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Research Article

Exact Penalization and Necessary Optimality Conditions for Multiobjective Optimization Problems with Equilibrium Constraints

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A calmness condition for a general multiobjective optimization problem with equilibrium constraints is proposed. Some exact penalization properties for two classes of multiobjective penalty problems are established and shown to be equivalent to the calmness condition. Subsequently, a Mordukhovich stationary necessary optimality condition based on the exact penalization results is obtained. Moreover, some applications to a multiobjective optimization problem with complementarity constraints and a multiobjective optimization problem with weak vector variational inequality constraints are given.

1. Introduction

In this paper, we consider a general multiobjective optimization problem with equilibrium constraints as follows:

(MOPEC)

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \in -\mathbb{R}^r_+, \\ & h(x) = 0_{\mathbb{R}^s}, \\ & 0_{\mathbb{R}^m} \in q(x) + Q(x), \quad x \in \Theta, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$, $f(x) = (f_1(x), f_2(x), \ldots, f_p(x))$, $g : \mathbb{R}^n \to \mathbb{R}^r$, $g(x) = (g_1(x), g_2(x), \ldots, g_r(x))$ $h : \mathbb{R}^n \to \mathbb{R}^s$, $h(x) = (h_1(x), h_2(x), \ldots, h_s(x))$, $q : \mathbb{R}^n \to \mathbb{R}^m$, and $q(x) = (q_1(x), q_2(x), \ldots, q_m(x))$ are vector-valued maps, $Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued map, and Θ is a nonempty and closed subset of $\mathbb{R}^n$. As usual, we denote by int Θ the interior of Θ and by $\text{gph}Q := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in Q(x)\}$ the graph of Q. Moreover, $\mathbb{R}^r_+$ denotes the nonnegative quadrant in $\mathbb{R}_r$, and $0_{\mathbb{R}^s}$ and $0_{\mathbb{R}^m}$ denote, respectively, the origins of $\mathbb{R}^s$ and $\mathbb{R}^m$. Throughout this paper, we assume that f is locally Lipschitz, g, h, and q are continuously Fréchet differentiable, Q is closed (i.e., gph$Q$ is closed in $\mathbb{R}^n \times \mathbb{R}^m$), and the feasible set $S := \{x \in \mathbb{R}^n \mid g(x) \in -\mathbb{R}^r_+, h(x) = 0_{\mathbb{R}^s}, 0_{\mathbb{R}^m} \in q(x) + Q(x), x \in \Theta\}$ of (MOPEC) is nonempty. Obviously, S is a closed subset of $\mathbb{R}^n$. Recall that a point $\widehat{x} \in S$ is said to be an efficient (resp. weak efficient) solution for (MOPEC) if and only if

$$f(x) - f(\widehat{x}) \notin -\mathbb{R}^p_+ \setminus \{0_{\mathbb{R}^p}\} \left(\text{resp.} - \text{int}\,\mathbb{R}^p_+\right), \quad \forall x \in S. \tag{2}$$

A point $\widehat{x} \in S$ is said to be a local efficient (resp. local weak efficient) solution for (MOPEC) if and only if there exists a neighborhood U of $\widehat{x}$ such that

$$f(x) - f(\widehat{x}) \notin -\mathbb{R}^p_+ \setminus \{0_{\mathbb{R}^p}\} \left(\text{resp.} - \text{int}\,\mathbb{R}^p_+\right), \quad \forall x \in S \cap U. \tag{3}$$

During the past few decades, there have been a lot of papers devoted to study the scalar optimization problem (i.e., the case $p = 1$) with equilibrium constraints, which plays an important role in engineering design, economic equilibria, operations research, and so on. It is well recognized that the scalar optimization problem with equilibrium constraints covers various classes of optimization-related problems and models arisen in practical applications, such as mathematical programs with geometric constraints, mathematical programs with complementarity constraints, and mathematical programs with variational inequality constraints. For more details, we refer to [1–4]. It is worth noting that when Q is a general closed set-valued map, even if $Q(x)$ is a fixed closed subset of $\mathbb{R}^m$ for all $x \in \mathbb{R}^n$, the general constraint system (1) fails to satisfy the standard linear independence constraint qualification and Mangasarian-Fromovitz constraint qualification at any feasible point [5]. Thus, it is a hard work to establish Karush-Kuhn-Tucker (in short, KKT) necessary optimality conditions for (MOPEC). Recently, by virtue of advanced tools of variational analysis and various coderivatives for set-valued maps developed in [6–8] and references therein, some necessary optimality conditions including the strong, Mordukhovich, Clarke, and Bouligand stationary conditions are obtained by using different reformulations under some generalized constraint qualifications. Simultaneously, Ye and Zhu [3] claimed that the Mordukhovich stationary (in short, M-stationary) condition is the strongest stationary condition except the strong stationary condition which is equivalent to the classical KKT condition, and proposed some new constraint qualifications for M-stationary conditions to hold.

It is well known that the penalization method is a very important and effective tool for dealing with optimization theories and numerical algorithms of constrained extremum problems. In scalar optimization with equality and inequality constraints, the classical exact penalty function with order 1 was extensively used to investigate optimality conditions and convergence analysis; see [6, 9, 10] and references therein. Clarke [6] derived some Fritz-John necessary optimality conditions for a constrained mathematical programming problem on a Banach space by virtue of exact penalty functions with order 1. Moreover, Burke [9] showed that the existence of an exact penalization function is equivalent to a calmness condition involving with the objective function and the equality and inequality constrained system. Subsequently, Flegel and Kanzow [4] demonstrated that the corresponding relationships still held in a generalized bilevel programming problem and a mathematical programming problem with complementarity constraints, respectively. Simultaneously, they obtained some KKT necessary optimality conditions by using exact penalty formulations and nonsmooth analysis. Recently, the classical penalization theory has been widely generalized by various kinds of Lagrangian functions, especially the augmented Lagrangian function, introduced by Rockafellar and Wets [7], and the nonlinear Lagrangian function, proposed by Rubinov et al. [11]. It has also been proved that the exactness of these types of penalty functions is equivalent to some generalized calmness conditions; see more details in [11, 12].

However, to the best of our knowledge, there are only a few papers devoted to study the penalty method for constrained multiobjective optimization problems, especially, for (MOPEC). Huang and Yang [13] first introduced a vector-valued nonlinear Lagrangian and penalty functions for multiobjective optimization problems with equality and inequality constraints and obtained some relationships between the exact penalization property and a generalized calmness-type condition. Moreover, Mordukhovich [8] and Bao et al. [14] investigated some more general optimization problems with equilibrium constraints by methods of modern variational analysis. It is worth noting that the standard Mangasarian-Fromovitz constraint qualification and error bound condition for a nonlinear programming problem with equality and inequality constraints implies the calmness condition; see [6, 15] for details. Taking into account this fact, it is necessary to further investigate the calmness condition and the penalty method for constrained multiobjective optimization problems.

The main motivation of this work is that there has been no study on the penalization method and M-stationary condition for (MOPEC) by using an appropriate calmness condition associated with the objective function and the constraint system. Although there have been many papers dealing with constrained multiobjective optimization problems, for example, [3, 8] and references therein, the KKT necessary optimality conditions are obtained under some generalized qualification conditions only involved with the constraint system. Inspired by the ideas reported in [3, 4, 6, 8, 13], we introduce a so-called (MOPEC-) calmness condition with order $\sigma > 0$ at a local efficient (weak efficient) solution associated with the objective function and the constraint system for (MOPEC) and show that the (MOPEC-) calmness condition can be implied by an error bound condition of the constraint system. Moreover, we establish some equivalent relationships between the exact penalization property with order σ and the (MOPEC-) calmness condition. Simultaneously, we apply a nonlinear scalar technical to obtain a KKT necessary optimality condition for (MOPEC) by using Mordukhovich generalized differentiation and the (MOPEC-) calmness condition with order 1.

The organization of this paper is as follows. In Section 2, we recall some basic concepts and tools generally used in variational analysis and set-valued analysis. In Section 3, we introduce a (MOPEC-) calmness condition for (MOPEC) and establish some relationships between the exact penalization property and the (MOPEC-) calmness condition. Moreover, we obtain a KKT necessary optimality condition under the (MOPEC-) calmness condition with order 1. In Section 4, we apply the obtained results to a multiobjective optimization problem with complementarity constraints and a multiobjective optimization problem with weak vector variational inequality constraints, respectively.

2. Notations and Preliminaries

Throughout this paper, all vectors are viewed as column vectors. Since all the norms on finite dimensional spaces are equivalent, we take specially the sum norm on $\mathbb{R}^n$ and the product space $\mathbb{R}^n \times \mathbb{R}^m$ for simplicity; that is, for all $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, $\|x\| = |x_1| + |x_2| + \cdots + |x_n|$, and, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, $\|(x, y)\| = \|x\| + \|y\|$. As usual, we denote by x^T the transposition of x and by $\langle x, y\rangle := x^T y$ the inner product of vectors x and y, respectively. For a given map $f : \mathbb{R}^n \to \mathbb{R}^p$ and a vector $\lambda \in \mathbb{R}^p$, the function $\langle \lambda, f\rangle : \mathbb{R}^n \to \mathbb{R}$ is defined by $\langle \lambda, f\rangle(x) := \langle \lambda, f(x)\rangle$ for all $x \in \mathbb{R}^n$. In general, we denote by $\mathscr{B}_{\mathbb{R}^n}$ the closed unit ball in $\mathbb{R}^n$ and by $\mathbb{B}(\hat{x}, r)$ the open ball with center at $\hat{x}$ and radius $r > 0$ for any $\hat{x} \in \mathbb{R}^n$.

