

Research Article

A Family of Novel Exact Solutions to $(2 + 1)$ -Dimensional KdV Equation

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Received 12 December 2013; Accepted 15 February 2014; Published 30 March 2014

Academic Editor: Tiecheng Xia

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We introduce two subequations with different independent variables for constructing exact solutions to nonlinear partial differential equations. In order to illustrate the efficiency and usefulness, we apply this method to $(2 + 1)$ -dimensional KdV equation, which was first derived by Boiti et al. (1986) using the idea of the weak Lax pair. As a result, we obtained many new exact solutions.

1. Introduction

Solving nonlinear partial differential equations (PDEs) has become increasingly important in many physical and mathematical fields, such as fluid physics, condensed matter, biophysics, plasma physics, nonlinear optics, quantum field theory, and particle physics [1–11], since many physical models could be reduced to PDEs which have a profuse mathematical structure. In recent years, with the help of computer, symbolic computation has enhanced a great progress in many subjects, including solving the PDEs. Under many scientists' works, there have been many papers and software about how to construct exact solutions to PDEs; Lou et al. [12] presented a computational method for constructing the explicit solutions to some complicated nonlinear wave equations by the phenomenon that soliton solutions could be expressed by a combination of tanh and sech functions. Malfliet [13] proposed a tanh function method which offers a systemic approach to construct soliton solutions to nonlinear wave equations. Recently, Parkes and Duffy put forward software named “ATFM” [14–16] on *Mathematica*. “ATFM” could output the required results directly after complicated algebraic computation. Li and Liu [17] developed another software program “RATH” for solving nonlinear partial differential equations on symbolic computation system *MAPLE*. Compared with “ATFM”, “RATH” is more automatic and

could obtain more solutions. Fan [18, 19] developed a uniform and direct method for constructing a numerous travelling wave solutions to nonlinear wave equations.

However, all these existing methods [12–22] could only construct exact solutions expressed by either hyperbolic functions tanh and sech or periodic functions tan, sin, and sn. et al. In this paper, by introducing two subequations with different independent variables, respectively, we will present a new method for constructing exact solutions. Some of these solutions are complexiton solutions which are first derived by Ma through Wronskian method in [23]. And especially some of them are multisoliton-like solutions. We should indicate that these complexiton solutions and multisoliton-like solutions are more general than the solutions obtained by many other methods because our solutions contain some arbitrary functions and constants. In some senses, the above methods [12–22] could be regarded as our special cases. This method could construct many kinds of exact solutions to nonlinear PDEs in a unified form, besides complexiton solutions, multisoliton-like solutions, periodic solutions and rational solutions.

The rest of this paper is arranged as follows: in Section 2, we mainly present our method; in Section 3, we will focus on the application of our method to a nonlinear partial differential equation, $(2 + 1)$ -dimensional KdV equation (also named the asymmetric Nizhnik-Novikov-Veselov (ANNV)

equation or BLMP (Boiti-Leon-Manna-Pempinelli) equation [24, 25]; in Section 4, a discussion about our solutions and the computer simulations will be provided. Finally, we will conclude the paper in the last section.

2. Summary of the Method

In this section, we would like to outline the main steps of our method as follows.

Step 1. Given a system of polynomial nonlinear PDE with constant coefficients,

$$H\left(U(X), U_{x_i}(X), U_{x_i x_j}(X), U_{x_i x_j x_k}(X), \dots\right) = 0, \quad (1)$$

$$(1 \leq i, j, k \leq n),$$

where X denotes some physical fields (x_1, x_2, x_3, \dots) .

Step 2. We introduce a new ansatz in terms of formal expansion in the following form:

$$U(X) = \sum_{0 \leq i, j \leq N} A_{ij} \phi^i(\xi) \psi^j(\eta), \quad (2)$$

where N is an integer determined by balancing the highest nonlinear terms and the highest-order partial derivative terms in the given nonlinear partial differential equation, and $\phi(\xi)$ and $\psi(\eta)$ satisfy two subequations with different independent variables separately

$$\frac{d\phi}{d\xi} = e_0 + e_2 \phi^2(\xi), \quad (3)$$

$$\frac{d\psi}{d\eta} = h_0 + h_2 \psi^2(\eta), \quad (4)$$

where e_0, e_2, h_0 , and h_2 are all arbitrary constants.

In the formal solutions (2), A_{ij} denotes arbitrary functions in terms of y and t . ξ and η denote combination of some arbitrary functions in terms of x, y , and t . All of these functions will be determined later.

Step 3. With the aid of *MAPLE*, substituting (2) into (1) along with the aid of (3) and (4) and setting all coefficients of $\phi^i(\xi)\psi^j(\eta)$, ($i = 0, 1, 2, \dots; j = 0, 1, 2, \dots$) of the resulting system to zero, we obtained an overdetermined nonlinear partial differential equation system with respect to A_{ij}, ξ, η , and their derivatives.

