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Research Article

On the Limit Cycles of a Class of Generalized Kukles Polynomial Differential Systems via Averaging Theory

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We apply the averaging theory of first and second order to a class of generalized Kukles polynomial differential systems to study the maximum number of limit cycles of these systems.

1. Introduction

One of the main topics in the theory of ordinary differential equations is the study of limit cycles: their existence, their number, and their stability. A limit cycle of a differential equation is an isolated periodic orbit in the set of all periodic orbits of the differential equation. The second part of the 16th Hilbert's problem [1] is related to the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. This problem and the Riemann conjecture are the only two problems on the list of Hilbert which have not been solved. Here we consider a very particular case of the sixteenth Hilbert problem. We study the upper bound of the generalized Kukles polynomial system

$$\dot{x} = -y, \qquad \dot{y} = Q(x, y), \tag{1}$$

where $Q(x, y)$ is a polynomial with real coefficients of degree n.

Kukles [2], in 1944, introduced the differential system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x^3 + a_5 x^2 y + a_6 xy^2 + a_7 y^3, \end{aligned} \tag{2}$$

and he gives the necessary and sufficient conditions in order that this system has a center at the origin. This cubic system without the term y^3 was also studied in [3] and the authors called it reduced. In [4] a description of the bifurcation of its critical period appears, and [5] presents the existence of reduced Kukles systems with five limit cycles. In the paper [6], the author studied the class of reduced Kukles systems under the cubic perturbation

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x - \frac{a+1}{a} x^2 + \frac{2-a}{1-a} y^2 + \frac{1}{a} x^3 + \varepsilon g(x, y), \\ g(x, y) &= b_{01} y + b_{11} xy + b_{02} y^2 + b_{12} xy^2 + b_{21} x^2 y + b_{03} y^3, \end{aligned} \tag{3}$$

where $\varepsilon > 0$ is small and $a > 2$ is a constant.

In [7] the author proves that some cubic systems of form (1) can have seven limit cycles. In [8], Chavarriga et al. studied the maximum number of small amplitude limit cycles for Kukles systems which can coexist with some invariant algebraic curves. Also they give a family of cubic Kukles systems

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + \lambda y + abx^2 - a\left(a^2 + 3c - b\lambda\right) xy - 2aby^2 \\ &\quad - b^2 c x^3 - b\left(b^2 - a^2 c - 2c^2 + bc\lambda\right) x^2 y \\ &\quad - b^2\left(-a^2 - 3c + b\lambda\right) xy^2 + b^3 y^3, \end{aligned} \tag{4}$$

with an invariant hyperbola $h(x, y) = 1+abx-b^2cx^2-b^3xy = 0$ with $b \neq 0$, which coexist with one or two small amplitude limit cycles. In [9] the author studied the maximum number of limit cycles of the generalized polynomial Liénard differential equations by using the first and second averaging method. In [10], Llibre and Mereu studied the maximum number of limit cycles of the Kukles polynomial differential systems

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - \sum_{k\geq 1}\varepsilon^k\left(f_n^k(x) + g_m^k(x)\,y + h_l^k(x)\,y^2 + d_0^k y^3\right),\end{aligned} \tag{5}$$

where for every k the polynomials $f_n^k(x)$, $g_m^k(x)$, and $h_l^k(x)$ have degrees n, m, and l, respectively, $d_0^k \neq 0$ is a real number, and ε is a small parameter.

In this work we study the maximum number of limit cycles given by averaging theory of first and second order, which can bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ perturbed inside the following class of generalized Kukles polynomial differential systems:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - \sum_{k\geq 1}\varepsilon^k\left(f_n^k(x) + g_m^k(x)\,y + h_l^k(x)\,y^2 + g_m^k(x)\,y^3\right),\end{aligned} \tag{6}$$

where for every k the polynomials $f_n^k(x)$, $g_m^k(x)$, and $h_l^k(x)$ have degrees n, m, and l, respectively, and ε is a small parameter. We have considered the same polynomial $g_m^k(x)$ as the coefficient of y and y^3. With this choice, we can apply the first and second order of the averaging method. If we consider the coefficients of y^i for $i = 0, 1, 2, 3$ as arbitrary polynomials, it is difficult to apply the second order averaging method, because to pass from the first order to the second order averaging method, we must put the averaged function of the first order $F_{10}(r)$ (see (8)) identically null. In this case the calculations of the averaged function of the second order $F_{20}(r)$ (see (9)) become difficult. If we replace d_0^k by $h_l^k(x)$ in the coefficient of y^3 of the differential systems (5), we can apply the first order averaging method but it is not easy to apply the second averaging method. To apply the second averaging method, we must put $F_{10}(r) = 0$ which is equivalent to cancel all coefficients of the polynomial $F_{10}(r)$. The conditions on the coefficient of $F_{10}(r)$ make the calculations of $F_{20}(r)$ difficult. We have used the averaging method for looking for the limit cycles of many classes of Liénard systems. Here we do the same for Kukles differential systems. Comparatively, with the results of the paper [10], we obtained more limit cycles than the results of this paper. More precisely our main result is the following.

Theorem 1. *Assume that for $k = 1, 2$ the polynomials $f_n^k(x)$, $g_m^k(x)$, and $h_l^k(x)$ have degrees n, m, and l, respectively, with $n, m, l \geq 1$. Then for $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the Kukles polynomial differential systems (6) bifurcating from the periodic orbits of the linear centre $\dot{x} = y$, $\dot{y} = -x$, using averaging theory*

(a) *of first order is*

(i) *no limit cycle for $m = 1$,*

(ii) *$[(m + 2)/2]$ limit cycles for $m \geq 2$,*

(b) *of second order is* $\max\{[n/2]+[(m+1)/2], [l/2]+[(m+3)/2], [(m + 2)/2]\}$,

where $[\cdot]$ denotes the integer part function.