The main tools for our study in this paper are the Mordukhovich generalized differentiation notions which are generally used in variational analysis and set-valued analysis; see more details in [6–8, 16] and references therein. Recall that $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be Fréchet differentiable at $\hat{x}$ if and only if there exists a matrix $A \in \mathbb{R}^{p\times n}$ such that

$$\lim_{x\to\hat{x}} \frac{\|f(x) - f(\hat{x}) - A(x-\hat{x})\|}{\|x-\hat{x}\|} = 0. \tag{4}$$

Obviously, A is uniquely determined by $\hat{x}$. As usual, A is called the Fréchet derivative of f at $\hat{x}$ and denoted by $\nabla f(\hat{x})$. If f is Fréchet differentiable at every $\hat{x} \in \mathbb{R}^n$, then f is said to be Fréchet differentiable on $\mathbb{R}^n$. f is said to be continuously Fréchet differentiable at $\hat{x}$ if and only if the map $\nabla f(\bullet) : \mathbb{R}^n \to \mathbb{R}^{p\times n}$ is continuous at $\hat{x}$. Specially, we denote by $(\nabla f(\hat{x}))^* : \mathbb{R}^p \to \mathbb{R}^n$ the adjoint operator of $\nabla f(\hat{x})$; that is, $\langle \nabla f(\hat{x})(x), y\rangle = \langle x, (\nabla f(\hat{x}))^*(y)\rangle$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. Moreover, f is said to be strictly differentiable at $\hat{x}$ if and only if

$$\lim_{x\to\hat{x}, u\to\hat{x}, x\neq u} \frac{\|f(x) - f(u) - \nabla f(\hat{x})(x-u)\|}{\|x-u\|} = 0. \tag{5}$$

Obviously, if f is continuously Fréchet differentiable at $\hat{x}$, then f is strictly differentiable at $\hat{x}$.

For a nonempty subset $S \subset \mathbb{R}^n$, the indicator function $\psi(\bullet, S) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $\psi(x, S) := 0$, $\forall x \in S$ and $\psi(x, S) := +\infty$, $\forall x \notin S$, and the distance function $d(\bullet, S) : \mathbb{R}^n \to \mathbb{R}$ is defined by $d(x, S) := \inf_{y\in S}\|x-y\|$ for all $x \in \mathbb{R}^n$, respectively. Given a point $\hat{x} \in S$, recall that the Fréchet normal cone $\widehat{N}(S, \hat{x})$ of S at $\hat{x}$, which is a convex, closed subset of $\mathbb{R}^n$ and consisted of all the Fréchet normals, has the form

$$\widehat{N}(S, \hat{x}) := \left\{x^* \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{S} \hat{x}} \frac{\langle x^*, x-\hat{x}\rangle}{\|x-\hat{x}\|} \le 0\right\}, \tag{6}$$

where $x \xrightarrow{S} \hat{x}$ means $x \in S$ and $x \to \hat{x}$. The Mordukhovich (or basic, limiting) normal cone of S at $\hat{x}$ is

$$\begin{aligned} N(S, \hat{x}) : = \Big\{x^* \in \mathbb{R}^n \mid \exists x_n \xrightarrow{S} \hat{x}, \exists x_n^* \longrightarrow x^* \\ \text{with } x_n^* \in \widehat{N}(S, x_n), \forall n \in \mathbb{N}\Big\}. \end{aligned} \tag{7}$$

Specially, if S is convex, then we have

$$\widehat{N}(S, \hat{x}) = N(S, \hat{x}) = \{x^* \in \mathbb{R}^n \mid \langle x^*, x-\hat{x}\rangle \le 0, \forall x \in S\}. \tag{8}$$

Let $h : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be an extended real-valued function and let $\hat{x} \in \operatorname{dom} h$, where $\operatorname{dom} h := \{x \in \mathbb{R}^n \mid h(x) < +\infty\}$ denotes the domain of h. The Fréchet subdifferential $\widehat{\partial} h(\hat{x})$ of h at $\hat{x}$ is defined in the geometric form by $\widehat{\partial} h(\hat{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, -1) \in \widehat{N}(\operatorname{epi} h, (\hat{x}, h(\hat{x})))\}$, or, equivalently, is defined in the analytical form by

$$\begin{aligned} &\widehat{\partial} h(\hat{x}) \\ &\quad := \left\{x^* \in \mathbb{R}^n \mid \liminf_{x\to\hat{x}, x\neq\hat{x}} \frac{h(x) - h(\hat{x}) - \langle x^*, x-\hat{x}\rangle}{\|x-\hat{x}\|} \ge 0\right\}. \end{aligned} \tag{9}$$

The Mordukhovich (or basic, limiting) subdifferential $\partial h(\hat{x})$ and singular subdifferential $\partial^\infty h(\hat{x})$ of h at $\hat{x}$ are defined, respectively, by $\partial h(\hat{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, -1) \in N(\operatorname{epi}\ h, (\hat{x}, h(\hat{x})))\}$ and $\partial^\infty h(\hat{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, 0) \in N(\operatorname{epi}\ h, (\hat{x}, h(\hat{x})))\}$. Clearly, we have $\widehat{\partial} h(\hat{x}) \subset \partial h(\hat{x})$ and

$$\begin{aligned} \partial h(\hat{x}) = \Big\{x^* \in \mathbb{R}^n \mid \exists x_n \xrightarrow{h} \hat{x}, \exists x_n^* \longrightarrow x^* \\ \text{with } x_n^* \in \widehat{\partial} h(x_n)\Big\}, \end{aligned} \tag{10}$$

where $x_n \xrightarrow{h} \hat{x}$ means $x_n \to \hat{x}$ and $h(x_n) \to h(\hat{x})$. Specially, for any $\hat{x} \in S$, it follows that $\widehat{\partial}\psi(\hat{x}, S) = \widehat{N}(S, \hat{x})$ and $\partial\psi(\hat{x}, S) = \partial^\infty\psi(\hat{x}, S) = N(S, \hat{x})$. Furthermore, if h is a convex function, then we have

$$\begin{aligned} &\widehat{\partial} h(\hat{x}) = \partial h(\hat{x}) \\ &\quad = \{x^* \in \mathbb{R}^n \mid \langle x^*, x-\hat{x}\rangle \le h(x) - h(\hat{x}), \forall x \in \mathbb{R}^n\}, \\ &\partial^\infty h(\hat{x}) \subset \{x^* \in \mathbb{R}^n \mid \langle x^*, x-\hat{x}\rangle \le 0, \forall x \in \operatorname{dom} h\} \\ &\quad = N(\operatorname{dom} h, \hat{x}). \end{aligned} \tag{11}$$

Recall that the Fréchet coderivative $\widehat{D}^* F(\hat{x}, \hat{y})$ and the Mordukhovich (or basic, limiting) coderivative $D^* F(\hat{x}, \hat{y})$ of the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ at $(\hat{x}, \hat{y}) \in \operatorname{gph} F$ are the set-valued maps from $\mathbb{R}^p$ to $\mathbb{R}^n$ defined, respectively, by

$$\begin{aligned} &\widehat{D}^* F(\hat{x}, \hat{y})(y^*) \\ &\quad := \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in \widehat{N}(\operatorname{gph} F, (\hat{x}, \hat{y}))\}, \\ &\qquad\qquad \forall y^* \in \mathbb{R}^p, \\ &D^* F(\hat{x}, \hat{y})(y^*) \\ &\quad := \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in N(\operatorname{gph} F, (\hat{x}, \hat{y}))\}, \\ &\qquad\qquad \forall y^* \in \mathbb{R}^p. \end{aligned} \tag{12}$$

Next, we collect some useful and important propositions and definitions for this paper.

Proposition 1 (see [8]). *For every nonempty subset $\Omega \subset \mathbb{R}^n$ and every $x \in \Omega$, we have*

(i) $\widehat{\partial} d(\bullet, \Omega)(x) = \mathscr{B}_{\mathbb{R}^n} \cap \widehat{N}(\Omega, x)$ *and* $\widehat{N}(\Omega, x) = \bigcup_{\lambda>0} \lambda\widehat{\partial} d(\bullet, \Omega)(x)$.

In addition, if Ω is closed, then we get

(ii) $\partial d(\bullet,\Omega)(x) \subset \mathscr{B}_{\mathbb{R}^n} \cap N(\Omega,x)$ *and* $N(\Omega,x) = \bigcup_{\lambda>0}\lambda\partial d(\bullet,\Omega)(x)$.

The following necessary optimality condition, called generalized Fermat rule, for a function to attain its local minimum is useful for our analysis.

Proposition 2 (see [7, 8]). *Let $\varphi : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function. If f attains a local minimum at $\hat{x} \in \mathbb{R}^n$, then $0_{\mathbb{R}^n} \in \hat{\partial} f(\hat{x})$ and $0_{\mathbb{R}^n} \in \partial f(\hat{x})$.*

We recall the following sum rule for the Mordukhovich subdifferential which is important in the sequel.