Step 4. We could determine ξ, η , and A_{ij} ($i = 0, 1, 2, \dots; j = 0, 1, 2, \dots$) by solving the overdetermined system with the symbolic computation system *MAPLE*.

Step 5. The solutions of system (3) are

$$\phi(\xi) = \begin{cases} \tanh(\xi), & \text{when } e_0 = 1, e_2 = -1; \\ \coth(\xi), & \text{when } e_0 = 1, e_2 = -1; \\ \tan(\xi), & \text{when } e_0 = 1, e_2 = 1; \\ \cot(\xi), & \text{when } e_0 = 1, e_2 = 1; \\ \frac{1}{\xi + \text{const}}, & \text{when } e_0 = 0, e_2 = -1, \end{cases} \quad (5)$$

and a similar result due to (4). Substituting the results in Step 4 and the expression of $\phi(\xi)$ and $\psi(\eta)$ into (2), we get formal exact solutions to system (1), including many new types of multisoliton-like solutions, complexiton solutions, periodic solutions, and rational solutions.

3. Exact Solutions to (2 + 1)-Dimensional KdV Equation

In this section, we will construct some new exact solutions to (2 + 1)-dimensional KdV equation [24–27] by using the method presented in Section 2 with the aid of symbolic computation system *MAPLE*.

The (2 + 1)-dimensional KdV equation could be written as

$$u_t - u_{xxx} - 3(uv)_x = 0, \quad u_x = v_y, \quad (6)$$

which was first derived by Boiti et al. [24] using the idea of the weak Lax pair. The equation system can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation [26].

By the balancing procedure, we could get $N = 2$. Because there are two dependent variables, the ansatz (2) can be rewritten as follows:

$$\begin{aligned} u &= A_0 + A_1 \phi(\xi) + A_2 \psi(\eta) + A_3 \phi^2(\xi) + A_4 \psi^2(\eta) \\ &\quad + A_5 \phi(\xi) \psi(\eta), \\ v &= B_0 + B_1 \phi(\xi) + B_2 \psi(\eta) + B_3 \phi^2(\xi) + B_4 \psi^2(\eta) \\ &\quad + B_5 \phi(\xi) \psi(\eta), \end{aligned} \quad (7)$$

where $\xi = p(x) + q(y, t)$, $\eta = k(x) + l(y, t)$, p and k are arbitrary functions with respect to x , and q and l are arbitrary functions in terms of y and t .

Substituting (7) into (6) and collecting all coefficients of polynomials in terms of $\phi^i \psi^j$ ($i = 0, 1, \dots; j = 0, 1, \dots$), and then setting each coefficient to zero with the aid of *MAPLE*, we deduce an overdetermined partial differential equation system in terms of A_{ij}, B_{ij}, p, q, k , and l .

With the aid of symbolic computation software *MAPLE* and solving the overdetermined partial differential equations, we get the following solutions.

Case 1. Consider

$$\begin{aligned} B_1 &= B_2 = B_5 = 0, & q &= F_2(y) + F_1(t), \\ A_5 &= A_1 = A_2 = 0, & A_4 &= -2C_3 e_2^2 \frac{d}{dy} F_2(y), \\ A_3 &= 2 \frac{C_1^2 (d/dy) F_2(y) h_2^2}{C_3}, \\ B_0 &= \frac{(d/dt) F_1(t)}{3C_3} - \frac{C_5 + 8C_1^3 h_0 h_2 + 8C_1 e_2 e_0 C_3^2}{6C_1}, \\ k &= C_1 x + C_2, & B_4 &= -2C_3^2 e_2^2, & B_3 &= -2C_1^2 h_2^2, \end{aligned}$$

$$A_0 = \frac{(d/dy) F_2(y) (-C_5 - 8C_1 e_2 e_0 C_3^2 + 8C_1^3 h_0 h_2)}{6C_3 C_1},$$

$$p = C_3 x + C_4, \quad l = \frac{C_1 F_1(t) - C_1 F_2(y) + C_3 (C_5 t + C_6)}{C_3}, \quad (8)$$

where $F_1(t)$ and $F_2(y)$ are arbitrary functions with t and y , respectively, and C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants.

Case 2. Consider

$$B_1 = B_2 = B_5 = 0, \quad A_1 = A_2 = A_5 = 0,$$

$$A_4 = -2C_3 e_2^2 \frac{d}{dy} F_2(y), \quad A_0 = -\frac{8}{3} C_3 e_0 e_2 \frac{d}{dy} F_2(y),$$

$$l = \frac{-C_1 F_2(y) + 8C_1^3 h_2 h_0 C_3 t + C_1 C_5 t + 8C_1 e_2 e_0 C_3^3 t + C_6 C_3}{C_3},$$

$$k = C_1 x + C_2, \quad B_4 = -2C_3^2 e_2^2,$$

$$A_3 = \frac{2C_1^2 (d/dy) F_2(y) h_2^2}{C_3}, \quad q = C_5 t + F_2(y),$$

$$p = C_3 x + C_4, \quad B_3 = -2C_1^2 h_2^2, \quad B_0 = \frac{C_5}{3C_3}, \quad (9)$$

where $F_1(t)$ and $F_2(y)$ are arbitrary functions with respect to t and y , respectively, C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants.

