delta<1\}}\left|u_n\right|^{p_M(x)-2}u_n\frac{\left(u_n-M\right)^+}{\delta}dx\\
&\qquad+\int_{\{(u_n-M)^+/\delta\geq1\}}\left|u_n\right|^{p_M(x)-2}u_n dx\\
&\quad\geq\frac{1}{\delta}\int_{\{M<u_n<M+\delta\}}\left|u_n\right|^{p_M(x)-2}u_n\left(u_n-M\right)dx\geq 0.
\end{aligned}\tag{86}$$

Note also that since the function γ is nondecreasing with $\gamma(0)=0$, we have

$$\begin{aligned}
&\int_{\partial\Omega}T_n\left(\gamma\left(u_n\right)\right)H_\delta\left(u_n-M\right)d\sigma\\
&\quad=\int_{\partial\Omega\cap\{(u_n-M)^+/\delta<1\}}T_n\left(\gamma\left(u_n\right)\right)H_\delta\left(u_n-M\right)d\sigma\\
&\qquad+\int_{\partial\Omega\cap\{(u_n-M)^+/\delta\geq1\}}T_n\left(\gamma\left(u_n\right)\right)H_\delta\left(u_n-M\right)d\sigma\\
&\quad=\int_{\partial\Omega\cap\{(u_n-M)^+/\delta<1\}}T_n\left(\gamma\left(u_n\right)\right)\frac{\left(u_n-M\right)^+}{\delta}d\sigma\\
&\qquad+\int_{\partial\Omega\cap\{(u_n-M)^+/\delta\geq1\}}T_n\left(\gamma\left(u_n\right)\right)d\sigma\\
&\quad=\frac{1}{\delta}\int_{\partial\Omega\cap\{0<u_n-M<\delta\}}T_n\left(\gamma\left(u_n\right)\right)\left(u_n-M\right)d\sigma\\
&\qquad+\int_{\partial\Omega\cap\{u_n-M\geq\delta\}}T_n\left(\gamma\left(u_n\right)\right)d\sigma\geq 0.
\end{aligned}\tag{87}$$

Then, (85) gives

$$\begin{aligned}
&\int_{\Omega}T_n\left(b\left(u_n\right)\right)H_\delta\left(u_n-M\right)dx\\
&\quad\leq\int_{\Omega}f_nH_\delta\left(u_n-M\right)dx,
\end{aligned}\tag{88}$$

which is equivalent to saying

$$\begin{aligned}
&\int_{\Omega}\left(T_n\left(b\left(u_n\right)\right)-T_n\left(b\left(M\right)\right)\right)H_\delta\left(u_n-M\right)dx\\
&\quad\leq\int_{\Omega}\left(f_n-T_n\left(b\left(M\right)\right)\right)H_\delta\left(u_n-M\right)dx.
\end{aligned}\tag{89}$$

We now let δ go to 0 in the above inequality to obtain

$$\begin{aligned}
&\int_{\Omega}\left(T_n\left(b\left(u_n\right)\right)-T_n\left(b\left(M\right)\right)\right)^+dx\\
&\quad\leq\int_{\Omega}\left(f_n-T_n\left(b\left(M\right)\right)\right)\operatorname{sign}_0^+\left(u_n-M\right)dx.
\end{aligned}\tag{90}$$

Choosing $M=b_0^{-1}(\|f_n\|_\infty)$ in the above inequality (since b is surjective), we obtain

$$\begin{aligned}
&\int_{\Omega}\left(T_n\left(b\left(u_n\right)\right)-T_n\left(\left\|f_n\right\|_\infty\right)\right)^+dx\\
&\quad\leq\int_{\Omega}\left(f_n-T_n\left(\left\|f_n\right\|_\infty\right)\right)\\
&\qquad\cdot\operatorname{sign}_0^+\left(u_n-b_0^{-1}\left(\left\|f_n\right\|_\infty\right)\right)dx.
\end{aligned}\tag{91}$$