Proposition 3 (see [8]). *Let $\varphi_1, \varphi_2 : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be proper lower semicontinuous functions and $x \in \operatorname{dom}\varphi_1 \cap \operatorname{dom}\varphi_2$. Suppose that the qualification condition*

$$\partial^{\infty}\varphi_1(x) \cap \left(-\partial^{\infty}\varphi_2(x)\right) = \{0_{\mathbb{R}^n}\}, \tag{13}$$

is fulfilled. Then one has

$$\partial\left(\varphi_1+\varphi_2\right)(x) \subset \partial\varphi_1(x) + \partial\varphi_2(x). \tag{14}$$

Specially, if either φ_1 or φ_2 is locally Lipschitz around x, then one always has

$$\partial\left(\varphi_1+\varphi_2\right)(x) \subset \partial\varphi_1(x) + \partial\varphi_2(x). \tag{15}$$

The following propositions of the scalarization of Mordukhovich coderivatives and the chain rule of Mordukhovich subdifferentials are important for this paper.

Proposition 4 (see [8, 16]). *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ be continuous around $\hat{x}$. Then*

$$\partial\langle y^*,\varphi\rangle(\hat{x}) \subset D^*\varphi(\hat{x})(y^*), \quad \forall y^* \in \mathbb{R}^p. \tag{16}$$

If in addition φ is locally Lipschitz around $\hat{x}$, then

$$D^*\varphi(\hat{x})(y^*) = \partial\langle y^*,\varphi\rangle(\hat{x}), \quad \forall y^* \in \mathbb{R}^p. \tag{17}$$

Proposition 5 (see [8, 16]). *Let the vector-valued map $H : \mathbb{R}^n \to \mathbb{R}^\ell$ be locally Lipschitz and let $h : \mathbb{R}^\ell \to \mathbb{R}$ be lower semicontinuous. If*

$$\begin{aligned} y^* \in \partial^{\infty} h(H(x)), \quad 0_{\mathbb{R}^n} \in D^*H(x)(y^*) \\ \text{implies } y^* = 0_{\mathbb{R}^\ell}, \end{aligned} \tag{18}$$

then

$$\partial h \circ H(x) \subset \{D^*H(x)(y^*) : y^* \in \partial h(H(x))\}. \tag{19}$$

Moreover, if H is strictly differentiable and h is locally Lipschitz, then one always has

$$\partial h \circ H(x) \subset \{(\nabla H(x))^*(y^*) : y^* \in \partial h(H(x))\}. \tag{20}$$

Finally in this section, we recall the following useful concept called nonlinear scalar function and some of its properties. For more details, we refer to [17–20].

Lemma 6. *Given $e = (1,1,\ldots,1) \in \operatorname{int}\mathbb{R}^p_+$, the nonlinear scalar function $\xi_e : \mathbb{R}^p \to \mathbb{R}$, defined by*

$$\xi_e(y) := \inf\{\alpha \in \mathbb{R} \mid y \in \alpha e - \mathbb{R}^p_+\}, \quad \forall y \in \mathbb{R}^p, \tag{21}$$

is convex, strictly $\operatorname{int}\mathbb{R}^p_+$-monotone, $\mathbb{R}^p_+$-monotone, nonnegative homogeneous, globally Lipschitz with modulus $d(e, bd\ \mathbb{R}^p_+)^{-1}$. Simultaneously, for every $\alpha \in \mathbb{R}$, it follows that $\xi_e(\alpha e) = \alpha$,

$$\begin{aligned} \{y \in \mathbb{R}^p \mid \xi_e(y) \le \alpha\} &= \alpha e - \mathbb{R}^p_+, \\ \{y \in \mathbb{R}^p \mid \xi_e(y) < \alpha\} &= \alpha e - \operatorname{int}\mathbb{R}^p_+. \end{aligned} \tag{22}$$

Furthermore, for every $\hat{y} \in \mathbb{R}^p$,

$$\partial\xi_e(\hat{y}) = \left\{\lambda \in \mathbb{R}^p_+ \mid \sum_{i=1}^{p}\lambda_i = 1, \langle\lambda,\hat{y}\rangle = \xi_e(\hat{y})\right\}. \tag{23}$$

Specially, one has

$$\partial\xi_e(0_{\mathbb{R}^p}) = \left\{\lambda \in \mathbb{R}^p_+ \mid \sum_{i=1}^{p}\lambda_i = 1\right\}. \tag{24}$$

3. Exact Penalization, Calmness Condition, and Necessary Optimality Condition for (MOPEC)

In this section, we focus our attention on establishing some equivalent properties between a multiobjective exact penalization and a calmness condition, called (MOPEC-) calmness, for (MOPEC). Simultaneously, we show that a local error bound condition associated merely with the constraint system, equivalently, a calmness condition of the parametric constraint system, implies the (MOPEC-) calmness condition. Subsequently, we apply a nonlinear scalar method to obtain a M-stationary necessary optimality condition under the (MOPEC-) calmness condition.

Consider the following parametric form of the feasible set S with parameter $(u,v,y,z) \in \mathbb{R}^{r+s+n+m}$:

$$\begin{aligned} g(x)+u \in -\mathbb{R}^r_+, \qquad h(x)+v = 0_{\mathbb{R}^s}, \\ z \in q(x) + Q(x+y), \quad x \in \Theta. \end{aligned} \tag{25}$$

Denote the corresponding feasible set by

$$\begin{aligned} S(u,v,y,z) : = \{x \in \mathbb{R}^n \mid g(x)+u \in -\mathbb{R}^r_+, h(x)+v = 0_{\mathbb{R}^s}, \\ z \in q(x)+Q(x+y), x \in \Theta\}. \end{aligned} \tag{26}$$

Obviously, for the set-valued map $S : \mathbb{R}^{r+s+n+m} \rightrightarrows \mathbb{R}^n$, we have $S = S(0_{\mathbb{R}^{r+s+n+m}})$.

We are now in the position to introduce a (MOPEC-) calmness concept for (MOPEC).

Definition 7. Given $\sigma > 0$ and $\hat{x} \in S$ being a local efficient (resp. local weak efficient) solution for (MOPEC), then (MOPEC) is said to be (MOPEC-) calm with order σ at $\hat{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that, for all $(u, v, y, z) \in \mathbb{B}(0_{\mathbb{R}^{r+s+n+m}}, \delta)$ and all $x \in S(u, v, y, z) \cap \mathbb{B}(\hat{x}, \delta)$, one has

$$f(x) + M\|(u, v, y, z)\|^{\sigma} e \notin f(\hat{x}) - \operatorname{int} \mathbb{R}_{+}^{p}. \tag{27}$$

Remark 8. Given $\sigma > 0$ and $\hat{x} \in S$ being a local efficient (resp. local weak efficient) solution for (MOPEC), we can also characterize the (MOPEC-) calmness condition by means of sequences. It is easy to verify that (MOPEC) is (MOPEC-) calm with order σ at $\hat{x}$ if and only if there exists $M > 0$ such that, for every sequence $\{(u_k, v_k, y_k, z_k)\} \subset \mathbb{R}^{r+s+n+m}$ with $(u_k, v_k, y_k, z_k) \to 0_{\mathbb{R}^{r+s+n+m}}$ and every sequence $\{x_k\} \subset \Theta$ satisfying $g(x_k) + u_k \in \mathbb{R}_{+}^{r}$, $h(x_k) + v_k = 0_{\mathbb{R}^s}$, $z_k \in g(x_k) + Q(x_k + y_k)$ and $x_k \to \hat{x}$, it holds that

$$f(x_k) + M\|(u_k, v_k, y_k, z_k)\|^{\sigma} e \notin f(\hat{x}) - \operatorname{int} \mathbb{R}_{+}^{p}. \tag{28}$$

Note that the (MOPEC-) calmness condition depends on not only the objective function but also the constraint system. In order to make up this deficiency, we propose the following local error bound notion for (MOPEC) associated merely with the constraint system.

Definition 9. Given $\sigma > 0$ and $\hat{x} \in S$, then the constraint system of (MOPEC) is said to have a local error bound with order σ at $\hat{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that, for all $(u, v, y, z) \in \mathbb{B}(0_{\mathbb{R}^{r+s+n+m}}, \delta) \setminus \{0_{\mathbb{R}^{r+s+n+m}}\}$ and all $x \in S(u, v, y, z) \cap \mathbb{B}(\hat{x}, \delta)$, one has

$$d(x, S)\mathscr{B}_{\mathbb{R}^p} \subset M\|(u, v, y, z)\|^{\sigma} e - \operatorname{int} \mathbb{R}_{+}^{p}. \tag{29}$$

Next, we show that the local error bound implies the (MOPEC-) calmness.