2. First and Second Order Averaging Method

In proof of our main result we use the averaging theory as it is presented in [11]. Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(x, t) + \varepsilon^2 F_2(x, t) + \varepsilon^3 R(x, t, \varepsilon), \tag{7}$$

where $F_1, F_2 : D\times\mathbb{R} \to \mathbb{R}^n$, $R : D\times\mathbb{R}\times(-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold.

(i) $F_1(\cdot, t) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R, and D_xF_1 are locally Lipschitz with respect to x, and R is differentiable with respect to ε. We define

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(z, s)\,ds, \tag{8}$$

$$F_{20}(z) = \frac{1}{T}\int_0^T \left[D_zF_1(z, s)\cdot\int_0^s F_1(z, t)\,dt + F_2(z, s)\right]ds. \tag{9}$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + F_{20}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$.

Then, for $|\varepsilon| > 0$ sufficiently small there exists *a* T-periodic solution $\varphi(\cdot, \varepsilon)$ of system (7) such that $\varphi(0, \varepsilon) = a_\varepsilon$.

The expression $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} : V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10} + \varepsilon F_{20}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ become mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the averaging theory of first order.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the averaging theory of second order. For more information about the averaging theory see [12, 13].

3. Proof of Theorem 1

3.1. Proof of Statement (a) of Theorem 1. In order to apply the first order averaging method we write system (6) with $k = 1$, in polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$,

$r > 0$. If we take $f_n^1(x) = \sum_{i=0}^n a_i x^i$, $g_m^1(x) = \sum_{i=0}^m b_i x^i$, and $h_l^1(x) = \sum_{i=0}^l c_i x^i$, system (6) can be written as follows:

$$\begin{aligned} \dot{r} &= -\varepsilon \sin\theta P(r,\theta), \\ \dot{\theta} &= -1 - \frac{\varepsilon}{r}\cos\theta P(r,\theta), \end{aligned} \tag{10}$$

where

$$\begin{aligned} P(r,\theta) = {} & \sum_{i=0}^{n} a_i r^i \cos^i\theta + \sum_{i=0}^{m} b_i r^{i+1}\cos^i\theta\sin\theta \\ & + \sum_{i=0}^{m} b_i r^{i+3}\cos^i\theta\sin^3\theta + \sum_{i=0}^{l} c_i r^{i+2}\cos^i\theta\sin^2\theta. \end{aligned} \tag{11}$$

If we take θ as a new independent variable, system (10) becomes

$$\frac{dr}{d\theta} = \varepsilon\sin\theta P(r,\theta) + o(\varepsilon^2) = \varepsilon F_1(r,\theta) + o(\varepsilon^2). \tag{12}$$

By using the notation introduced in Section 2 we have that

$$F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi}\sin\theta P(r,\theta)\,d\theta. \tag{13}$$

We know that

$$\int_0^{2\pi}\cos^i\theta\sin^j\theta\,d\theta = \begin{cases} 0 & \text{if } i \text{ odd or } j \text{ is odd} \\ I_{i,j} \neq 0 & \text{if } i \text{ is even and } j \text{ even.} \end{cases} \tag{14}$$

Let k be a positive integer. We define $\mathrm{ev}(k)$ as the largest even integer less than or equal to k, and $\mathrm{od}(k)$ as the largest odd integer less than or equal to k.

Hence

$$F_{10}(r) = \frac{1}{2\pi}\sum_{\substack{i=0\\ i \text{ even}}}^{\mathrm{ev}(m)} b_i I_{i,2} r^{i+1} + \frac{1}{2\pi}\sum_{\substack{i=0\\ i \text{ even}}}^{\mathrm{ev}(m)} b_i I_{i,4} r^{i+3}. \tag{15}$$

We obtain

$$\begin{aligned} F_{10}(r) = \frac{r}{2\pi}\Big[& b_0 I_{0,2} + (b_2 I_{2,2} + b_0 I_{0,4}) r^2 + (b_4 I_{4,2} + b_2 I_{2,4}) r^4 \\ & + \cdots + (b_{\mathrm{ev}(m)} I_{\mathrm{ev}(m),2} + b_{\mathrm{ev}(m)-2} I_{\mathrm{ev}(m)-2,4}) r^{\mathrm{ev}(m)} \\ & + b_{\mathrm{ev}(m)} r^{\mathrm{ev}(m)+2}\Big]. \end{aligned} \tag{16}$$

For the case $m = 1$, we obtain that

$$\begin{aligned} F_{10}(r) &= \frac{b_0 r}{2\pi}\left(I_{0,2} + I_{0,4} r^2\right) \\ &= \frac{b_0 r}{2}\left(1 + \frac{3}{4}r^2\right). \end{aligned} \tag{17}$$

There is no positive root for $F_{10}(r)$.

For $m \geq 2$, the polynomial $F_{10}(r)$ has at most $[(m+2)/2]$ positive roots. Hence (a) of Theorem 1 is proved.