Case 3. Consider

$$B_1 = B_2 = B_5 = 0, \quad A_1 = A_2 = A_5 = 0,$$

$$q = F_2(y) + F_1(t), \quad A_4 = -2C_3 e_2^2 \frac{d}{dy} F_2(y),$$

$$A_0 = -\frac{8}{3} C_3 e_0 e_2 \frac{d}{dy} F_2(y),$$

$$l = \left(C_1 F_1(t) - C_1 F_2(y) + 8 \left(C_1 e_2 e_0 C_3^2 t + C_1^3 h_2 h_0 t + \frac{1}{8} C_5 \right) C_3 \right) C_3^{-1}, \quad (10)$$

$$B_0 = \frac{(d/dt) F_1(t)}{3C_3}, \quad A_3 = 2 \frac{C_1^2 (d/dy) F_2(y) h_2^2}{C_3},$$

$$k = C_1 x + C_2, \quad B_4 = -2C_3^2 e_2^2, \quad p = C_3 x + C_4,$$

$$B_3 = -2C_1^2 h_2^2,$$

where $F_1(t)$ and $F_2(y)$ are arbitrary functions with respect to t and y , respectively, C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants.

Case 4. Consider

$$B_1 = B_2 = B_5 = 0, \quad A_1 = A_2 = A_5 = 0,$$

$$A_3 = -4 \frac{C_1 (d/dy) F_2(y) e_2 e_0 h_2^2}{\sqrt{-2e_2 e_0 h_2 h_0}},$$

$$A_4 = -2 \frac{C_1 h_2 (d/dy) F_2(y) e_2^2 h_0}{\sqrt{-2e_2 e_0 h_2 h_0}},$$

$$q(y, t) = -\frac{C_1^3 h_2 h_0 \sqrt{-2e_2 e_0 h_2 h_0}}{e_2 e_0} t + F_2(y),$$

$$p(x) = -\frac{\sqrt{-2e_2 e_0 h_2 h_0} C_1 x}{2e_2 e_0} + C_3, \quad (11)$$

$$l(y, t) = \frac{2e_2 e_0 F_2(y) + C_5 \sqrt{-2e_2 e_0 h_2 h_0}}{\sqrt{-2e_2 e_0 h_2 h_0}},$$

$$A_0 = -\frac{14C_1 h_2 (d/dy) F_2(y) e_2 e_0 h_0}{3},$$

$$B_4 = \frac{e_2 h_0 C_1^2 h_2}{e_0}, \quad k = C_1 x + C_2, \quad B_3 = -2C_1^2 h_2^2,$$

$$B_0 = -\frac{C_1^2 h_2 h_0}{3},$$

where $F_2(y)$ is an arbitrary function with respect to y ; C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants.

Case 5. Consider

$$B_1 = B_2 = B_5 = 0, \quad A_1 = A_2 = A_5 = 0,$$

$$k(x) = C_1 x + C_2, \quad B_3 = -2C_1^2 h_2^2, \quad q(y, t) = F_2(y),$$

$$A_3 = -\frac{\delta C_1 h_2 (d/dy) F_2(y)}{h_0}, \quad p(x) = \frac{\delta C_1 x}{e_2 e_0} + C_3,$$

$$A_0 = -\frac{(d/dy) F_2(y) (C_6 - 24C_1^3 h_0 h_2) e_2 e_0}{6\delta C_1^2},$$

$$B_4 = 4 \frac{e_2 h_0 C_1^2 h_2}{e_0}, \quad A_4 = -2 \frac{e_2 \delta C_1 (d/dy) F_2(y)}{e_0},$$

$$B_0 = \frac{C_6 + 8C_1^3 h_0 h_2}{6C_1}, \quad l = \frac{-e_2 e_0 F_2(y) + (C_6 t + C_7) \delta}{\delta}, \quad (12)$$

where $F_2(y)$ is an arbitrary function with respect to y ; C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants; δ denotes $\sqrt{-2e_2 e_0 h_2 h_0}$.

Case 6. Consider

$$\begin{aligned}
 B_1 = B_2 = B_5 = 0, \quad A_1 = A_2 = A_5 = 0, \\
 A_3 = 2 \frac{C_1^2 C_5 h_2^2}{C_3}, \quad B_0 = -\frac{2}{3} e_2 e_0 C_3^2 - \frac{4}{3} C_1^2 h_2 h_0, \\
 k = C_1 x + C_2, \quad l = \frac{-C_1 C_5 y + 4 C_1 e_2 e_0 C_3^3 t + C_7 C_3}{C_3}, \\
 B_4 = -2 C_3^2 e_2^2, \quad q(y, t) = C_5 y + C_6, \\
 p = C_3 x + C_4, \quad A_4 = -2 C_3 e_2^2 C_5, \\
 A_0 = \frac{-6 C_5 C_3^2 e_2 e_0 + 4 C_5 h_0 h_2 C_1^2}{3 C_3}, \quad B_3 = -2 C_1^2 h_2^2,
 \end{aligned} \tag{13}$$

where C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants.