Theorem 10. *Let $\hat{x} \in S$ be a local efficient (resp. local weak efficient) solution for (MOPEC). If the constraint system of (MOPEC) has a local error bound with order σ at $\hat{x}$, then (MOPEC) is (MOPEC-) calm with order σ at $\hat{x}$.*

Proof. Since $\hat{x} \in S$ is a local efficient (resp. local weak efficient) solution for (MOPEC) and $S = S(0_{\mathbb{R}^{r+s+n+m}})$, it immediately follows that

$$f(x) + M\|(u, v, y, z)\|^{\sigma} e \notin f(\hat{x}) - \operatorname{int} \mathbb{R}_{+}^{p} \tag{30}$$

holds for all $x \in S(u, v, y, z) \cap \mathbb{B}(\hat{x}, \delta)$ with $(u, v, y, z) = 0_{\mathbb{R}^{r+s+n+m}}$ and sufficiently small $\delta > 0$. Thus, we only need to prove the case $(u, v, y, z) \neq 0_{\mathbb{R}^{r+s+n+m}}$. Assume that (MOPEC) is not (MOPEC-) calm with order σ at $\hat{x}$. Then, for every $k \in \mathbb{N}$, there exist $(u_k, v_k, y_k, z_k) \in \mathbb{B}(0_{\mathbb{R}^{r+s+n+m}}, 1/k) \setminus \{0_{\mathbb{R}^{r+s+n+m}}\}$ and $x_k \in S(u_k, v_k, y_k, z_k) \cap \mathbb{B}(\hat{x}, 1/k)$ such that

$$f(x_k) + k\|(u_k, v_k, y_k, z_k)\|^{\sigma} e \in f(\hat{x}) - \operatorname{int} \mathbb{R}_{+}^{p}. \tag{31}$$

Since S is nonempty and closed, there exists a projection $P(x_k, S)$ of x_k onto S such that $d(x_k, S) = \|x_k - P(x_k, S)\|$ for all $k \in \mathbb{N}$. Note that $x_k \to \hat{x}$ and $\hat{x} \in S$. Then it follows that

$$\begin{aligned} \|P(x_k, S) - \hat{x}\| &\leq \|P(x_k, S) - x_k\| + \|x_k - \hat{x}\| \\ &= d(x_k, S) + \|x_k - \hat{x}\| \longrightarrow 0. \end{aligned} \tag{32}$$

Together with $\hat{x} \in S$ being a local efficient (resp. local weak efficient) solution for (MOPEC), there exists some $N_1 \in \mathbb{N}$ such that

$$\begin{aligned} f(P(x_k, S)) - f(\hat{x}) \notin -\mathbb{R}_{+}^{p} \setminus \{0_{\mathbb{R}^p}\} \left(\text{resp.} - \operatorname{int} \mathbb{R}_{+}^{p}\right), \\ \forall k \geq N_1. \end{aligned} \tag{33}$$

Moreover, since f is locally Lipschitz, there exist $L > 0$ and $N_2 \in \mathbb{N}$ such that

$$\|f(x_k) - f(P(x_k, S))\| \leq L\|x_k - P(x_k, S)\|, \quad \forall k \geq N_2. \tag{34}$$

By (31) and (33), we have for all $k \geq N_1$

$$\begin{aligned} &f(P(x_k, S)) - f(x_k) \\ &\quad = f(P(x_k, S)) - f(\hat{x}) \\ &\qquad + (f(\hat{x}) - f(x_k)) \notin k\|(u_k, v_k, y_k, z_k)\|^{\sigma} e - \operatorname{int} \mathbb{R}_{+}^{p}. \end{aligned} \tag{35}$$

Together with $d(x_k, S) = \|x_k - P(x_k, S)\|$ and (34), we can conclude that

$$\begin{aligned} d(x_k, S)\mathscr{B}_{\mathbb{R}^p} \not\subset \frac{k}{L}\|(u_k, v_k, y_k, z_k)\|^{\sigma} e - \operatorname{int} \mathbb{R}_{+}^{p}, \\ \forall k \geq \max\{N_1, N_2\}. \end{aligned} \tag{36}$$

This is a contradiction to the assumption that (MOPEC) has a local error bound with order σ at $\hat{x}$ since $k/L \to +\infty$, $(u_k, v_k, y_k, z_k) \neq 0_{\mathbb{R}^{r+s+n+m}}$, $(u_k, v_k, y_k, z_k) \to 0_{\mathbb{R}^{r+s+n+m}}$, $x_k \in S(u_k, v_k, y_k, z_k)$, and $x_k \to \hat{x}$. □

Remark 11. Specially, if we consider the case $p = 1$ for every given $\sigma > 0$ and $\hat{x} \in S$, then Definition 9 reduces to the fact that there exist $\delta > 0$ and $M > 0$ such that, for all $(u, v, y, z) \in \mathbb{B}(0_{\mathbb{R}^{r+s+n+m}}, \delta) \setminus \{0_{\mathbb{R}^{r+s+n+m}}\}$ and all $x \in S(u, v, y, z) \cap \mathbb{B}(\hat{x}, \delta)$, one has

$$d(x, S) < M\|(u, v, y, z)\|^{\sigma}. \tag{37}$$

It is worth noting that this condition is essentially sufficient and necessary for the situation $p > 1$. Clearly, the necessity holds. In fact, since $(1/p)e \in \mathscr{B}_{\mathbb{R}^p}$, it follows that

$$d(x, S)\frac{1}{p}e \in M\|(u, v, y, z)\|^{\sigma} e - \operatorname{int} \mathbb{R}_{+}^{p}, \tag{38}$$

for all $(u, v, y, z) \in \mathbb{B}(0_{\mathbb{R}^{r+s+n+m}}, \delta) \setminus \{0_{\mathbb{R}^{r+s+n+m}}\}$ and all $x \in S(u, v, y, z) \cap \mathbb{B}(\hat{x}, \delta)$. Moreover, ξ_e is nonnegative homogeneous and $\xi_e((1/p)e) = 1/p$. By Lemma 6, we have $\xi_e(d(x, S)(1/p)e) < M\|(u, v, y, z)\|^{\sigma}$, which implies $d(x, S) < pM\|(u, v, y, z)\|^{\sigma}$. For the sufficiency, since ξ_e is continuous and $\mathscr{B}_{\mathbb{R}^p}$ is compact, there exists some $m \in \mathbb{R}$ such that

$$m = \max_{w \in \mathscr{B}_{\mathbb{R}^p}} \xi_e(w). \tag{39}$$

Obviously, $(1/p)e \in \mathscr{B}_{\mathbb{R}^p}$ and $\xi_e((1/p)e) = 1/p > 0$; then we have $m > 0$. Thus, we get from the nonnegative homogeneity of ξ_e that, for all $(u, v, y, z) \in \mathbb{B}(0_{\mathbb{R}^{r+s+n+m}}, \delta) \setminus \{0_{\mathbb{R}^{r+s+n+m}}\}$, all $x \in S(u, v, y, z) \cap \mathbb{B}(\hat{x}, \delta)$ and all $w \in \mathscr{B}_{\mathbb{R}^p}$,

$$\xi_e(d(x, S)w) = d(x, S)\xi_e(w) < mM\|(u, v, y, z)\|^\sigma. \quad (40)$$

By Lemma 6, we have

$$d(x, S)w \in mM\|(u, v, y, z)\|^\sigma e - \operatorname{int}\mathbb{R}^p_+, \quad (41)$$

which implies

$$d(x, S)\mathscr{B}_{\mathbb{R}^p} \subset mM\|(u, v, y, z)\|^\sigma e - \operatorname{int}\mathbb{R}^p_+. \quad (42)$$

Furthermore, recall that a set-valued map $\Psi : \mathbb{R}^t \rightrightarrows \mathbb{R}^n$ is said to be calm with order $\sigma > 0$ at $(\overline{x}, \overline{y}) \in \operatorname{gph}\Psi$ if and only if there exist neighborhoods U of $\overline{x}$ and V of $\overline{y}$ and a real number $\ell > 0$ such that

$$\Psi(x) \cap V \subset \Psi(\overline{x}) + \ell\|x - \overline{x}\|^\sigma \mathscr{B}_{\mathbb{R}^n}, \quad \forall x \in U. \quad (43)$$

Then we can immediately obtain the following characterization of local error bounds for the constraint system of (MOPEC) based on the arguments in Remark 11.

Proposition 12. *Given $\sigma > 0$ and $\hat{x} \in S$, then the following assertions are equivalent.*

(i) *The constraint system of (MOPEC) has a local error bound with order σ at $\hat{x}$.*

(ii) *There exist $\delta > 0$ and $M > 0$ such that, for all $(u, v, y, z) \in \mathbb{B}(0_{\mathbb{R}^{r+s+n+m}}, \delta) \setminus \{0_{\mathbb{R}^{r+s+n+m}}\}$ and all $x \in S(u, v, y, z) \cap \mathbb{B}(\hat{x}, \delta)$,*

$$d(x, S(0_{\mathbb{R}^{r+s+n+m}})) < M\|(u, v, y, z)\|^\sigma. \quad (44)$$

(iii) *The set-valued map $S : \mathbb{R}^{r+s+n+m} \rightrightarrows \mathbb{R}^n$, defined by (26), is calm with order σ at $(0_{\mathbb{R}^{r+s+n+m}}, \hat{x})$.*

Proof. As discussed in Remark 11, (i) is equivalent to (ii). We only need to prove the equivalence of (ii) and (iii). In fact, it follows from the definition of calmness for a set-valued map that (ii) is obviously equivalent to the calmness with order σ at $(0_{\mathbb{R}^{r+s+n+m}}, \hat{x})$ of the set-valued map S. □

As we know, there have been many papers devoted to investigate the calmness of a set-valued map Ψ (which is equivalent to the metric subregularity of its converse Ψ^{-1}). For more details, we refer to [21–24] and references therein. It has been shown in Remark 11 and Proposition 12 that there have been no differences between the scalar ($p = 1$) and the multiobjective ($p > 1$) settings when we only consider the calmness or the local error bound for the constraint system of (MOPEC). However, if we pay attention to the weaker (MOPEC-) calmness, we cannot negative the differences between them.