For any $n>\|f\|_1/\text{meas}(\Omega)$, we have

$$\begin{aligned}
&\int_{\Omega}\left(f_n-T_n\left(\left\|f_n\right\|_\infty\right)\right)\operatorname{sign}_0^+\left(u_n-b_0^{-1}\left(\left\|f_n\right\|_\infty\right)\right)dx\\
&\quad=\int_{\Omega}\left(f_n-\left\|f_n\right\|_\infty\right)\operatorname{sign}_0^+\left(u_n-b_0^{-1}\left(\left\|f_n\right\|_\infty\right)\right)dx\\
&\quad\leq 0.
\end{aligned}\tag{92}$$

Then, (91) gives

$$\int_{\Omega}\left(T_n\left(b\left(u_n\right)\right)-\left\|f_n\right\|_\infty\right)^+dx\leq 0,\quad\forall n>\frac{\|f\|_1}{\text{meas}\left(\Omega\right)}.\tag{93}$$

Hence, for all $n>\|f\|_1/\text{meas}(\Omega)$, we have $(T_n(b(u_n))-\|f_n\|_\infty)^+=0$ a.e. in Ω, which implies that

$$T_n\left(b\left(u_n\right)\right)\leq\left\|f_n\right\|_\infty\quad\forall n>\frac{\|f\|_1}{\text{meas}\left(\Omega\right)}.\tag{94}$$

Let us remark that as u_n is a weak solution of (39), then $(-u_n)$ is a weak solution to the following problem:

$$(\widetilde{P}_n)\begin{cases}-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\widetilde{a}_i\left(x,\frac{\partial u_n}{\partial x_i}\right)+T_n\left(\widetilde{b}(u_n)\right)+\varepsilon|u_n|^{p_M(x)-2}u_n=\widetilde{f}_n & \text{in } \Omega,\\ \sum_{i=1}^{N}\widetilde{a}_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\eta_i=-\gamma(u_n) & \text{on } \partial\Omega,\end{cases} \tag{95}$$

where $\widetilde{a}_i(x,\xi)=-a_i(x,-\xi)$, $\widetilde{b}(s)=-b(-s)$, and $\widetilde{f}_n=-f_n$.

According to (94) we deduce that

$$T_n(-b(u_n))\le\|f_n\|_\infty\quad\forall n>\frac{\|f\|_1}{\text{meas}(\Omega)}. \tag{96}$$

Therefore

$$T_n(b(u_n))\ge-\|f_n\|_\infty\quad\forall n>\frac{\|f\|_1}{\text{meas}(\Omega)}. \tag{97}$$

It follows from (94) and (97) that, for all $n>\|f\|_1/\text{meas}(\Omega)$, $|T_n(b(u_n))|\le\|f_n\|_\infty$ which implies

$$|b(u_n)|\le\|f_n\|_\infty\le\frac{\|f\|_1}{\text{meas}(\Omega)}\quad\text{a.e. in }\Omega. \tag{98}$$

We can now use the Lebesgue dominated convergence theorem to get

$$\lim_{n\to+\infty}\int_\Omega T_n(b(u_n))T_k(u_n-\varphi)\,dx=\int_\Omega b(u)T_k(u-\varphi)\,dx. \tag{99}$$

By using again the Lebesgue dominated convergence theorem, we obtain

$$\int_{\partial\Omega}T_n(\gamma(u_n))T_k(u_n-\varphi)\,d\sigma\longrightarrow\int_{\partial\Omega}\gamma(u)T_k(u-\varphi)\,d\sigma. \tag{100}$$

For the third term of (79), let us prove that

$$\liminf_{n\to+\infty}\varepsilon\int_\Omega|u_n|^{p_M(x)-2}u_nT_k(u_n-\varphi)\,dx\ge0\quad\text{for }\varepsilon\longrightarrow0. \tag{101}$$

We have

$$\int_\Omega|u_n|^{p_M(x)-2}u_nT_k(u_n-\varphi)\,dx=\int_\Omega\left(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\right)T_k(u_n-\varphi)\,dx+\int_\Omega|\varphi|^{p_M(x)-2}\varphi T_k(u_n-\varphi)\,dx. \tag{102}$$