3.2. Proof of Statement (b) of Theorem 1. For proving statement (b) of Theorem 1 we will use the second-order averaging theory. If we write

$$\begin{aligned} f_n^1(x) &= \sum_{i=0}^{n} a_i x^i, & f_n^2(x) &= \sum_{i=0}^{n} \tilde{a}_i x^i, \\ g_m^1(x) &= \sum_{i=0}^{m} b_i x^i, & g_m^2(x) &= \sum_{i=0}^{m} \tilde{b}_i x^i, \\ h_l^1(x) &= \sum_{i=0}^{l} c_i x^i, & h_l^2(x) &= \sum_{i=0}^{l} \tilde{c}_i x^i \end{aligned} \tag{18}$$

then system (6) with $k = 2$ in polar coordinates (r,θ), $r > 0$, becomes

$$\begin{aligned} \dot{r} &= -\varepsilon\sin\theta P(r,\theta) - \varepsilon^2\sin\theta K(r,\theta), \\ \dot{\theta} &= -1 - \frac{\varepsilon}{r}\cos\theta P(r,\theta) - \frac{\varepsilon^2}{r}\cos\theta K(r,\theta), \end{aligned} \tag{19}$$

where

$$\begin{aligned} K(r,\theta) = {} & \sum_{i=0}^{n} \tilde{a}_i r^i\cos^i\theta + \sum_{i=0}^{m} \tilde{b}_i r^{i+1}\cos^i\theta\sin\theta \\ & + \sum_{i=0}^{m} \tilde{b}_i r^{i+3}\cos^i\theta\sin^3\theta + \sum_{i=0}^{l} \tilde{c}_i r^{i+2}\cos^i\theta\sin^2\theta. \end{aligned} \tag{20}$$

Taking θ as the new independent variable system, (19) can be written as

$$\frac{dr}{d\theta} = \varepsilon F_1(r,\theta) + \varepsilon^2 F_2(r,\theta) + o(\varepsilon^3), \tag{21}$$

where $F_1(r,\theta) = \sin\theta P(r,\theta)$ and $F_2(r,\theta) = \sin\theta K(r,\theta) - (\cos\theta\sin\theta/r)P^2(r,\theta)$.

In order to apply the averaging theory of second order, F_{10} must be identically zero. Therefore from (16), F_{10} is identically zero if and only if $b_i = 0$ for i even.

Now we determine the corresponding function

$$F_{20} = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{d}{dr}F_1(r,\theta)\cdot\int_0^{\theta}F_1(r,\phi)\,d\phi + F_2(r,\theta)\right]d\theta. \tag{22}$$

For this we compute

$$\begin{aligned} & \frac{d}{dr}F_1(r,\theta) \\ & \quad = \sum_{i=1}^{n} i a_i r^{i-1}\cos^i\theta\sin\theta \\ & \qquad + \sum_{\substack{i=1\\ i \text{ odd}}}^{\mathrm{od}(m)} (i+1) b_i r^i\cos^i\theta\sin^2\theta \end{aligned}$$

$$+\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+3)\,b_i r^{i+2}\cos^i\theta\sin^4\theta$$

$$+\sum_{i=0}^{l}(i+2)\,c_i r^{i+1}\cos^i\theta\sin^3\theta,$$

$$\begin{aligned}\int_0^\theta F_1(r,\phi)\,d\phi &= \sum_{j=0}^{n} a_j r^j \int_0^\theta \cos^j\phi\sin\phi\,d\phi \\ &+ \sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)} b_j r^{j+1}\int_0^\theta \cos^j\phi\sin^2\phi\,d\phi \\ &+ \sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)} b_j r^{j+3}\int_0^\theta \cos^j\phi\sin^4\phi\,d\phi \\ &+ \sum_{j=0}^{l} c_j r^{j+2}\int_0^\theta \cos^j\phi\sin^3\phi\,d\phi \\ &= \sum_{j=0}^{n} a_j r^j \frac{1}{j+1}\left(1-\cos^{j+1}\theta\right) \\ &+ \sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)} b_j r^{j+1}\left(\sum_{k=1}^{(j+3)/2}\alpha_{k,j}\sin((2k-1)\theta)\right) \\ &+ \sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)} b_j r^{j+3}\left(\sum_{k=1}^{(j+5)/2}\beta_{k,j}\sin((2k-1)\theta)\right) \\ &+ \sum_{\substack{j=0\\ j\text{ even}}}^{\text{ev}(l)} c_j r^{j+2}\left(\gamma_{1,j}+\sum_{k=2}^{(j+6)/2}\gamma_{k,j}\cos((2k-3)\theta)\right) \\ &+ \sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(l)} c_j r^{j+2}\left(\delta_{1,j}+\sum_{k=2}^{(j+5)/2}\delta_{k,j}\cos((2k-2)\theta)\right),\end{aligned} \tag{23}$$

where $\alpha_{k,j}$, $\beta_{k,j}$, $\gamma_{k,j}$, and $\delta_{k,j}$ are constants.

The integral $\int_0^{2\pi}(d/dr)F_1(r,\theta)(\int_0^\theta F_1(r,\phi)d\phi)d\theta$ will be given in several lemmas.

Lemma 2. *The integral*

$$\int_0^{2\pi}\left(\sum_{i=1}^{n} i a_i r^{i-1}\cos^i\theta\sin\theta\right)\left(\int_0^\theta F_1(r,\phi)\,d\phi\right)d\theta \tag{24}$$

is, in the variable r, the polynomial

$$\sum_{\substack{i=2\\ i\,even}}^{ev(n)}\sum_{\substack{j=1\\ j\,odd}}^{od(m)} i a_i b_j r^{i+j}A_{ij} + \sum_{\substack{i=2\\ i\,even}}^{ev(n)}\sum_{\substack{j=1\\ j\,odd}}^{od(m)} i a_i b_j r^{i+j+2}B_{ij}, \tag{25}$$

where A_{ij} and B_{ij} are real constants.