We now give the following equivalent characterizations of two classes of multiobjective penalty problems and the (MOPEC-) calmness condition.

Theorem 13. *Let $\hat{x} \in S$ be a local efficient (resp. local weak efficient) solution for (MOPEC). Then the following assertions are equivalent.*

(i) *(MOPEC) is (MOPEC-) calm with order $\sigma > 0$ at $\hat{x}$.*

(ii) *There exists some $\hat{\rho} > 0$ such that, for any $\rho \geq \hat{\rho}$, $(\hat{x}, 0_{\mathbb{R}^{n+m}})$ is a local efficient (resp. local weak efficient) solution for the following multiobjective penalty problem with order σ:*

$(MPP)_I$

$$\begin{aligned} \min \quad & f(x) + \rho(\|g_+(x)\| + \|h(x)\| + \|(y, z)\|)^\sigma e, \\ \text{s.t.} \quad & z \in q(x) + Q(x + y), \\ & x \in \Theta, (y, z) \in \mathbb{R}^{n+m}, \end{aligned} \quad (45)$$

where $g_+(x) := (\max\{g_1(x), 0\}, \max\{g_2(x), 0\}, \ldots, \max\{g_r(x), 0\})$.

(iii) *There exists some $\hat{\mu} > 0$ such that, for any $\mu \geq \hat{\mu}$, $\hat{x}$ is a local efficient (resp. local weak efficient) solution for the following multiobjective penalty problem with order σ:*

$(MPP)_{II}$

$$\begin{aligned} \min \quad & f(x) \\ & + \mu[\|g_+(x)\| + \|h(x)\| + d((x, -q(x)), gphQ)]^\sigma e, \\ \text{s.t.} \quad & x \in \Theta. \end{aligned} \quad (46)$$

Proof. We only prove the case for $\hat{x}$ being a local weak efficient solution since the proof of the case for $\hat{x}$ being a local efficient solution is similar.

(i)⇒(ii). Suppose to the contrary that, for every $k \in \mathbb{N}$, there exists $(x_k, y_k, z_k) \in \mathbb{B}((\hat{x}, 0_{\mathbb{R}^{n+m}}), 1/k)$ with $x_k \in \Theta$ and $z_k \in q(x_k) + Q(x_k + y_k)$ such that

$$\begin{aligned} f(x_k) + k(\|g_+(x_k)\| + \|h(x_k)\| & \\ + \|(y_k, z_k)\|)^\sigma e \in f(\hat{x}) - \operatorname{int}\mathbb{R}^p_+. & \end{aligned} \quad (47)$$

Take $u_k = -g_+(x_k)$ and $v_k = -h(x_k)$. Then it follows that $g(x_k) + u_k \in -\mathbb{R}^r_+$ and $h(x_k) + v_k = 0_{\mathbb{R}^s}$. Together with $z_k \in q(x_k) + Q(x_k + y_k)$ and $x_k \in \Theta$, we get $x_k \in S(u_k, v_k, y_k, z_k)$ for all $k \in \mathbb{N}$. Moreover, by (47), we have

$$f(x_k) + k\|(u_k, v_k, y_k, z_k)\|^\sigma e \in f(\hat{x}) - \operatorname{int}\mathbb{R}^p_+. \quad (48)$$

Note that $x_k \to \hat{x}$, $g(\hat{x}) \in -\mathbb{R}^r_+$, $h(\hat{x}) = 0_{\mathbb{R}^s}$, and g and h are continuously Fréchet differentiable. Then it follows that $u_k = g_+(x_k) \to 0_{\mathbb{R}^r}$ and $v_k = h(x_k) \to 0_{\mathbb{R}^s}$. Together with $(y_k, z_k) \to 0_{\mathbb{R}^{n+m}}$ and (48), this is a contradiction to the (MOPEC-) calmness with order σ of (MOPEC) at $\hat{x}$.

(ii)⇒(i). Suppose that (MOPEC) is not (MOPEC-) calm with order $\sigma > 0$ at $\hat{x}$. Then, for every $k \in \mathbb{N}$, there exist $(u_k, v_k, y_k, z_k) \in \mathbb{B}(0_{\mathbb{R}^{r+s+n+m}}, 1/k)$ and $x_k \in S(u_k, v_k, y_k, z_k) \cap \mathbb{B}(\hat{x}, 1/k)$ such that (31) holds. Since $x_k \in S(u_k, v_k, y_k, z_k)$, it follows that $x_k \in \Theta$, $g(x_k) + u_k \in -\mathbb{R}^r_+$, $h(x_k) + v_k = 0_{\mathbb{R}^s}$ and

$z_k \in q(x_k) + Q(x_k + y_k)$; that is, $(x_k + y_k, z_k - q(x_k)) \in \text{gph}Q$ for all $k \in \mathbb{N}$. Thus, we have

$$\begin{aligned}&\|g_+(x_k)\| + \|h(x_k)\| \\ &\quad \le \|g(x_k) - (g(x_k) + u_k)\| + \|h(x_k) - (h(x_k) + v_k)\| \\ &\quad = \|u_k\| + \|v_k\|, \quad \forall k \in \mathbb{N},\end{aligned} \tag{49}$$

which implies $(\|g_+(x_k)\| + \|h(x_k)\| + \|(y_k, z_k)\|)^\sigma \le \|(u_k, v_k, y_k, z_k)\|^\sigma$, $\forall k \in \mathbb{N}$. Together with (31), we get

$$\begin{aligned}&f(x_k) + k(\|g_+(x_k)\| + \|h(x_k)\| + \|(y_k, z_k)\|)^\sigma e \\ &\quad = f(x_k) + k\|(u_k, v_k, y_k, z_k)\|^\sigma e \\ &\qquad + k\left[(\|g_+(x_k)\| + \|h(x_k)\| + \|(y_k, z_k)\|)^\sigma \right. \\ &\qquad \left. - \|(u_k, v_k, y_k, z_k)\|^\sigma\right] e \in f(\hat{x}) \\ &\qquad - \operatorname{int}\mathbb{R}^p_+ - \operatorname{int}\mathbb{R}^p_+ \\ &\quad = f(\hat{x}) - \operatorname{int}\mathbb{R}^p_+, \quad \forall k \in \mathbb{N}.\end{aligned} \tag{50}$$

This shows that the multiobjective penalty problem $(\text{MPP})_\text{I}$ with order σ does not admit a local exact penalization at $(\hat{x}, 0_{\mathbb{R}^{n+m}})$ since $x_k \in \Theta$, $z_k \in q(x_k) + Q(x_k + y_k)$, and $(x_k, y_k, z_k) \to (\hat{x}, 0_{\mathbb{R}^{n+m}})$.

(i)⇒(iii). Assume that, for every $k \in \mathbb{N}$, there exists $x_k \in \Theta \cap \mathbb{B}(\hat{x}, 1/k)$ such that

$$\begin{aligned}&f(x_k) \\ &\quad + k[\|g_+(x_k)\| + \|h(x_k)\| + d((x_k, -q(x_k)), \text{gph}Q)]^\sigma e \\ &\quad - f(\hat{x}) \in -\operatorname{int}\mathbb{R}^p_+.\end{aligned} \tag{51}$$

Note that

$$d((x_k, -q(x_k)), \text{gph}Q) = \inf_{\beta \in Q(\alpha)} \|(x_k, -q(x_k)) - (\alpha, \beta)\|. \tag{52}$$

Thus, for every $k \in \mathbb{N}$, there exists $(\alpha_k, \beta_k) \in \mathbb{R}^{n+m}$ with $\beta_k \in Q(\alpha_k)$ such that

$$\begin{aligned}&\|(x_k, -q(x_k)) - (\alpha_k, \beta_k)\| \\ &\quad \le \left(1 + \frac{1}{k}\right) d((x_k, -q(x_k)), \text{gph}Q).\end{aligned} \tag{53}$$

Take $u_k = -g_+(x_k)$, $v_k = -h(x_k)$, $y_k = \alpha_k - x_k$, and $z_k = q(x_k) + \beta_k$. Then it follows that $g(x_k) + u_k \in -\mathbb{R}^r_+$, $h(x_k) + v_k = 0_{\mathbb{R}^s}$, and $z_k \in q(x_k) + Q(x_k + y_k)$, which implies $x_k \in S(u_k, v_k, y_k, z_k)$ since $x_k \in \Theta$, and

$$\begin{aligned}&(\|u_k\| + \|v_k\| + \|(y_k, z_k)\|)^\sigma \\ &\quad \le \left(1 + \frac{1}{k}\right)^\sigma [\|g_+(x_k)\| + \|h(x_k)\| \\ &\qquad + d((x_k, -q(x_k)), \text{gph}Q)]^\sigma.\end{aligned} \tag{54}$$