Since the quantity $(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi)T_k(u_n-\varphi)$ is nonnegative and since for all x in Ω, the application $\xi\mapsto|\xi|^{p_M(x)-2}\xi$ is continuous, we have

$$\left(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\right)T_k(u_n-\varphi)\longrightarrow\left(|u|^{p_M(x)-2}u-|\varphi|^{p_M(x)-2}\varphi\right)T_k(u-\varphi)\quad\text{a.e. in }\Omega \tag{103}$$

and by Fatou's lemma, it follows that

$$\liminf_{n\to+\infty}\int_\Omega\left(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\right)\cdot T_k(u_n-\varphi)\,dx\ge\int_\Omega\left(|u|^{p_M(x)-2}u-|\varphi|^{p_M(x)-2}\varphi\right)T_k(u-\varphi)\,dx. \tag{104}$$

We have

$$\begin{aligned}\int_\Omega\left||\varphi|^{p_M(x)-2}\varphi\right|dx&=\int_\Omega|\varphi|^{p_M(x)-1}\varphi dx\\&\le\int_\Omega(\|\varphi\|_\infty)^{p_M(x)-1}dx\\&\le\int_{\{\|\varphi\|_\infty\le1\}}(\|\varphi\|_\infty)^{p_M(x)-1}dx+\int_{\{\|\varphi\|_\infty>1\}}(\|\varphi\|_\infty)^{p_M(x)-1}dx\\&\le\text{meas}(\Omega)+(\|\varphi\|_\infty)^{p_M^+-1}\text{meas}(\Omega)\\&<+\infty.\end{aligned} \tag{105}$$

Hence, $|\varphi|^{p_M(x)-2}\varphi\in L^1(\Omega)$.

Since $T_k(u_n-\varphi)$ converges weakly-* to $T_k(u-\varphi)$ in $L^\infty(\Omega)$ and $|\varphi|^{p_M(x)-2}\varphi\in L^1(\Omega)$, it follows that

$$\lim_{n\to+\infty}\int_\Omega|\varphi|^{p_M(x)-2}\varphi T_k(u_n-\varphi)\,dx=\int_\Omega|\varphi|^{p_M(x)-2}\varphi T_k(u-\varphi)\,dx. \tag{106}$$

By adding (104) and (106), we get

$$\liminf_{n\to+\infty}\int_\Omega |u_n|^{p_M(x)-2}u_nT_k(u_n-\varphi)\,dx \geq \int_\Omega |u|^{p_M(x)-2}uT_k(u-\varphi)\,dx. \tag{107}$$

Since

$$\int_\Omega |u|^{p_M(x)-2}uT_k(u-\varphi)\,dx \leq k\int_\Omega |u|^{p_M(x)-1}\,dx < +\infty, \tag{108}$$

thus we get

$$\liminf_{n\to+\infty}\varepsilon\int_\Omega |u_n|^{p_M(x)-2}u_nT_k(u_n-\varphi)\,dx \geq 0 \quad \text{for } \varepsilon \longrightarrow 0. \tag{109}$$

Combining (80), (81), (99), (100), and (109) we obtain

$$\sum_{i=1}^N\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-\varphi)\,dx + \int_\Omega b(u)T_k(u-\varphi)\,dx + \int_{\partial\Omega}\gamma(u)T_k(u-\varphi)\,d\sigma \leq \int_\Omega fT_k(u-\varphi)\,dx. \tag{110}$$

Then, u is an entropy solution of problem (1). That completes the proof of Theorem 6.

We now state the uniqueness result of entropy solution.

Theorem 12. *Assume that (4)–(12) hold true and let u be an entropy solution of (1). Then, u is unique.*

Proof. The proof is done in two steps.

Step 1 (a priori estimates). We consider the following.

Lemma 13. *Assume (4)–(12) and $f \in L^1(\Omega)$. Let u be an entropy solution of (1). Then*

$$\sum_{i=1}^N\int_{\{|u|\leq k\}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx \leq \frac{k}{C_3}\|f\|_1 \tag{111}$$

and there exists a positive constant C_9 such that

$$\|b(u)\|_1 \leq C_9\cdot\text{meas}(\Omega) + \|f\|_1. \tag{112}$$

Proof. Let us take $\varphi = 0$ in the entropy inequality (36).

(i) By the fact that $\int_\Omega b(u)T_k(u)dx + \int_{\partial\Omega}\gamma(u)T_k(u)d\sigma \geq 0$, using (7), we get (111).