Case 7. Consider

$$\begin{aligned}
 A_1 = A_2 = A_3 = A_5 = 0, \quad B_1 = B_2 = B_3 = B_5 = 0, \\
 p(x) = C_1 x + C_2, \quad l(y, t) = l(y, t), \\
 k = k(x), \quad B_0 = \frac{(d/dt) F_1(t) + 3 C_3 C_1}{3 C_1}, \\
 q(y, t) = F_2(y) + F_1(t), \\
 a_0 = \frac{(-3 C_3 - 8 C_1^2 e_2 e_0) (d/dy) F_2(y)}{3 C_1}, \\
 A_4 = -2 C_1 e_2^2 \frac{d}{dy} F_2(y), \quad B_4 = -2 C_1^2 e_2^2,
 \end{aligned} \tag{14}$$

where $F_1(t)$ and $F_2(y)$ are arbitrary functions with respect to t and y , respectively; C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants.

With the aid of (7) and (5), we could obtain many new exact solutions to $(2 + 1)$ -dimensional KdV equation. There are too many solutions to cite one by one, we only select Case 1, for instance. We omit the other cases here which could be obtained by the similar way as in Case 1. Corresponding to Case 1, we have 16 families solutions in what follows.

Family 1. For $e_0 = 1, e_2 = -1, h_0 = 1$, and $h_2 = -1$, then, we have the following solution to $(2 + 1)$ -dimensional KdV equations

$$\begin{aligned}
 u_1(x, y, t) \\
 = \frac{(d/dy) F_2(y) (-C_5 + 8 C_1 C_3^2 - 8 C_1^3)}{6 C_3 C_1} \\
 + 2 \frac{C_1^2 (d/dy) F_2(y)}{C_3} \phi^2 (C_3 x + C_4 + F_2(y) + F_1(t))
 \end{aligned}$$

$$\begin{aligned}
 & -2 C_3 \frac{d}{dy} F_2(y) \psi^2 \\
 & \times \left(C_1 x + C_2 + \frac{C_1 F_1(t) - C_1 F_2(y) + C_3 (C_5 t + C_6)}{C_3} \right), \\
 v_1(x, y, t) \\
 = & \frac{(d/dy) F_1(t)}{3 C_3} + \frac{C_5 - 8 C_1^3 - 8 C_1 C_3^2}{6 C_1} \\
 & - 2 C_1^2 \phi^2 (C_3 x + C_4 + F_2(y) + F_1(t)) \\
 & - 2 C_3^2 \psi^2 \\
 & \times \left(C_1 x + C_2 + \frac{C_1 F_1(t) - C_1 F_2(y) + C_3 (C_5 t + C_6)}{C_3} \right),
 \end{aligned} \tag{15}$$

where $F_1(t)$ is an arbitrary function with respect to t , $F_2(y)$ is an arbitrary function with respect to y , C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants, $\phi \in \{\tanh, \coth\}$, and $\psi \in \{\tanh, \coth\}$.

In fact, (15) indicates four families solutions to $(2 + 1)$ -dimensional KdV equation. If we choose $\phi = \tanh$ and $\psi = \tanh$, we get some multisoliton-like solutions which are simulated by computer in Section 4.

Family 2. For $e_0 = 1, e_2 = -1, h_0 = 1$, and $h_2 = 1$, then, we can get four families solutions to $(2 + 1)$ -dimensional KdV equation

$$\begin{aligned}
 u_2(x, y, t) \\
 = & \frac{(d/dy) F_2(y) (-C_5 + 8 C_1 C_3^2 + 8 C_1^3)}{6 C_3 C_1} \\
 & + 2 \frac{C_1^2 (d/dy) F_2(y)}{C_3} \phi^2 (C_3 x + C_4 + F_2(y) + F_1(t)) \\
 & - 2 C_3 \frac{d}{dy} F_2(y) \psi^2 \\
 & \times \left(C_1 x + C_2 + \frac{C_1 F_1(t) - C_1 F_2(y) + C_3 (C_5 t + C_6)}{C_3} \right), \\
 v_2(x, y, t) \\
 = & \frac{(d/dt) F_1(t)}{3 C_3} + \frac{C_5 - 8 C_1^3 + 8 C_1 C_3^2}{6 C_1} \\
 & - 2 C_1^2 \phi^2 (C_3 x + C_4 + F_2(y) + F_1(t)) \\
 & - 2 C_3^2 \psi^2 \\
 & \times \left(C_1 x + C_2 + \frac{C_1 F_1(t) - C_1 F_2(y) + C_3 (C_5 t + C_6)}{C_3} \right),
 \end{aligned} \tag{16}$$

where $F_1(t)$ is an arbitrary function with respect to t , $F_2(y)$ is arbitrary function with respect to y , C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants, $\phi \in \{\tanh, \coth\}$, and $\psi \in \{\tan, \cot\}$.