Connecting $e \in \operatorname{int} \mathbb{R}^p_+$, (51), and (54), we have for any $k \in \mathbb{N}$

$$\begin{aligned}&f(x_k) + \frac{k^{\sigma+1}}{(k+1)^\sigma}\|(u_k, v_k, y_k, z_k)\|^\sigma e - f(\hat{x}) \\ &\quad = f(x_k) + k[\|g_+(x_k)\| + \|h(x_k)\| \\ &\qquad + d((x_k, -q(x_k)), \text{gph}Q)]^\sigma e - f(\hat{x}) \\ &\qquad + \frac{k^{\sigma+1}}{(k+1)^\sigma}\left\{(\|u_k\| + \|v_k\| + \|(y_k, z_k)\|)^\sigma - \left(1 + \frac{1}{k}\right)^\sigma \right. \\ &\qquad \times [\|g_+(x_k)\| + \|h(x_k)\| \\ &\qquad \left. + d((x_k, -q(x_k)), \text{gph}Q)]^\sigma \right\} e \\ &\quad \in -\operatorname{int}\mathbb{R}^p_+ - \operatorname{int}\mathbb{R}^p_+ = -\operatorname{int}\mathbb{R}^p_+.\end{aligned} \tag{55}$$

Moreover, it follows from [25, Lemma 3.21] and (51) that for any $\lambda \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$ we get

$$\begin{aligned}&[\|g_+(x_k)\| + \|h(x_k)\| + d((x_k, -q(x_k)), \text{gph}Q)]^\sigma \\ &\quad \le \frac{1}{k}\sum_{i=1}^p \lambda_i (f_i(\hat{x}) - f_i(x_k)), \quad \forall k \in \mathbb{N}.\end{aligned} \tag{56}$$

Note that f is locally Lipschitz and $x_k \to \hat{x}$. Then it follows that $[\|g_+(x_k)\| + \|h(x_k)\| + d((x_k, -q(x_k)), \text{gph}Q)]^\sigma \to 0$, which implies $(u_k, v_k, y_k, z_k) \to 0_{\mathbb{R}^{r+s+n+m}}$ from (54). Together with $k^{\sigma+1}/(k+1)^\sigma \to +\infty$, $x_k \in S(u_k, v_k, y_k, z_k)$, $x_k \to \hat{x}$, and (55), this is a contradiction to the (MOPEC-) calmness with order σ of (MOPEC) at $\hat{x}$.

(iii)⇒(i). Assume that (MOPEC) is not (MOPEC-) calm with order $\sigma > 0$ at $\hat{x}$. Then, by the same argument to the proof of (ii)⇒(i), it follows that, for every $k \in \mathbb{N}$, there exist $(u_k, v_k, y_k, z_k) \in \mathbb{B}(0_{\mathbb{R}^{r+s+n+m}}, 1/k)$ and $x_k \in \mathbb{B}(\hat{x}, 1/k)$ with $x_k \in \Theta$, $g(x_k) + u_k \in -\mathbb{R}^r_+$, $h(x_k) + v_k = 0_{\mathbb{R}^s}$ and $z_k \in q(x_k) + Q(x_k + y_k)$; that is, $(x_k + y_k, z_k - q(x_k)) \in \text{gph}Q$ such that (31) holds. Thus, we have

$$\begin{aligned}&[\|g_+(x_k)\| + \|h(x_k)\| + d((x_k, -q(x_k)), \text{gph}Q)]^\sigma \\ &\quad \le [\|g(x_k) - (g(x_k) + u_k)\| \\ &\qquad + \|h(x_k) - (h(x_k) + v_k)\| \\ &\qquad + \|(x_k, -q(x_k)) - (x_k + y_k, z_k - q(x_k))\|]^\sigma \\ &\quad = \|(u_k, v_k, y_k, z_k)\|^\sigma, \quad \forall k \in \mathbb{N}.\end{aligned} \tag{57}$$

Together with (31) and $e \in \operatorname{int} \mathbb{R}^p_+$, we get

$$\begin{aligned} & f(x_k) + k\left[\|g_+(x_k)\| + \|h(x_k)\| \right. \\ & \qquad \left. + d\left((x_k, -q(x_k)), \mathrm{gphQ}\right)\right]^{\sigma} e \\ & = f(x_k) + k\|(u_k, v_k, y_k, z_k)\|^{\sigma} e \\ & \quad + k\left(\left[\|g_+(x_k)\| + \|h(x_k)\| + d\left((x_k, -q(x_k)), \mathrm{gphQ}\right)\right]^{\sigma}\right. \\ & \quad \left. - \|(u_k, v_k, y_k, z_k)\|^{\sigma}\right) e \in f(\hat{x}) - \operatorname{int}\mathbb{R}^p_+ - \operatorname{int}\mathbb{R}^p_+ \\ & = f(\hat{x}) - \operatorname{int}\mathbb{R}^p_+, \quad \forall k \in \mathbb{N}, \end{aligned} \tag{58}$$

which implies that the multiobjective penalty problem $(\mathrm{MPP})_{\mathrm{II}}$ with order σ does not admit a local exact penalization at $\hat{x}$ since the sequence $\{x_k\} \subset \Theta$ and $x_k \to \hat{x}$. □

It is well known that a calmness condition with order 1 for standard nonlinear programming can lead to a KKT condition. In fact, we can also obtain a M-stationary condition for (MOPEC) under the (MOPEC-) calmness condition with order 1. To this end, we need the following generalized Fermat rule for a multiobjective optimization problem with an abstract constraint, which is established by applying the nonlinear scalar function in Lemma 6.

Lemma 14. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^p$ be a locally Lipschitz vector-valued map, and let $\Omega \subset \mathbb{R}^n$ be a nonempty and closed subset. If $\hat{x} \in \Omega$ is a local weak efficient solution for the multiobjective optimization problem*

$$\begin{aligned} \min \quad & \phi(x) \\ \text{s.t.} \quad & x \in \Omega, \end{aligned} \tag{59}$$

then there exists some $\lambda \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$ such that

$$0_{\mathbb{R}^n} \in \sum_{i=1}^{p} \partial \langle \lambda, \phi \rangle (\hat{x}) + N(\Omega, \hat{x}). \tag{60}$$

Proof. Define the function $\Phi : \mathbb{R}^n \to \mathbb{R}$ by

$$\Phi(x) := \xi_e\left(\phi(x) - \phi(\hat{x})\right) + \psi(\Omega, x), \quad \forall x \in \mathbb{R}^n. \tag{61}$$

Since $\hat{x} \in \Omega$ is a local weak efficient solution, Φ attains a local minimum at $\hat{x}$. Otherwise, there exists a sequence $\{x_n\} \subset \mathbb{R}^n$ converging to $\hat{x}$ such that $\Phi(x_n) < 0$ since $\Phi(\hat{x}) = 0$. Then we have $\{x_n\} \subset \Omega$ and $\Phi(x_n) = \xi_e(\phi(x_n) - \phi(\hat{x})) < 0$. Together with Lemma 6, we get

$$\phi(x_n) - \phi(\hat{x}) \in -\operatorname{int}\mathbb{R}^p_+, \quad \forall n \in \mathbb{N}. \tag{62}$$

This is a contradiction to $\hat{x} \in \Omega$ being a local weak efficient solution since $\{x_n\} \subset \Omega$ and $x_n \to \hat{x}$. Note that ϕ is locally Lipschitz and Ω is closed. It follows from Propositions 2, 3, and 5 and Lemma 6 that

$$\begin{aligned} 0_{\mathbb{R}^n} & \in \partial\Phi(\hat{x}) \subset \partial\xi_e\left(\phi(\bullet) - \phi(\hat{x})\right)(\hat{x}) + \partial\psi(\Omega, \bullet)(\hat{x}) \\ & \subset \left\{D^*\phi(\hat{x})(\lambda) : \lambda \in \partial\xi_e\left(0_{\mathbb{R}^p}\right)\right\} + N(\Omega, \hat{x}) \\ & = \left\{D^*\phi(\hat{x})(\lambda) : \lambda \in \mathbb{R}^p_+, \sum_{i=1}^{p}\lambda_i = 1\right\} + N(\Omega, \hat{x}). \end{aligned} \tag{63}$$

Therefore, there exists some $\lambda \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$ such that

$$0_{\mathbb{R}^n} \in D^*\phi(\hat{x})(\lambda) + N(\Omega, \hat{x}). \tag{64}$$

By Proposition 4, it follows that

$$0_{\mathbb{R}^n} \in \partial\langle\lambda, \phi\rangle(\hat{x}) + N(\Omega, \hat{x}). \tag{65}$$

This completes the proof. □

Next, we show that the (MOPEC-) calmness condition with order 1 is sufficient to establish a M-stationary condition for (MOPEC).