Proof. We have that

(a_1)

$$\int_0^{2\pi}\left(\sum_{i=1}^{n} i a_i r^{i-1}\cos^i\theta\sin\theta\right)\cdot\left(\sum_{j=0}^{n} a_j r^j\frac{1}{j+1}\left(1-\cos^{j+1}\theta\right)\right)d\theta = 0, \tag{26}$$

(b_1)

$$\begin{aligned}&\int_0^{2\pi}\left(\sum_{i=1}^{n} i a_i r^{i-1}\cos^i\theta\sin\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)}\sum_{k=1}^{(j+3)/2} b_j r^{j+1}\alpha_{k,j}\sin((2k-1)\theta)\right)d\theta \\ &= \sum_{\substack{i=2\\ i\text{ even}}}^{\text{ev}(n)}\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)} i a_i b_j r^{i+j}A_{ij},\end{aligned} \tag{27}$$

where $A_{ij} = \int_0^{2\pi}\cos^i\theta\sin\theta(\sum_{k=1}^{(j+3)/2}\alpha_{k,j}\sin((2k-1)\theta))d\theta \neq 0$ for $i \geqslant 2$ even and $j \geqslant 1$ odd,

(c_1)

$$\begin{aligned}&\int_0^{2\pi}\left(\sum_{i=1}^{n} i a_i r^{i-1}\cos^i\theta\sin\theta\right)\cdot\left(\sum_{\substack{j=0\\ j\text{ even}}}^{\text{ev}(l)} c_j r^{j+2}\left(\gamma_{1,j}+\sum_{k=2}^{(j+6)/2}\gamma_{k,j}\cos((2k-3)\theta)\right)\right)d\theta \\ &= 0,\end{aligned} \tag{28}$$

(d_1)

$$\begin{aligned}&\int_0^{2\pi}\left(\sum_{i=1}^{n} i a_i r^{i-1}\cos^i\theta\sin\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(l)} c_j r^{j+2}\left(\delta_{1,j}+\sum_{k=2}^{(j+5)/2}\delta_{k,j}\cos((2k-2)\theta)\right)\right)d\theta \\ &= 0,\end{aligned} \tag{29}$$

(e$_1$)

$$\int_0^{2\pi}\left(\sum_{i=1}^{n} i a_i r^{i-1}\cos^i\theta\sin\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)}\sum_{k=1}^{(j+5)/2} b_j r_k^{j+3}\beta_{k,j}\sin((2k-1)\theta)\right)d\theta = \sum_{\substack{i=2\\ i\text{ even}}}^{\text{ev}(n)}\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)} i a_i b_j r^{i+j+2} B_{ij}, \tag{30}$$

where $B_{ij} = \int_0^{2\pi}\cos^i\theta\sin\theta(\sum_{k=1}^{(j+5)/2}\beta_{k,j}\sin((2k-1)\theta)d\theta) \neq 0$ for $i \geqslant 2$ even and $j \geqslant 1$ odd.

We have that the sum of the integrals (a$_1$)–(e$_1$) is polynomial (3.6). This ends the proof of the lemma. □

Lemma 3. *The integral*

$$\int_0^{2\pi}\left(\sum_{i=0}^{m}(i+1)b_i r^i\cos^i\theta\sin^2\theta\right)\left(\int_0^{\theta}F_1(r,\phi)\,d\phi\right)d\theta \tag{31}$$

is, in the variable r, the polynomial

$$-\sum_{\substack{1\le i\le od(m)\\ 0\le j\le ev(n)\\ i\text{ odd and } j\text{ even}}}\frac{i+1}{j+1}a_j b_i r^{i+j} I_{i+j+1,2} + \sum_{\substack{i=1\\ i\text{ odd}}}^{od(m)}\sum_{\substack{j=0\\ j\text{ even}}}^{ev(l)}(i+1)b_i c_j r^{i+j+2}D_{ij}, \tag{32}$$

where $I_{i+j+1,2}$ and D_{ij} are real constants.

Proof. We have that

(a$_2$)

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+1)b_i r^i\cos^i\theta\sin^2\theta\right)\cdot\left(\sum_{j=0}^{n}a_j r^j\frac{1}{j+1}\left(1-\cos^{j+1}\theta\right)\right)d\theta = -\sum_{\substack{1\le i\le\text{od}(m)\\ 0\le j\le\text{ev}(n)\\ i\text{ odd and } j\text{ even}}}\frac{i+1}{j+1}a_j b_i r^{i+j} I_{i+j+1,2}, \tag{33}$$

where $I_{i+j+1,2} = \int_0^{2\pi}(\cos^{i+j+1}\theta\sin^2\theta)d\theta \neq 0$ for i odd and j even.

(b$_2$)

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+1)b_i r^i\cos^i\theta\sin^2\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)}\sum_{k=1}^{(j+3)/2} b_j r^{j+1}\alpha_{k,j}\sin((2k-1)\theta)\right)d\theta = 0, \tag{34}$$

(c$_2$)

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+1)b_i r^i\cos^i\theta\sin^2\theta\right)\cdot\left(\sum_{\substack{j=0\\ j\text{ even}}}^{\text{ev}(l)} c_j r^{j+2}\left(\gamma_{1,j}+\sum_{k=2}^{(j+6)/2}\gamma_{k,j}\cos((2k-3)\theta)\right)\right)d\theta = \sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}\sum_{\substack{j=0\\ j\text{ even}}}^{\text{ev}(l)}(i+1)b_i c_j r^{i+j+2}D_{ij}, \tag{35}$$

where $D_{ij} = \int_0^{2\pi}\cos^i\theta\sin^2\theta(\gamma_{1,j} + \sum_{k=2}^{(j+6)/2}\gamma_{k,j}\cos((2k-3)\theta))d\theta \neq 0$ for $i \geqslant 1$ odd and $j \geqslant 0$ even,