In fact, (16) also indicates four families solutions to (2+1)-dimensional KdV equation. If we choose $\phi = \tanh$ and $\psi = \tan$, we could get some complexiton-like solutions to (2+1)-dimensional KdV equations. If we select $e_0 = 1$, $e_2 = 1$, $h_0 = 1$, and $h_2 = -1$, we get other four families solutions to (2+1)-dimensional KdV equation similar to (16).

Family 3. For $e_0 = 1$, $e_2 = 1$, $h_0 = 1$, and $h_2 = 1$, then we have four families solutions to (2+1)-dimensional KdV equation

$$\begin{aligned} u_3(x, y, t) &= \frac{(d/dy) F_2(y) (-C_5 - 8C_1C_3^2 + 8C_1^3)}{6C_3C_1} \\ &+ 2 \frac{C_1^2 (d/dy) F_2(y)}{C_3} \phi^2 (C_3x + C_4 + F_2(y) + F_1(t)) \\ &- 2C_3 \frac{d}{dy} F_2(y) \psi^2 \\ &\times \left(C_1x + C_2 + \frac{C_1F_1(t) - C_1F_2(y) + C_3(C_5t + C_6)}{C_3} \right), \\ v_3(x, y, t) &= \frac{(d/dy) F_1(t)}{3C_3} + \frac{C_5 - 8C_1^3 - 8C_1C_3^2}{6C_1} \\ &- 2C_1^2 \phi^2 (C_3x + C_4 + F_2(y) + F_1(t)) \\ &- 2C_3^2 \psi^2 \\ &\times \left(C_1x + C_2 + \frac{C_1F_1(t) - C_1F_2(y) + C_3(C_5t + C_6)}{C_3} \right), \end{aligned} \quad (17)$$

where $F_1(t)$ is an arbitrary function with respect to t , $F_2(y)$ is an arbitrary function with respect to y , C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants, $\phi \in \{\tan, \cot\}$, and $\psi \in \{\tan, \cot\}$.

In fact, (17) also denotes four families periodic solutions to (2+1)-dimensional KdV equation.

Family 4. For $e_0 = 0$, $e_2 = -1$, $h_0 = 1$, and $h_2 = -1$, then we have two families soliton-like solution to (2+1)-dimensional KdV equation

$$\begin{aligned} u_4(x, y, t) &= \frac{(d/dy) F_2(y) (-C_5 - 8C_1^3)}{6C_3C_1} \\ &+ 2 \frac{C_1^2 (d/dy) F_2(y)}{C_3} \left(\frac{1}{C_3x + C_4 + F_2(y) + F_1(t)} \right)^2 \end{aligned}$$

$$\begin{aligned} &- 2C_3 \frac{d}{dy} F_2(y) \psi^2 \\ &\times \left(C_1x + C_2 + \frac{C_1F_1(t) - C_1F_2(y) + C_3(C_5t + C_6)}{C_3} \right), \\ v_4(x, y, t) &= \frac{(d/dy) F_1(t)}{3C_3} + \frac{C_5 + 8C_1^3}{6C_1} \\ &- 2C_1^2 \left(\frac{1}{C_3x + C_4 + F_2(y) + F_1(t)} \right)^2 \\ &- 2C_3^2 \psi^2 \\ &\times \left(C_1x + C_2 + \frac{C_1F_1(t) - C_1F_2(y) + C_3(C_5t + C_6)}{C_3} \right), \end{aligned} \quad (18)$$

where $F_1(t)$ is an arbitrary function with respect to t , $F_2(y)$ is an arbitrary function with respect to y , C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants, and $\psi \in \{\tanh, \coth\}$. Other two solutions similar to (18) could be obtained with $e_0 = 1$, $e_2 = -1$, $h_0 = 0$, and $h_2 = -1$.

Family 5. For $e_0 = 0$, $e_2 = -1$, $h_0 = 1$, and $h_2 = 1$, then, we have two families solutions to (2+1)-dimensional KdV equation

$$\begin{aligned} u_5(x, y, t) &= \frac{(d/dy) F_2(y) (-C_5 + 8C_1^3)}{6C_3C_1} \\ &+ 2 \frac{C_1^2 (d/dy) F_2(y)}{C_3} \left(\frac{1}{C_3x + C_4 + F_2(y) + F_1(t)} \right)^2 \\ &- 2C_3 \frac{d}{dy} F_2(y) \psi^2 \\ &\times \left(C_1x + C_2 + \frac{C_1F_1(t) - C_1F_2(y) + C_3(C_5t + C_6)}{C_3} \right), \\ v_5(x, y, t) &= \frac{(d/dy) F_1(t)}{3C_3} + \frac{C_5 - 8C_1^3}{6C_1} \\ &- 2C_1^2 \left(\frac{1}{C_3x + C_4 + F_2(y) + F_1(t)} \right)^2 \\ &- 2C_3^2 \psi^2 \\ &\times \left(C_1x + C_2 + \frac{C_1F_1(t) - C_1F_2(y) + C_3(C_5t + C_6)}{C_3} \right), \end{aligned} \quad (19)$$