(ii) Also, using the fact that $\sum_{i=1}^N\int_\Omega a_i(x,\partial u/\partial x_i)(\partial/\partial x_i)T_k(u)dx+\int_{\partial\Omega}\gamma(u)T_k(u)d\sigma \geq 0$, relation (36) gives

$$\int_\Omega b(u)T_k(u)\,dx \leq \int_\Omega f(x)T_k(u)\,dx. \tag{113}$$

By (113), we deduce that

$$\int_{\{|u|\leq k\}} b(u)T_k(u)\,dx + \int_{\{|u|>k\}} b(u)T_k(u)\,dx \leq k\|f\|_1 \tag{114}$$

which imply that

$$\int_{\{|u|>k\}} b(u)T_k(u)\,dx \leq k\|f\|_1 \tag{115}$$

or

$$\int_{\{u>k\}} b(u)\,dx + \int_{\{u<-k\}} -b(u)\,dx \leq \|f\|_1. \tag{116}$$

Therefore,

$$\int_{\{|u|>k\}} |b(u)|\,dx \leq \|f\|_1. \tag{117}$$

So, we obtain

$$\int_\Omega |b(u)|\,dx = \int_{\{|u|\leq k\}} |b(u)|\,dx + \int_{\{|u|>k\}} |b(u)|\,dx \leq \int_{\{|u|\leq k\}} |b(u)|\,dx + \|f\|_1. \tag{118}$$

Since the function b is nondecreasing, then

$$\int_{\{|u|\leq k\}} |b(u)|\,dx \leq \max\{b(k);|b(-k)|\}\cdot\text{meas}(\Omega). \tag{119}$$

Consequently, there exists a constant $C_9 = \max\{b(k);|b(-k)|\}$ such that

$$\|b(u)\|_1 \leq C_9\cdot\text{meas}(\Omega) + \|f\|_1. \tag{120}$$

□

Lemma 14. *Assume (4)–(12) and let $f \in L^1(\Omega)$. If u is an entropy solution of (1), then there exists a constant D which depends on f and Ω such that*

$$\text{meas}(\{|u|>k\}) \leq \frac{D}{\min\{b(k),|b(-k)|\}}, \quad \forall k>0 \tag{121}$$

and a constant $D' > 0$ which depends on f and Ω such that

$$\text{meas}\left(\left\{\left|\frac{\partial u}{\partial x_i}\right|>k\right\}\right) \leq \frac{D'}{k^{1/(p_M^-)'}}, \quad \forall k\geq 1. \tag{122}$$

Proof. (i) For any $k>0$, relation (112) gives

$$\int_{\{|u|>k\}} \min\{b(k),|b(-k)|\}\,dx \leq \int_{\{|u|>k\}} |b(u)|\,dx \leq C_9\cdot\text{meas}(\Omega) + \|f\|_1. \tag{123}$$

Therefore,

$$\min\{b(k),|b(-k)|\}\cdot\text{meas}(\{|u|>k\}) \leq C_9\cdot\text{meas}(\Omega) + \|f\|_1 = D; \tag{124}$$

that is,

$$\text{meas}\,(\{|u| > k\}) \leq \frac{D}{\min\{b(k), |b(-k)|\}}. \tag{125}$$

(ii) See [7] for the poof of (122). □

Lemma 15. *Assume (4)–(12) and let $f \in L^1(\Omega)$. If u is an entropy solution of (1), then*

$$\lim_{h\to+\infty}\int_\Omega |f|\,\chi_{\{|u|>h-t\}}dx = 0, \tag{126}$$

where $h > 0$ and $t > 0$.

Proof. Since the function b is surjective, according to Lemma 14-(121), we have

$$\lim_{h\to+\infty}\text{meas}\,(\{|u| > h-t\}) = 0 \tag{127}$$

and as $f \in L^1(\Omega)$, it follows by using the Lebesgue dominated convergence theorem that

$$\lim_{h\to+\infty}\int_\Omega |f|\,\chi_{\{|u|>h-t\}}dx = 0. \tag{128}$$

□

Lemma 16. *Assume (4)–(12) and let $f \in L^1(\Omega)$. If u is an entropy solution of (1), then there exists a positive constant K such that*

$$\rho_{p_i'(\cdot)}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\chi_F\right) \leq K, \quad \forall i = 1,\dots,N, \tag{129}$$

where $F = \{h < |u| \leq h+k\}$, $h > 0$, $k > 0$.