(d$_2$)

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+1)b_i r^i\cos^i\theta\sin^2\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(l)} c_j r^{j+2}\left(\delta_{1,j}+\sum_{k=2}^{(j+5)/2}\delta_{k,j}\cos((2k-2)\theta)\right)\right)d\theta = 0, \tag{36}$$

(e$_2$)

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+1)b_i r^i\cos^i\theta\sin^2\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)}\sum_{k=1}^{(j+5)/2} b_j r^{j+3}\beta_{k,j}\sin((2k-1)\theta)\right)d\theta = 0. \tag{37}$$

We have that the sum of the integrals (a_2)–(e_2) is polynomial (32). This ends the proof of the lemma. □

Lemma 4. *The integral*

$$\int_0^{2\pi}\left(\sum_{i=0}^{l}(i+2)\,c_i r^{i+1}\cos^i\theta\sin^3\theta\right)\left(\int_0^{\theta}F_1(r,\phi)\,d\phi\right)d\theta \tag{38}$$

is, in the variable r, the polynomial

$$\sum_{\substack{i=0\\ i\,even}}^{ev(l)}\sum_{\substack{j=1\\ j\,odd}}^{od(m)}(i+2)\,c_i b_j r^{i+j+2}E_{ij} + \sum_{\substack{i=0\\ i\,even}}^{ev(l)}\sum_{\substack{j=1\\ j\,odd}}^{od(m)}(i+2)\,c_i b_j r^{i+j+4}F_{ij}, \tag{39}$$

where E_{ij} and F_{ij} are real constants.

Proof. We have that

(a_3)

$$\int_0^{2\pi}\left(\sum_{i=0}^{l}(i+2)\,c_i r^{i+1}\cos^i\theta\sin^3\theta\right)\cdot\left(\sum_{j=0}^{n}a_j r^j\frac{1}{J+1}\left(1-\cos^{j+1}\theta\right)\right)d\theta=0, \tag{40}$$

(b_3)

$$\int_0^{2\pi}\left(\sum_{i=0}^{l}(i+2)\,c_i r^{i+1}\cos^i\theta\sin^3\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{od(m)}\sum_{k=1}^{(j+3)/2}b_j r^{j+1}\alpha_{k,j}\sin((2k-1)\theta)\right)d\theta \tag{41}$$

$$=\sum_{\substack{i=0\\ i\text{ even}}}^{ev(l)}\sum_{\substack{j=1\\ j\text{ odd}}}^{od(m)}(i+2)\,c_i b_j r^{i+j+2}E_{ij},$$

where $E_{ij}=\int_0^{2\pi}\cos^i\theta\sin^3\theta(\sum_{k=1}^{(j+3)/2}\alpha_{k,j}\sin((2k-1)\theta))\neq 0$ for $i\geqslant 0$ even and $j\geqslant 1$ odd,

(c_3)

$$\int_0^{2\pi}\left(\sum_{i=0}^{l}(i+2)\,c_i r^{i+1}\cos^i\theta\sin^3\theta\right)\cdot\left(\sum_{\substack{j=0\\ j\text{ even}}}^{ev(l)}c_j r^{j+2}\left(\gamma_{1,j}+\sum_{k=2}^{(j+6)/2}\gamma_{k,j}\cos((2k-3)\theta)\right)\right)d\theta=0, \tag{42}$$

(d_3)

$$\int_0^{2\pi}\left(\sum_{i=0}^{l}(i+2)\,c_i r^{i+1}\cos^i\theta\sin^3\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{od(l)}c_j r^{j+2}\left(\delta_{1,j}+\sum_{k=2}^{(j+5)/2}\delta_{k,j}\cos((2k-2)\theta)\right)\right)d\theta=0, \tag{43}$$

(e_3)

$$\int_0^{2\pi}\left(\sum_{i=0}^{l}(i+2)\,c_i r^{i+1}\cos^i\theta\sin^3\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{od(m)}\sum_{k=1}^{(j+5)/2}b_j r^{j+3}\beta_k^j\sin((2k-1)\theta)\right)d\theta \tag{44}$$

$$=\sum_{\substack{i=0\\ i\text{ even}}}^{ev(l)}\sum_{\substack{j=1\\ j\text{ odd}}}^{od(m)}(i+2)\,c_i b_j r^{i+j+4}F_{ij},$$

where $F_{ij}=\int_0^{2\pi}\cos^i\theta\sin^3\theta(\sum_{k=1}^{(j+5)/2}\beta_{k,j}\sin((2k-1)\theta))\neq 0$ for $i\geqslant 0$ even and $j\geqslant 1$ odd.

We have that the sum of the integrals (a_3)–(e_3) is polynomial (39). This completes the proof of the lemma. □

Lemma 5. *The integral*

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\,odd}}^{od(m)}(i+3)\,b_i r^{i+2}\cos^i\theta\sin^4\theta\right)\left(\int_0^{\theta}F_1(r,\phi)\,d\phi\right)d\theta \tag{45}$$

is, in the variable r, the polynomial

$$-\sum_{\substack{1\le i\le od(m)\\ 0\le j\le ev(n)\\ i\,odd\,and\,j\,even}}\frac{i+3}{j+1}a_j b_i r^{i+j+2}I_{i+j+1,4} + \sum_{\substack{i=1\\ i\,odd}}^{od(m)}\sum_{\substack{j=0\\ j\,even}}^{ev(l)}(i+1)\,b_i c_j r^{i+j+4}H_{ij}, \tag{46}$$

where $I_{i+j+1,4}$ and H_{ij} are real constants.