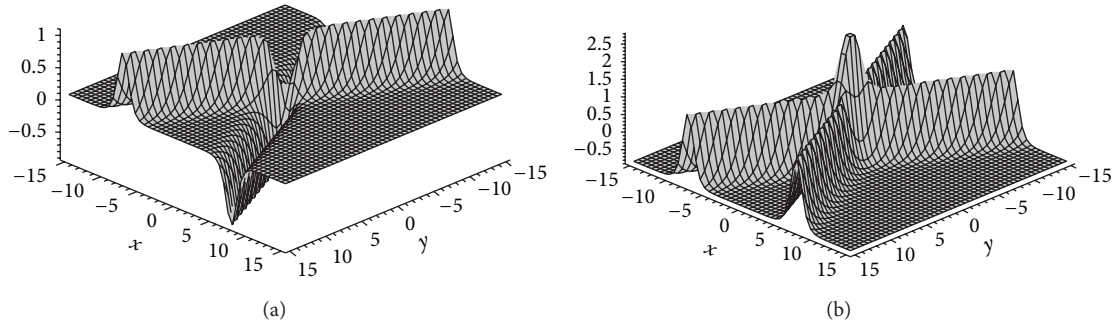


FIGURE 1: A computer simulation about two-soliton solution collision at $t = 0$ in Family 1, with parameters $F_1(t) = t$, $F_2(y) = -0.5y$, $C_1 = C_3 = C_5 = 1$, $C_2 = C_4 = 0$, $\phi = \tanh$, and $\psi = \tanh$.

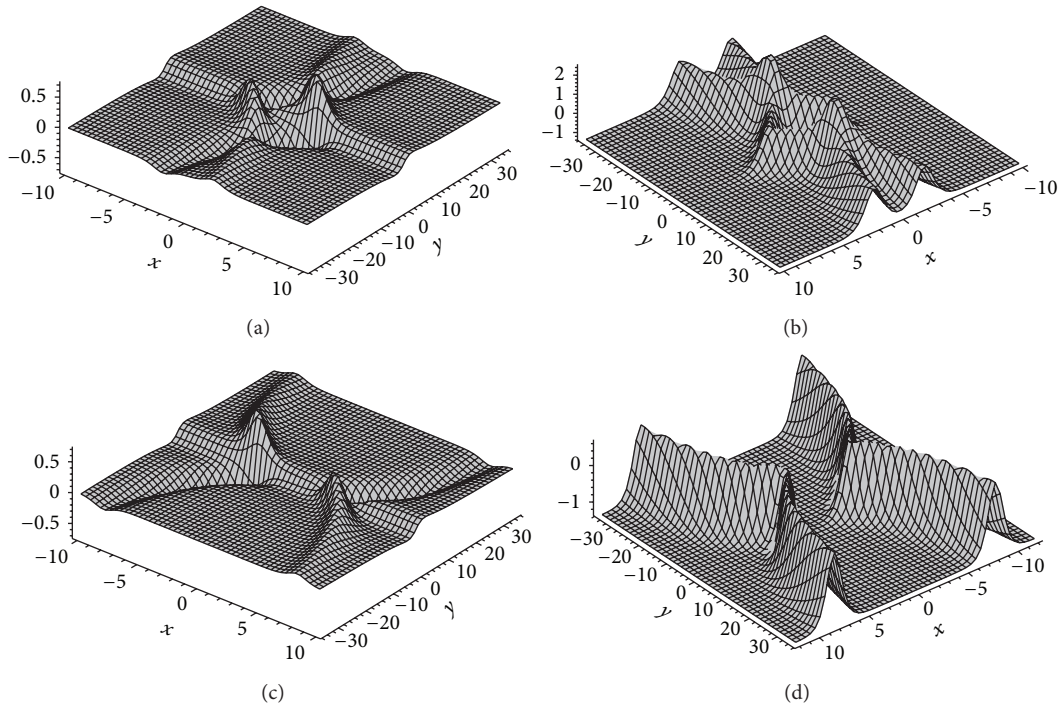


FIGURE 2: The plots of u_1 and v_1 in Family 1, with parameters $F_1(t) = t$, $F_2(y) = \ln(y^2 + 10)$, $C_1 = C_3 = 1$, $C_5 = -2$, $C_2 = C_4 = C_6 = 0$, $\phi = \tanh$, and $\psi = \tanh$ at ((a) and (b)) $t = -5$ and ((c) and (d)) $t = 1$.

where $F_1(t)$ is an arbitrary function with respect to t , $F_2(y)$ is an arbitrary function with respect to y , C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants, and $\psi \in \{\tan, \cot\}$. Other two solutions similar to (19) could be obtained with $e_0 = 1$, $e_2 = 1$, $h_0 = 0$, and $h_2 = -1$.