Proof. Let $\varphi = T_h(u)$ as test function in the entropy inequality (36). We get

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k\left(u-T_h(u)\right)dx \\
&\quad+\int_\Omega b(u)\,T_k\left(u-T_h(u)\right)dx \\
&\quad+\int_{\partial\Omega}\gamma(u)\,T_k\left(u-T_h(u)\right)d\sigma \\
&\leq\int_\Omega fT_k\left(u-T_h(u)\right)dx.
\end{aligned} \tag{130}$$

Thus,

$$\sum_{i=1}^N\int_{\{h<|u|\leq h+k\}} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}dx \leq k\,\|f\|_1 \tag{131}$$

and using (7), we have

$$\int_F\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx \leq \frac{k}{C_3}\|f\|_1, \quad \forall i = 1,\dots,N. \tag{132}$$

Consequently,

$$\rho_{p_i'(\cdot)}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\chi_F\right) \leq K, \quad \forall i = 1,\dots,N. \tag{133}$$

□

Step 2 (uniqueness of entropy solution). Let $h > 0$ and u, v be two entropy solutions of (1). We write the entropy inequality corresponding to the solution u, with $T_h(v)$ as test function, and to the solution v, with $T_h(u)$ as test function to get

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k\left(u-T_h(v)\right)dx \\
&\quad+\int_\Omega b(u)\,T_k\left(u-T_h(v)\right)dx \\
&\quad+\int_{\partial\Omega}\gamma(u)\,T_k\left(u-T_h(v)\right)d\sigma \\
&\leq\int_\Omega fT_k\left(u-T_h(v)\right)dx, \\
&\sum_{i=1}^N\int_\Omega a_i\left(x,\frac{\partial v}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k\left(v-T_h(u)\right)dx \\
&\quad+\int_\Omega b(v)\,T_k\left(v-T_h(u)\right)dx \\
&\quad+\int_{\partial\Omega}\gamma(v)\,T_k\left(v-T_h(u)\right)d\sigma \\
&\leq\int_\Omega fT_k\left(v-T_h(u)\right)dx.
\end{aligned} \tag{134}$$

Upon addition, we get

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k\left(u-T_h(v)\right)dx \\
&\quad+\sum_{i=1}^N\int_\Omega a_i\left(x,\frac{\partial v}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k\left(v-T_h(u)\right)dx \\
&\quad+\int_\Omega b(u)\,T_k\left(u-T_h(v)\right)dx \\
&\quad+\int_\Omega b(v)\,T_k\left(v-T_h(u)\right)dx \\
&\quad+\int_{\partial\Omega}\gamma(u)\,T_k\left(u-T_h(v)\right)d\sigma \\
&\quad+\int_{\partial\Omega}\gamma(v)\,T_k\left(v-T_h(u)\right)d\sigma \\
&\leq\int_\Omega f\left[T_k\left(u-T_h(v)\right)+T_k\left(v-T_h(u)\right)\right]dx.
\end{aligned} \tag{135}$$

Define the following sets:

$$\begin{aligned} &E_1 \{|u-v| \leq k; |v| \leq h\}; \\ &E_2 E_1 \cap \{|u| \leq h\}, \\ &E_3 E_1 \cap \{|u| > h\}. \end{aligned} \tag{136}$$

We start with the first integral in (135). We have

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{|u-T_h(v)|\leq k\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k\left(u - T_h(v)\right) dx \\ &\quad = \sum_{i=1}^{N} \int_{\{|u-T_h(v)|\leq k\}\cap\{|v|\leq h\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k\left(u - T_h(v)\right) dx \\ &\quad + \sum_{i=1}^{N} \int_{\{|u-T_h(v)|\leq k\}\cap\{|v|> h\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k\left(u - T_h(v)\right) dx \\ &\quad = \sum_{i=1}^{N} \int_{\{|u-v|\leq k\}\cap\{|v|\leq h\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial (u-v)}{\partial x_i} dx \\ &\quad + \sum_{i=1}^{N} \int_{\{|u-h\,\mathrm{sign}(v)|\leq k\}\cap\{|v|> h\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial u}{\partial x_i} dx \\ &\quad \geq \sum_{i=1}^{N} \int_{E_1} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}(u-v)\, dx \geq \sum_{i=1}^{N} \int_{E_2} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}(u-v)\, dx \\ &\quad + \sum_{i=1}^{N} \int_{E_3} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}(u-v)\, dx. \end{aligned} \tag{137}$$