Proof. We have that

(a_4)

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+3)\,b_i r^{i+2}\cos^i\theta\sin^4\theta\right)\cdot\left(\sum_{j=0}^{n}a_j r^j\frac{1}{J+1}\left(1-\cos^{j+1}\theta\right)\right)d\theta = -\sum_{\substack{1\le i\le \text{od}(m)\\ 0\le j\le \text{ev}(n)\\ i\text{ odd and } j\text{ even}}}\frac{i+3}{j+1}a_j b_i r^{i+j+2}I_{i+j+1,4}, \tag{47}$$

where $I_{i+j+1,4}=\int_0^{2\pi}(\cos^{i+j+1}\theta\sin^4\theta)d\theta\neq 0$ for i odd and j even,

(b_4)

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+3)\,b_i r^{i+2}\cos^i\theta\sin^4\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)}\sum_{k=1}^{(j+3)/2}b_j r^{j+1}\alpha_{k,j}\sin((2k-1)\theta)\right)d\theta=0, \tag{48}$$

(c_4)

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+3)\,b_i r^{i+2}\cos^i\theta\sin^4\theta\right)\cdot\left(\sum_{\substack{j=0\\ j\text{ even}}}^{\text{ev}(l)}c_j r^{j+2}\left(\gamma_{1,j}+\sum_{k=2}^{(j+6)/2}\gamma_{k,j}\cos((2k-3)\theta)\right)\right)d\theta = \sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}\sum_{\substack{j=0\\ j\text{ even}}}^{\text{ev}(l)}(i+1)\,b_i c_j r^{i+j+4}H_{ij}, \tag{49}$$

where $H_{ij}=\int_0^{2\pi}\cos^i\theta\sin^4\theta(\gamma_{1,j}+\sum_{k=2}^{(j+6)/2}\gamma_{k,j}\cos((2k-3)\theta))d\theta\neq 0$ for $i\geqslant 1$ odd and $j\geqslant 0$ even,

(d_4)

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+3)\,b_i r^{i+2}\cos^i\theta\sin^4\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(l)}c_j r^{j+2}\left(\delta_{1,j}+\sum_{k=2}^{(j+5)/2}\delta_{k,j}\cos((2k-2)\theta)\right)\right)d\theta = 0, \tag{50}$$

(e_4)

$$\int_0^{2\pi}\left(\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}(i+3)\,b_i r^{i+2}\cos^i\theta\sin^4\theta\right)\cdot\left(\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)}\sum_{k=1}^{(j+3)/2}b_j r^{j+3}\beta_{k,j}\sin((2k-1)\theta)\right)d\theta=0. \tag{51}$$

We have that the sum of the integrals (a_4)–(e_4) is polynomial (46). This ends the proof of the lemma. □

From Lemmas 2–5 we have that

$$\begin{aligned}\int_0^{2\pi}&\frac{d}{dr}F_1(r,\theta)\left(\int_0^{\theta}F_1(r,\phi)\,d\phi\right)d\theta\\ &=\sum_{\substack{i=2\\ i\text{ even}}}^{\text{ev}(n)}\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)}ia_ib_jr^{i+j}A_{ij}+\sum_{\substack{i=2\\ i\text{ even}}}^{\text{ev}(n)}\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)}ia_ib_jr^{i+j+2}B_{ij}\\ &\quad-\sum_{\substack{1\le i\le \text{od}(m)\\ 0\le j\le \text{ev}(n)\\ i\text{ odd and } j\text{ even}}}\frac{i+1}{j+1}a_jb_ir^{i+j}I_{i+j+1,2}\\ &\quad+\sum_{\substack{i=1\\ i\text{ odd}}}^{\text{od}(m)}\sum_{\substack{j=0\\ j\text{ even}}}^{\text{ev}(l)}(i+1)\,b_ic_jr^{i+j+2}D_{ij}\\ &\quad+\sum_{\substack{i=0\\ i\text{ even}}}^{\text{ev}(l)}\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)}(i+2)\,c_ib_jr^{i+j+2}E_{ij}\\ &\quad+\sum_{\substack{i=0\\ i\text{ even}}}^{\text{ev}(l)}\sum_{\substack{j=1\\ j\text{ odd}}}^{\text{od}(m)}(i+2)\,c_ib_jr^{i+j+4}F_{ij}\end{aligned}$$

$$-\sum_{\substack{1\le i\le \mathrm{od}(m)\\ 0\le j\le \mathrm{ev}(n)\\ i \text{ odd and } j \text{ even}}} \frac{i+3}{j+1} a_j b_i r^{i+j+2} I_{i+j+1,4} + \sum_{\substack{i=1\\ i \text{ odd}}}^{\mathrm{od}(m)} \sum_{\substack{j=0\\ j \text{ even}}}^{\mathrm{ev}(l)} (i+1) b_i c_j r^{i+j+4} H_{ij}. \quad (52)$$

Now we calculate the integral $\int_0^{2\pi} F_2(r,\theta)d\theta$.

$$\int_0^{2\pi} F_2(r,\theta)\,d\theta = \int_0^{2\pi} \sin\theta K(r,\theta)\,d\theta - \int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} P^2(r,\theta)\,d\theta. \quad (53)$$