Family 6. For $e_0 = 0$, $e_2 = -1$, $h_0 = 0$, and $h_2 = -1$, then, we have rational solutions with two arbitrary functions to (2+1)-dimensional KdV equation

$$u_6(x, y, t) = \frac{C_5 (d/dy) F_2(y)}{6C_3 C_1} + 2 \frac{C_1^2 (d/dy) F_2(y)}{C_3} \left(\frac{1}{C_3 x + C_4 + F_2(y) + F_1(t)} \right)^2$$

$$v_6(x, y, t) = -2C_3 \frac{d}{dy} F_2(y) \times \left(1 \left(C_1 x + C_2 + \frac{C_1 F_1(t) - C_1 F_2(y) + C_3 (C_5 t + C_6)}{C_3} \right)^{-1} \right)^2$$

$$= \frac{(d/dy) F_1(t)}{3C_3} + \frac{C_5}{6C_1} - 2C_1^2 \left(\frac{1}{C_3 x + C_4 + F_2(y) + F_1(t)} \right)^2$$

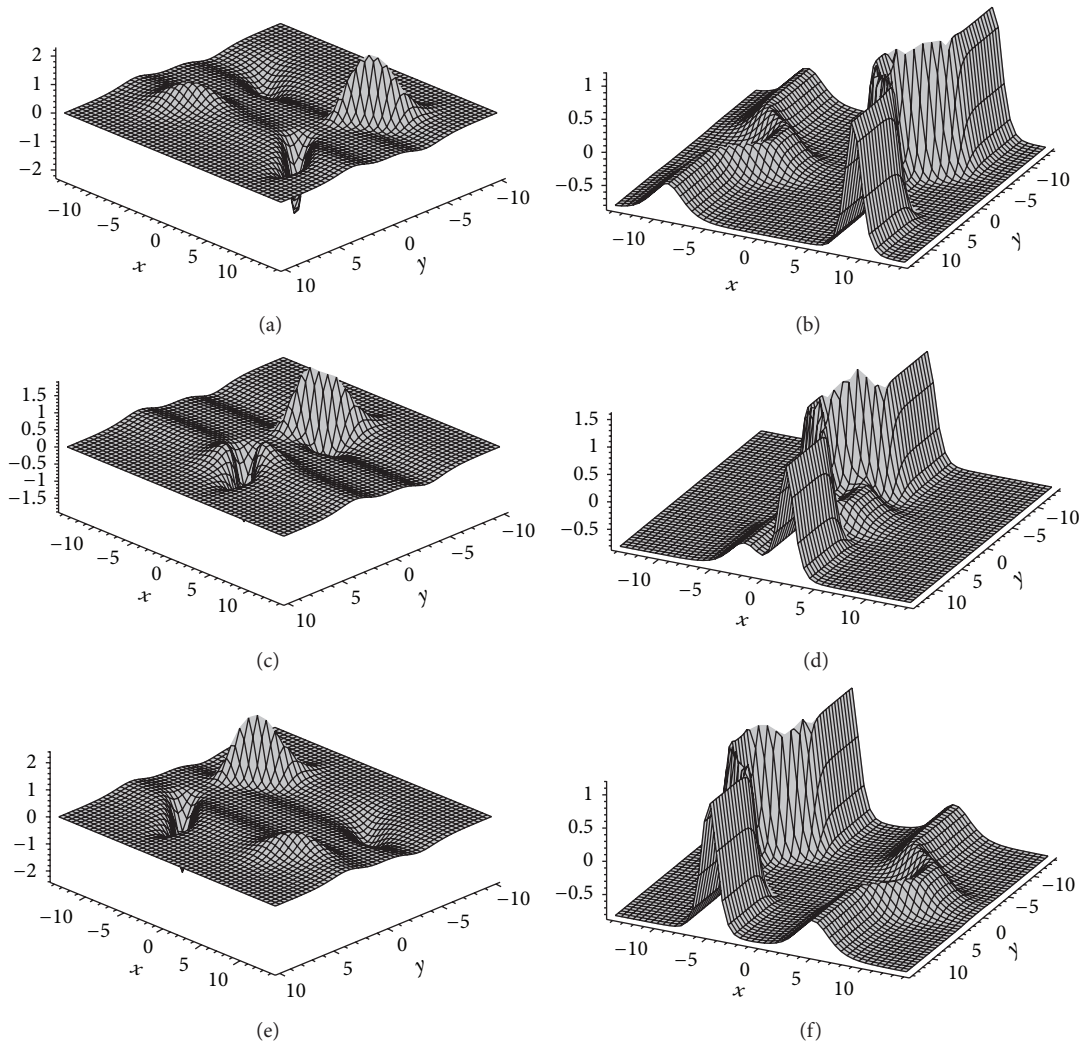


FIGURE 3: The plots of u_1, v_1 in Family 1 with parameters $F_1(t) = t, F_2(y) = 4\text{sech}(0.1y^2), C_1 = 0.5, C_3 = 1, C_5 = -1, C_2 = C_4 = C_6 = 0, \phi = \tanh$, and $\psi = \tanh$ at ((a) and (b)) $t = -9$, ((c) and (d)) $t = -2$, and ((e) and (f)) $t = 5$.