Then, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{|u-T_h(v)|\leq k\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k\left(u - T_h(v)\right) dx \\ &\quad \geq \sum_{i=1}^{N} \int_{E_2} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}(u-v)\, dx \\ &\quad - \sum_{i=1}^{N} \int_{E_3} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial v}{\partial x_i} dx. \end{aligned} \tag{138}$$

According to (5) and the Hölder type inequality, we have

$$\begin{aligned} &\left|\sum_{i=1}^{N} \int_{E_3} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial v}{\partial x_i} dx\right| \\ &\quad \leq C_1 \sum_{i=1}^{N} \int_{E_3} \left(j_i(x) + \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\right) \left|\frac{\partial v}{\partial x_i}\right| dx \\ &\quad \leq C_1 \sum_{i=1}^{N} \left(\left|j_i\right|_{p_i'(\cdot)} + \left\|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\right|_{p_i'(\cdot),\{h<|u|\leq h+k\}}\right) \\ &\quad \cdot \left|\frac{\partial v}{\partial x_i}\right|_{p_i(\cdot),\{h-k<|v|\leq h\}}, \end{aligned} \tag{139}$$

where

$$\begin{aligned} &\left\|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\right|_{p_i'(\cdot),\{h<|u|\leq h+k\}} \\ &\quad = \left\|\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\right\|_{L^{p_i'(\cdot)}(\{h<|u|\leq h+k\})}. \end{aligned} \tag{140}$$

Thanks to relation (18) and Lemma 16, the quantity $(|j_i|_{p_i'(\cdot)} + \||\partial u/\partial x_i|^{p_i(x)-1}|_{p_i'(\cdot),\{h<|u|\leq h+k\}})$ is finite for all $i = 1, \ldots, N$.

According to Lemma 15, the quantity $|\partial v/\partial x_i|_{p_i(\cdot),\{h-k<|v|\leq h\}}$ converges to zero as h goes to infinity. Consequently, the last integral of (138) converges to zero as h goes to infinity. Then,

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{|u-T_h(v)|\leq k\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k\left(u - T_h(v)\right) dx \\ &\quad \geq I_h + \sum_{i=1}^{N} \int_{E_2} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}(u-v)\, dx, \end{aligned} \tag{141}$$

with $\lim_{h\to+\infty} I_h = 0$.

We may adopt the same procedure to treat the second term in (135) to obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{|v-T_h(u)|\leq k\}} a_i\left(x, \frac{\partial v}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k\left(v - T_h(u)\right) dx \\ &\quad \geq J_h - \sum_{i=1}^{N} \int_{E_2} a_i\left(x, \frac{\partial v}{\partial x_i}\right) \frac{\partial}{\partial x_i}(u-v)\, dx, \end{aligned} \tag{142}$$

with $\lim_{h\to+\infty} J_h = 0$.

For the other terms in the left-hand side of (135), we denote

$$\begin{aligned} K_h &= \int_{\Omega} b(u)\, T_k\left(u - T_h(v)\right) dx \\ &\quad + \int_{\Omega} b(v)\, T_k\left(v - T_h(u)\right) dx, \\ L_h &= \int_{\partial\Omega} \gamma(u)\, T_k\left(u - T_h(v)\right) d\sigma \\ &\quad + \int_{\partial\Omega} \gamma(v)\, T_k\left(v - T_h(u)\right) d\sigma. \end{aligned} \tag{143}$$

We have

$$b(u)T_k(u-T_h(v)) \longrightarrow b(u)T_k(u-v) \quad \text{a.e. in } \Omega \text{ since } h \longrightarrow +\infty, \tag{144}$$
$$|b(u)T_k(u-T_h(v))| \le k|b(u)| \in L^1(\Omega).$$

Then, by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{h\to+\infty}\int_\Omega b(u)T_k(u-T_h(v))\,dx &= \int_\Omega b(u)T_k(u-v)\,dx,\\ \lim_{h\to+\infty}\int_\Omega b(v)T_k(v-T_h(u))\,dx &= \int_\Omega b(v)T_k(v-u)\,dx.\end{aligned} \tag{145}$$