Theorem 15. *Suppose that $\hat{x} \in S$ is a local weak efficient solution for (MOPEC) and (MOPEC) is (MOPEC-) calm with order 1 at $\hat{x}$. Then $\hat{x}$ is a M-stationary point for (MOPEC); that is, there exist $\lambda \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$, $\beta \in \mathbb{R}^r_+$, $\gamma \in \mathbb{R}^s$, $\tau > 0$, and $(x^*, y^*) \in \mathbb{R}^{n+m}$ with $x^* \in D^*Q(\hat{x}, -q(\hat{x}))(y^*)$ such that*

$$\begin{aligned} 0_{\mathbb{R}^n} & \in \partial\langle\lambda, f\rangle(\hat{x}) + \sum_{i=1}^{r}\beta_i\nabla g_i(\hat{x}) + \sum_{i=1}^{s}\gamma_i\nabla h_i(\hat{x}) \\ & \quad + \tau\left(x^* + \left(\nabla q(\hat{x})\right)^*\left(y^*\right)\right) + N(\Theta, \hat{x}), \\ & \beta_i g_i(\hat{x}) = 0, \quad \forall i = 1, 2, \ldots, r. \end{aligned} \tag{66}$$

Proof. Since $\hat{x} \in S$ is a local weak efficient solution for (MOPEC) and (MOPEC) is (MOPEC-) calm with order 1 at $\hat{x}$, it follows from Theorem 13 (i)⇔(iii) that there exists some $\hat{\mu} > 0$ such that $\hat{x}$ is a local weak efficient solution for the multiobjective penalty problem $(\mathrm{MPP})_{\mathrm{II}}$ with order 1. For simplicity, let the real-valued function $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{T}(x) = \|g_+(x)\| + \|h(x)\| + d\left((x, -q(x)), \mathrm{gphQ}\right), \\ \forall x \in \mathbb{R}^n. \end{aligned} \tag{67}$$

Note that f is locally Lipschitz, g, h, and q are continuously Fréchet differentiable, and Q is closed. Then $\mathcal{T}$ is locally Lipschitz and the penalty function $f(\bullet) + \hat{\mu}\mathcal{T}(\bullet)e : \mathbb{R}^n \to \mathbb{R}^p$ is also locally Lipschitz. Together with the closedness of Θ and Lemma 14, it follows that there exists some $\lambda \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$ such that

$$0_{\mathbb{R}^n} \in \partial\langle\lambda, f(\bullet) + \hat{\mu}\mathcal{T}(\bullet)e\rangle(\hat{x}) + N(\Theta, \hat{x}). \tag{68}$$

Moreover, by using $\langle\lambda, e\rangle = \sum_{i=1}^p \lambda_i = 1$ and Proposition 3, we have

$$\partial\langle\lambda, f(\bullet) + \hat{\mu}\mathcal{T}(\bullet)e\rangle(\hat{x}) \subset \partial\langle\lambda, f\rangle(\hat{x}) + \hat{\mu}\partial\mathcal{T}(\hat{x}), \tag{69}$$

$$\begin{aligned} \partial\mathcal{T}(\hat{x}) & \subset \partial\|g_+(\bullet)\|(\hat{x}) + \partial\|h(\bullet)\|(\hat{x}) \\ & \quad + \partial d\left((\bullet, -q(\bullet)), \mathrm{gphQ}\right)(\hat{x}). \end{aligned} \tag{70}$$

Note that g, h, and q are continuously Fréchet differentiable and Q is closed. Then it follows from Propositions 1 (ii), 4, and 5 that, for all $i \in \{1, 2, \ldots, r\}$,

$$\partial\max\{0, g_i(\bullet)\}(\hat{x}) = \begin{cases} \{0_{\mathbb{R}^n}\}, & \text{if } g_i(\hat{x}) < 0, \\ [0, 1]\nabla g_i(\hat{x}), & \text{if } g_i(\hat{x}) = 0, \end{cases} \tag{71}$$

for all $i \in \{1, 2, \ldots, s\}$,

$$
\begin{aligned}
&\partial \left|h_i(\bullet)\right|(\widehat{x}) = [-1, 1]\, \nabla h_i(\widehat{x}),\\
&\partial d\left((\bullet, -q(\bullet)), \mathrm{gph} Q\right)(\widehat{x})\\
&\quad \subset \left\{\left(\mathcal{I}_{\mathbb{R}^n}, -\nabla q(\widehat{x})\right)^* (x^*, -y^*) : (x^*, -y^*)\right.\\
&\qquad \left.\in N\left(\mathrm{gph} Q, (-\widehat{x}, q(\widehat{x}))\right)\right\},
\end{aligned} \tag{72}
$$

where $\mathcal{I}_{\mathbb{R}^n}$ denotes the identity map from $\mathbb{R}^n$ to itself. Let $\mathcal{J}(\widehat{x}) := \{i \in \{1, 2, \ldots, r\} \mid g_i(\widehat{x}) = 0\}$ be the set of active constraints of g at $\widehat{x}$. Then we can conclude from (70)–(72) that

$$
\begin{aligned}
\partial \mathcal{T}(\widehat{x}) \subset &\sum_{i \in \mathcal{J}(\widehat{x})} [0, 1]\, \nabla g_i(\widehat{x}) + \sum_{i=1}^{s} [-1, 1]\, \nabla h_i(\widehat{x})\\
&+ \left\{x^* + \left(\nabla q(\widehat{x})\right)^* (y^*) : (x^*, -y^*)\right.\\
&\qquad \left.\in N\left(\mathrm{gph} Q, (-\widehat{x}, q(\widehat{x}))\right)\right\}.
\end{aligned} \tag{73}
$$

Together with (68) and (69), there exist $\overline{\beta}_i \geq 0$ with $i \in \mathcal{J}(\widehat{x})$, $\overline{\gamma} \in \mathbb{R}^s$, and $(x^*, -y^*) \in N(\mathrm{gph} Q, (-\widehat{x}, q(\widehat{x})))$; that is, $x^* \in D^* Q(-\widehat{x}, q(\widehat{x}))(y^*)$, such that

$$
\begin{aligned}
0_{\mathbb{R}^n} \in\ &\partial \langle \lambda, f \rangle (\widehat{x})\\
&+ \widehat{\mu}\Bigg(\sum_{i \in \mathcal{J}(\widehat{x})} \overline{\beta}_i \nabla g_i(\widehat{x}) + \sum_{i=1}^{s} \overline{\gamma}_i \nabla h_i(\widehat{x})\\
&\qquad + \left(x^* + \left(\nabla q(\widehat{x})\right)^* (y^*)\right)\Bigg) + N(\Theta, \widehat{x})\\
=\ &\partial \langle \lambda, f \rangle (\widehat{x}) + \sum_{i \in \mathcal{J}(\widehat{x})} \widehat{\mu} \overline{\beta}_i \nabla g_i(\widehat{x})\\
&+ \sum_{i=1}^{s} \widehat{\mu}\, \overline{\gamma}_i \nabla h_i(\widehat{x}) + \widehat{\mu}\left(x^* + \left(\nabla q(\widehat{x})\right)^* (y^*)\right) + N(\Theta, \widehat{x}).
\end{aligned} \tag{74}
$$

Take $\beta \in \mathbb{R}_+^r$ with $\beta_i = \widehat{\mu}\overline{\beta}_i$, $i \in \mathcal{J}(\widehat{x})$, and $\beta_i = 0$, $i \in \{1, 2, \ldots, r\} \setminus \mathcal{J}(\widehat{x})$, $\gamma \in \mathbb{R}^s$ with $\gamma = \widehat{\mu}\,\overline{\gamma}$ and $\tau = \widehat{\mu}$. Then we have

$$
\begin{aligned}
0_{\mathbb{R}^n} \in\ &\partial \langle \lambda, f \rangle (\widehat{x}) + \sum_{i=1}^{r} \beta_i \nabla g_i(\widehat{x})\\
&+ \sum_{i=1}^{s} \gamma_i \nabla h_i(\widehat{x}) + \tau\left(x^* + \left(\nabla q(\widehat{x})\right)^* (y^*)\right) + N(\Theta, \widehat{x}),\\
&\beta_i g_i(\widehat{x}) = 0, \quad i = 1, 2, \ldots, r.
\end{aligned} \tag{75}
$$

This completes the proof. □

Combining Proposition 12 and Theorem 15, we immediately have the following corollary.

Corollary 16. *Let $\widehat{x} \in S$ be a local efficient solution for (MOPEC). Suppose that the constraint system of (MOPEC) has a local error bound with order 1 at $\widehat{x}$, or, equivalently, the set-valued map $S : \mathbb{R}^{r+s+n+m} \rightrightarrows \mathbb{R}^n$, defined by (26), is calm with order 1 at $(0_{\mathbb{R}^{r+s+n+m}}, \widehat{x})$. Then $\widehat{x}$ is a M-stationary point for (MOPEC).*

Remark 17. Recently, Kanzow and Schwartz [26] discussed the enhanced Fritz-John conditions for a smooth scalar optimization problem with equilibrium constraints and proposed some new constraint qualifications for the enhanced M-stationary condition. In particular, they obtained some sufficient conditions for the existence of a local error bound for the constraint system and the exactness of penalty functions with order 1 by using an appropriate condition. Subsequently, Ye and Zhang [27] extended Kanzow and Schwartz's results to the nonsmooth case. It is worth noting that the exactness of the penalty function with order 1 in [26, 27] was established by using various qualification conditions, which were actually sufficient for the local error bound property of the constraint system; see [28, 29] for more details. However, just as shown in Theorem 13, the exactness for the two types of multiobjective penalty functions with order σ is obtained by means of the equivalent (MOPEC-) calmness condition, which is associated with not only the objective function but also the constraint system. Simultaneously, it follows from Theorem 10 and Proposition 12 that the (MOPEC-) calmness condition is weaker than the local error bound property of the constraint system.

4. Applications

The main purpose of this section is to apply the obtained results for (MOPEC) to a multiobjective optimization problem with complementarity constraints (in short, (MOPCC)) and a multiobjective optimization problem with weak vector variational inequality constraints (in short, (MOPWVVI)) and establish corresponding calmness conditions and M-stationary conditions.