$$-2C_3^2 \left(1 \left(C_1x + C_2 + \frac{C_1F_1(t) - C_1F_2(y) + C_3(C_5t + C_6)}{C_3} \right)^{-1} \right)^2, \quad (20)$$

where $F_1(t)$ is an arbitrary function with respect to t , $F_2(y)$ is an arbitrary function with respect to y , and C_1, C_2, C_3, C_4, C_5 , and C_6 are all integral constants.

4. Some Special Localized Solutions

In this section, we will show some special types of localized structure of our solutions. If we select different idiographic functions to replace the arbitrary functions appeared in our solutions, we could reveal many new structure of the solutions to $(2 + 1)$ -dimensional KdV equation.

If we select $F_1(t) = t, F_2(y) = -0.5y, C_1 = C_3 = C_5 = 1, C_2 = C_4 = 0, \phi = \tanh$, and $\psi = \tanh$ in Family 1, we could get a 2-soliton solution to $(2 + 1)$ -dimensional KdV equation. On the beeline

$$x - 1.5y = 0, \quad (21)$$

where $u_1 = 0$ means the two waves are counteracting each other and the biggest value v_1 means the two waves are building up each other. We show the collision of the two waves at $t = 0$ in Figures 1(a) and 1(b).

If we select $F_1(t) = t, F_2(y) = \ln(y^2 + 10), C_1 = C_3 = 1, C_5 = -2, C_2 = C_4 = C_6 = 0, \phi = \tanh$, and $\psi = \tanh$ in Family 1, we could obtain a 2-dromion solution to $(2 + 1)$ -dimensional KdV equation. The dromion shown by Figure 2 is driven by two curved-line solitons, and the curved lines have the forms

$$\begin{aligned} x + \ln(y^2 + 10) &= 0, \\ x - \ln(y^2 + 10) &= 0. \end{aligned} \quad (22)$$

We show the collision of the two waves at $t = -5$ in Figures 2(a) and 2(b) and $t = 1$ in Figures 2(c) and 2(d). From the four figures, we could find that Family 1 reduces to bidirectional dromion solutions along x -axes.

Noted that the waves promulgation have different directions with different parameters.

If we select $F_1(t) = t$, $F_2(y) = 4\text{sech}(0.1y^2)$, $C_1 = 0.5$, $C_3 = 1$, $C_5 = -1$, $C_2 = C_4 = C_6 = 0$, $\phi = \tanh$, and $\psi = \tanh$ in Family 1, we could obtain another 2-dromion solution to $(2 + 1)$ -dimensional KdV equation. The dromion shown by Figure 3 is driven by two-curved line solitons and the curved lines have the forms

$$\begin{aligned}x + 4\text{sech}(0.1y^2) &= 0, \\x - 4\text{sech}(0.1y^2) &= 0.\end{aligned}\quad (23)$$

This is also a bidirectional dromion solution to $(2 + 1)$ -dimensional KdV equation along x -axes. Figures 3(a) and 3(b) denote the shape of u_1 and v_1 at $t = -9$, Figures 3(c) and 3(d) denote the shape of u_1 and v_1 at $t = -2$, and Figures 3(e) and 3(f) denote the shape of u_1 and v_1 at $t = 5$. We could easily observe the lower dromion which controls by C_1 travelling along x -axes from negative to positive direction, and the higher dromion which controls by C_3 travelling along x -axes from positive to negative direction.

5. Conclusion

In this paper, we have presented a uniform method for constructing exact solutions to the $(2 + 1)$ -dimensional KdV equation by using two subequations with different variables. Some of these exact solutions are multisoliton-like solutions which have arbitrary functions in 2-soliton solutions, and some of them are complexiton solutions admitting both hyperbolic and periodic function in a solution to PDEs.

From the above discussion, we conclude that two-soliton solutions could be obtained by introducing two subequations. In fact, if we use N subequations, then N soliton solutions will be obtained. We see that further research on the subject is needed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by National Science Foundation of China (51109031, 51379033, 51239002, 51221961, and J1103110), Doctoral Foundation of Ministry of Education of China under the Grant 20100041120037, the Fundamental Research Funds for the Central Universities (DUT12LK52, DUT12LK34), and 973 Project (2013CB036101, 2010CB32700).

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