Then,

$$\lim_{h\to+\infty} K_h = \int_\Omega (b(u)-b(v))T_k(u-v)\,dx. \tag{146}$$

In the same way, we get

$$\lim_{h\to+\infty} L_h = \int_{\partial\Omega} (\gamma(u)-\gamma(v))T_k(u-v)\,d\sigma. \tag{147}$$

Now, consider the right-hand side of inequality (135). We have

$$\begin{aligned}&\lim_{h\to+\infty} f(T_k(u-T_h(v))+T_k(v-T_h(u))) = 0 \quad \text{a.e. in } \Omega,\\ &|f(x)(T_k(u-T_h(v))+T_k(v-T_h(u)))| \le 2k|f| \in L^1(\Omega).\end{aligned} \tag{148}$$

By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{h\to+\infty}\int_\Omega f(x)(T_k(u-T_h(v))+T_k(v-T_h(u)))\,dx = 0. \tag{149}$$

After passing to the limit as h goes to $+\infty$ in (135), we get

$$\begin{aligned}&\sum_{i=1}^N \int_{\{|u-v|\le k\}}\left(a_i\left(x,\frac{\partial u}{\partial x_i}\right) - a_i\left(x,\frac{\partial v}{\partial x_i}\right)\right)\frac{\partial}{\partial x_i}(u-v)\,dx\\ &+\int_\Omega (b(u)-b(v))T_k(u-v)\,dx + \int_{\partial\Omega}(\gamma(u) - \gamma(v))T_k(u-v)\,d\sigma \le 0.\end{aligned} \tag{150}$$

Since b, γ, and $a_i(x,\cdot)$ are monotone, then

$$\int_\Omega (b(u)-b(v))T_k(u-v)\,dx = 0, \tag{151}$$

$$\int_{\partial\Omega} (\gamma(u)-\gamma(v))T_k(u-v)\,d\sigma = 0, \tag{152}$$

$$\int_{\{|u-v|\le k\}}\sum_{i=1}^N\left(a_i\left(x,\frac{\partial u}{\partial x_i}\right) - a_i\left(x,\frac{\partial v}{\partial x_i}\right)\right)\frac{\partial}{\partial x_i}(u-v)\,dx = 0. \tag{153}$$

According to (6), we deduce from (153) that

$$u-v=c \quad \text{a.e. } x\in\Omega, \text{ where } c \text{ is a real constant.} \tag{154}$$

For x fixed in $\partial\Omega$, $s \mapsto |s|^{p(x)-2}s$ is nondecreasing and vanishes at 0. Then,

$$(\gamma(u(x))-\gamma(v(x)))T_k(u(x)-v(x)) \ge 0, \quad \forall x\in\partial\Omega,\ \forall k>0. \tag{155}$$

Now, by inequality above and (152), we deduce that for all $k\in\mathbb{N}^*$ there exists $C_k\subset\partial\Omega$ with $\text{meas}(C_k)=0$ such that, for all $x\in\partial\Omega\setminus C_k$,

$$(\gamma(u(x))-\gamma(v(x)))T_k(u(x)-v(x)) = 0. \tag{156}$$

Therefore,

$$(\gamma(u(x))-\gamma(v(x)))(u(x)-v(x)) = 0, \quad \forall x\in\partial\Omega\setminus\bigcup_{k\in\mathbb{N}^*}C_k. \tag{157}$$

As $p^- > 1$, the following relation is true for any $\xi,\eta\in\mathbb{R}$, $\xi\ne\eta$ (cf. [14]):

$$\left(|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta\right)(\xi-\eta) > 0. \tag{158}$$

From inequality above and (157), we get

$$u-v=0 \quad \text{a.e. on } \partial\Omega. \tag{159}$$

Finally as

$$\begin{aligned}u-v&=c \quad \text{a.e. in } \Omega,\\ u-v&=0 \quad \text{a.e. on } \partial\Omega,\end{aligned} \tag{160}$$

it follows that

$$u=v \quad \text{a.e. in } \Omega. \tag{161}$$

That completes the proof of Theorem 12.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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