First we calculate $\int_0^{2\pi} \sin\theta K(r,\theta)d\theta$. Noting that $K(r,\theta)$ is given by (20), we have

$$\begin{aligned}\int_0^{2\pi} \sin\theta K(r,\theta)\,d\theta ={}& \sum_{i=0}^{n} \tilde{a}_i r^i \int_0^{2\pi} \cos^i\theta \sin\theta\,d\theta + \sum_{i=0}^{m} \tilde{b}_i r^{i+1} \int_0^{2\pi} \cos^i\theta \sin^2\theta\,d\theta \\ &+ \sum_{i=0}^{m} \tilde{b}_i r^{i+3} \int_0^{2\pi} \cos^i\theta \sin^4\theta\,d\theta + \sum_{i=0}^{l} \tilde{c}_i r^{i+2} \int_0^{2\pi} \cos^i\theta \sin^3\theta\,d\theta.\end{aligned} \quad (54)$$

Hence

$$\int_0^{2\pi} \sin\theta K(r,\theta)\,d\theta = \sum_{\substack{i=0\\ i \text{ even}}}^{\mathrm{ev}(m)} \tilde{b}_i r^{i+1} I_{i,2} + \sum_{\substack{i=0\\ i \text{ even}}}^{\mathrm{ev}(m)} \tilde{b}_i r^{i+3} I_{i,4}. \quad (55)$$

Now noting that $P(r,\theta)$ is given by (11), we compute

$$\begin{aligned}&\int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} P^2(r,\theta)\,d\theta \\ &= \int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} \left(\sum_{i=0}^{n} a_i r^i \cos^i\theta\right)^2 d\theta + \int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} \left(\sum_{\substack{i=1\\ i \text{ odd}}}^{\mathrm{od}(m)} b_i r^{i+1} \cos^i\theta \sin\theta\right)^2 d\theta \\ &+ \int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} \left(\sum_{i=0}^{l} c_i r^{i+2} \cos^i\theta \sin^2\theta\right)^2 d\theta + \int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} \left(\sum_{\substack{i=1\\ i \text{ odd}}}^{\mathrm{od}(m)} b_i r^{i+3} \cos^i\theta \sin^3\theta\right)^2 d\theta \\ &+ 2\int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} \left(\sum_{i=0}^{n} a_i r^i \cos^i\theta\right) \cdot \left(\sum_{\substack{j=1\\ j \text{ odd}}}^{\mathrm{od}(m)} b_j r^{j+1} \cos^j\theta \sin\theta\right) d\theta \\ &+ 2\int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} \left(\sum_{i=0}^{n} a_i r^i \cos^i\theta\right) \cdot \left(\sum_{\substack{j=1\\ j \text{ odd}}}^{\mathrm{od}(m)} b_j r^{j+3} \cos^j\theta \sin^3\theta\right) d\theta \\ &+ 2\int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} \left(\sum_{i=0}^{n} a_i r^i \cos^i\theta\right) \cdot \left(\sum_{j=0}^{l} c_j r^{j+2} \cos^j\theta \sin^2\theta\right) d\theta \\ &+ 2\int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} \left(\sum_{\substack{i=1\\ i \text{ odd}}}^{\mathrm{od}(m)} b_i r^{i+1} \cos^i\theta \sin\theta\right) \cdot \left(\sum_{\substack{j=1\\ j \text{ odd}}}^{\mathrm{od}(m)} b_j r^{j+3} \cos^j\theta \sin^3\theta\right) d\theta \\ &+ 2\int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} \left(\sum_{\substack{i=1\\ i \text{ odd}}}^{\mathrm{od}(m)} b_i r^{i+1} \cos^i\theta \sin\theta\right) \cdot \left(\sum_{j=0}^{l} c_j r^{j+2} \cos^j\theta \sin^2\theta\right) d\theta \\ &+ 2\int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} \left(\sum_{\substack{i=1\\ i \text{ odd}}}^{\mathrm{od}(m)} b_i r^{i+3} \cos^i\theta \sin^3\theta\right) \cdot \left(\sum_{j=0}^{l} c_j r^{j+2} \cos^j\theta \sin^2\theta\right) d\theta.\end{aligned} \quad (56)$$

Hence

$$\int_0^{2\pi} \frac{\sin\theta\cos\theta}{r} P^2(r,\theta)\,d\theta = 2 \sum_{\substack{0\le i\le \mathrm{ev}(n)\\ 1\le j\le \mathrm{od}(m)\\ i \text{ even and } j \text{ odd}}} a_i b_j r^{i+j} I_{i+j+1,2}$$

$$
\begin{aligned}
&+ 2 \sum_{\substack{0\le i\le \mathrm{ev}(n)\\ 1\le j\le \mathrm{od}(m)\\ i \text{ even and } j \text{ odd}}} a_i b_j r^{i+j+2} I_{i+j+1,4}\\
&+ 2 \sum_{\substack{1\le i\le \mathrm{od}(m)\\ 0\le j\le \mathrm{ev}(l)\\ i \text{ odd and } j \text{ even}}} b_i c_j r^{i+j+2} I_{i+j+1,4}\\
&+ 2 \sum_{\substack{1\le i\le \mathrm{od}(m)\\ 0\le j\le \mathrm{ev}(l)\\ i \text{ odd and } j \text{ even}}} b_i c_j r^{i+j+4} I_{i+j+1,6}.
\end{aligned} \tag{57}
$$

From (55) and (57) we obtain

$$
\begin{aligned}
\int_0^{2\pi} F_2(r,\theta)\,d\theta = {} & \sum_{\substack{i=0\\ i \text{ even}}}^{\mathrm{ev}(m)} \widetilde{b}_i r^{i+1} I_{i,2} + \sum_{\substack{i=0\\ i \text{ even}}}^{\mathrm{ev}(m)} \widetilde{b}_i r^{i+3} I_{i,4}\\
&- 2 \sum_{\substack{0\le i\le \mathrm{ev}(n)\\ 1\le j\le \mathrm{od}(m)\\ i \text{ even and } j \text{ odd}}} a_i b_j r^{i+j} I_{i+j+1,2}\\
&- 2 \sum_{\substack{0\le i\le \mathrm{ev}(n)\\ 1\le j\le \mathrm{od}(m)\\ i \text{ even and } j \text{ odd}}} a_i b_j r^{i+j+2} I_{i+j+1,4}\\
&- 2 \sum_{\substack{1\le i\le \mathrm{od}(m)\\ 0\le j\le \mathrm{ev}(l)\\ i \text{ odd and } j \text{ even}}} b_i c_j r^{i+j+2} I_{i+j+1,4}\\
&- 2 \sum_{\substack{1\le i\le \mathrm{od}(m)\\ 0\le j\le \mathrm{ev}(l)\\ i \text{ odd and } j \text{ even}}} b_i c_j r^{i+j+4} I_{i+j+1,6}.
\end{aligned} \tag{58}
$$

Then F_{20} is the polynomial

$$
\begin{aligned}
F_{20}(r) = \frac{1}{2\pi} \Bigg(& \sum_{\substack{i=2\\ i \text{ even}}}^{\mathrm{ev}(n)} \sum_{\substack{j=1\\ j \text{ odd}}}^{\mathrm{od}(m)} i a_i b_j r^{i+j} A_{ij}\\
&+ \sum_{\substack{i=2\\ i \text{ even}}}^{\mathrm{ev}(n)} \sum_{\substack{j=1\\ j \text{ odd}}}^{\mathrm{od}(m)} i a_i b_j r^{i+j+2} B_{ij}\\
&- \sum_{\substack{1\le i\le \mathrm{od}(m)\\ 0\le j\le \mathrm{ev}(n)\\ i \text{ odd and } j \text{ even}}} \frac{i+1}{j+1} a_j b_i r^{i+j} I_{i+j+1,2}\\
&+ \sum_{\substack{i=1\\ i \text{ odd}}}^{\mathrm{od}(m)} \sum_{\substack{j=0\\ j \text{ even}}}^{\mathrm{ev}(l)} (i+1)\, b_i c_j r^{i+j+2} D_{ij}\\
&+ \sum_{\substack{i=0\\ i \text{ even}}}^{\mathrm{ev}(l)} \sum_{\substack{j=1\\ j \text{ odd}}}^{\mathrm{od}(m)} (i+2)\, c_i b_j r^{i+j+2} E_{ij}\\
&+ \sum_{\substack{i=0\\ i \text{ even}}}^{\mathrm{ev}(l)} \sum_{\substack{j=1\\ j \text{ odd}}}^{\mathrm{od}(m)} (i+2)\, c_i b_j r^{i+j+4} F_{ij}\\
&- \sum_{\substack{1\le i\le \mathrm{od}(m)\\ 0\le j\le \mathrm{ev}(n)\\ i \text{ odd and } j \text{ even}}} \frac{i+3}{j+1} a_j b_i r^{i+j+2} I_{i+j+1,4}\\
&+ \sum_{\substack{i=1\\ i \text{ odd}}}^{\mathrm{od}(m)} \sum_{\substack{j=0\\ j \text{ even}}}^{\mathrm{ev}(l)} (i+1)\, b_i c_j r^{i+j+4} H_{ij}\\
&+ \sum_{\substack{i=0\\ i \text{ even}}}^{\mathrm{ev}(m)} \widetilde{b}_i r^{i+1} I_{i,2} + \sum_{\substack{i=0\\ i \text{ even}}}^{\mathrm{ev}(m)} \widetilde{b}_i r^{i+3} I_{i,4}\\
&- 2 \sum_{\substack{0\le i\le \mathrm{ev}(n)\\ 1\le j\le \mathrm{od}(m)\\ i \text{ even and } j \text{ odd}}} a_i b_j r^{i+j} I_{i+j+1,2}\\
&- 2 \sum_{\substack{0\le i\le \mathrm{ev}(n)\\ 1\le j\le \mathrm{od}(m)\\ i \text{ even and } j \text{ odd}}} a_i b_j r^{i+j+2} I_{i+j+1,4}\\
&- 2 \sum_{\substack{1\le i\le \mathrm{od}(m)\\ 0\le j\le \mathrm{ev}(l)\\ i \text{ odd and } j \text{ even}}} b_i c_j r^{i+j+2} I_{i+j+1,4}\\
&- 2 \sum_{\substack{1\le i\le \mathrm{od}(m)\\ 0\le j\le \mathrm{ev}(l)\\ i \text{ odd and } j \text{ even}}} b_i c_j r^{i+j+4} I_{i+j+1,6} \Bigg).
\end{aligned} \tag{59}
$$

Note that in order to find the positive roots of F_{20} after dividing by r, we must find the zeros of a polynomial in the variable r^2 of degree equal to the $\max\{((\mathrm{ev}(n) + \mathrm{od}(m) + 2) - 1)/2, ((\mathrm{ev}(l) + \mathrm{od}(m) + 4) - 1)/2, ((\mathrm{ev}(m) + 3) - 1)/2\}$, we conclude that F_{20} has at most $\max\{[n/2] + [(m+1)/2], [l/2] + [(m+3)/2], [(m+2)/2]\}$ positive roots. Hence (b) of Theorem 1 is proved.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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