4.1. Applications to (MOPCC). In this subsection, we consider the following multiobjective optimization problem with complementarity constraints:

(MOPCC)

$$
\begin{aligned}
\min \quad & f(x)\\
\text{s.t.} \quad & g(x) \in -\mathbb{R}_+^r,\\
& h(x) = 0_{\mathbb{R}^s},\\
& G(x) \in \mathbb{R}_+^l, H(x) \in \mathbb{R}_+^l, G(x)^T H(x) = 0,\\
& x \in \Theta,
\end{aligned} \tag{76}
$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$ is locally Lipschitz, $g : \mathbb{R}^n \to \mathbb{R}^r$, $h : \mathbb{R}^n \to \mathbb{R}^s$, $G, H : \mathbb{R}^n \to \mathbb{R}^l$ are continuously Fréchet differentiable, and Θ is a nonempty and closed subset of $\mathbb{R}^n$. As usual, we denote

$$\begin{aligned} I_{0+} &:= \{i \mid G_i(\hat{x}) = 0, H_i(\hat{x}) > 0\}, \\ I_{00} &:= \{i \mid G_i(\hat{x}) = 0, H_i(\hat{x}) = 0\}, \\ I_{+0} &:= \{i \mid G_i(\hat{x}) > 0, H_i(\hat{x}) = 0\}. \end{aligned} \tag{77}$$

Obviously, the feasible set $\widehat{S} := \{x \in \mathbb{R}^n \mid g(x) \in -\mathbb{R}^r_+, h(x) = 0_{\mathbb{R}^s}, G(x) \in \mathbb{R}^l_+, H(x) \in \mathbb{R}^l_+, G(x)^T H(x) = 0, x \in \Theta\}$ is a closed subset of $\mathbb{R}^n$. It is easy to verify that (MOPCC) can be reformulated as a special case of (MOPEC) if we let $m = 2l$,

$$q(x) := \begin{pmatrix} G_1(x) \\ H_1(x) \\ \vdots \\ G_l(x) \\ H_l(x) \end{pmatrix}, \qquad Q(x) := C^l, \ \forall x \in \mathbb{R}^n, \tag{78}$$

where $C := \{(a,b) \in \mathbb{R}^2 \mid 0 \leq -a \perp -b \geq 0\}$. Note that Q is constant and equals to C^l. Then the parametric form $\widehat{S}(u,v,y,z)$ of $\widehat{S}$ with parameter $(u,v,y,z) \in \mathbb{R}^{r+s+2l}$ is

$$\begin{aligned} \widehat{S}(u,v,y,z) = \{ & x \in \Theta \mid g(x) + u \in -\mathbb{R}^r_+, h(x) + v = 0_{\mathbb{R}^s}, \\ & G(x) + y \in \mathbb{R}^l_+, H(x) + z \in \mathbb{R}^l_+, \\ & (G(x) + y)^T (H(x) + z) = 0 \}. \end{aligned} \tag{79}$$

Clearly, for the set-valued map $\widehat{S} : \mathbb{R}^{r+s+2l} \rightrightarrows \mathbb{R}^n$, one has $\widehat{S}(0_{\mathbb{R}^{r+s+2l}}) = \widehat{S}$.

Inspired by Definitions 7 and 9, we give the following concepts, called (MOPCC-) calm and local error bound, for (MOPCC).

Definition 18. Given $\sigma > 0$ and $\hat{x} \in \widehat{S}$ being a local efficient (resp. local weak efficient) solution for (MOPCC), then (MOPCC) is said to be (MOPCC-) calm with order σ at $\hat{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that, for all $(u,v,y,z) \in \mathbb{B}(0_{\mathbb{R}^{r+s+2l}}, \delta)$ and all $x \in \widehat{S}(u,v,y,z) \cap \mathbb{B}(\hat{x}, \delta)$, one has

$$f(x) + M\|(u,v,y,z)\|^\sigma e \notin f(\hat{x}) - \operatorname{int} \mathbb{R}^p_+. \tag{80}$$

Definition 19. Given $\sigma > 0$ and $\hat{x} \in S$, then the constraint system of (MOPCC) is said to have a local error bound with order σ at $\hat{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that, for all $(u,v,y,z) \in \mathbb{B}(0_{\mathbb{R}^{r+s+2l}}, \delta) \setminus \{0_{\mathbb{R}^{r+s+2l}}\}$ and all $x \in \widehat{S}(u,v,y,z) \cap \mathbb{B}(\hat{x}, \delta)$, one has

$$d(x, \widehat{S}) \mathscr{B}_{\mathbb{R}^p} \subset M\|(u,v,y,z)\|^\sigma e - \operatorname{int} \mathbb{R}^p_+. \tag{81}$$

Similarly, it follows from Theorem 10 that if the constraint system of (MOPCC) has a local error bound with order σ at $\hat{x}$, then (MOPCC) is (MOPCC-) calm with order σ at $\hat{x}$. Moreover, by Proposition 12, the constraint system of (MOPCC) has a local error bound with order σ at $\hat{x}$ if and only if the set-valued map $\widehat{S} : \mathbb{R}^{r+s+2l} \rightrightarrows \mathbb{R}^n$ is calm with order σ at $(0_{\mathbb{R}^{r+s+2l}}, \hat{x})$. Specially, if we take $p = 1$ and $\sigma = 1$, then Definitions 18 and 19 reduce to Definitions 3.3 and 3.6 in [4], respectively. Simultaneously, the corresponding results to Propositions 3.4 and 3.7 in [4] also hold.

As mentioned in the introduction, there have been various stationary concepts proposed for (MOPCC). Here we only recall the notion of the M-stationary point.

Definition 20 (see [4]). A point $\hat{x} \in \widehat{S}$ is called a M-stationary point of (MOPCC) if and only if there exists a Lagrange multiplier $\lambda^* = (\lambda^f, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+r+s+2l}$ with $\lambda^f \in \mathbb{R}^p_+$ and $\sum_{i=1}^p \lambda_i^f = 1$ such that

$$\begin{aligned} & 0_{\mathbb{R}^n} \in \partial \langle \lambda^f, f \rangle (\hat{x}) + \sum_{i=1}^r \lambda_i^g \nabla g_i(\hat{x}) + \sum_{i=1}^s \lambda_i^h \nabla h_i(\hat{x}) \\ & \qquad - \sum_{i=1}^l \left[\lambda_i^G \nabla G_i(\hat{x}) + \lambda_i^H \nabla H_i(\hat{x}) \right] + N(\Theta, \hat{x}), \\ & \lambda_i^G = 0, \quad \forall i \in I_{+0}, \quad \lambda_i^G \in \mathbb{R}, \quad \forall i \in I_{0+}; \\ & \lambda_i^H = 0, \quad \forall i \in I_{0+}, \quad \lambda_i^H \in \mathbb{R}, \quad \forall i \in I_{+0}, \\ & \text{either } \lambda_i^G > 0, \quad \lambda_i^H > 0 \quad \text{or} \quad \lambda_i^G \lambda_i^H = 0, \quad \forall i \in I_{00}, \\ & \lambda^g \in \mathbb{R}^r_+, \quad \lambda_i^g g_i(\hat{x}) = 0, \quad \forall i = 1, 2, \ldots, r. \end{aligned} \tag{82}$$

The following formula for the Mordukhovich normal cone of the set C is useful in the sequel.

Lemma 21 (see [4]). *For every $(a,b) \in C$, we have*

$$N(C,(a,b)) = \begin{cases} (d_1, d_2) \mid d_1 \in \mathbb{R}, d_2 = 0 & \textit{if } a = 0 > b \\ \qquad d_1 = 0, d_2 \in \mathbb{R} & \textit{if } a < 0 = b \\ \qquad \textit{either } d_1 > 0, & \\ \qquad\quad d_2 > 0 & \\ \qquad\quad \textit{or } d_1 d_2 = 0 & \textit{if } a = 0 = b. \end{cases} \tag{83}$$

We now apply Theorem 15 to establish a M-stationary condition for (MOPCC) by virtue of the (MOPCC-) calmness condition.

Theorem 22. *Suppose that $\hat{x} \in \widehat{S}$ is a local weak efficient solution for (MOPCC) and (MOPCC) is (MOPCC-) calm with order 1 at $\hat{x}$. Then $\hat{x}$ is a M-stationary point of (MOPCC).*

Proof. As stated above, (MOPCC) is equivalent to (MOPCC) with $m = 2l$ and q, Q given by (78). Note that the (MOPCC-) calmness of (MOPCC) implies the (MOPEC-) calmness of (MOPCC). Then it follows from Theorem 15 that there exist

$\lambda \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$, $\beta \in \mathbb{R}^r_+$, $\gamma \in \mathbb{R}^s$, $\tau > 0$ and $(x^*, y^*) \in \mathbb{R}^{n+2l}$ with $x^* \in D^*Q(\hat{x}, -q(\hat{x}))(y^*)$ such that

$$\begin{gathered} 0_{\mathbb{R}^n} \in \partial \langle \lambda, f\rangle (\hat{x}) + \sum_{i=1}^r \beta_i \nabla g_i(\hat{x}) + \sum_{i=1}^s \gamma_i \nabla h_i(\hat{x}) \\ + \tau\left(x^* + (\nabla q(\hat{x}))^*(y^*)\right) + N(\Theta, \hat{x}), \\ \beta_i g_i(\hat{x}) = 0, \quad \forall i = 1, 2, \ldots, r. \end{gathered} \tag{84}$$

Note that $Q(x) = C^l$ for all $x \in \mathbb{R}^n$. Then it follows that $\mathrm{gphQ} = \mathbb{R}^n \times C^l$ and

$$\begin{aligned} N&(\mathrm{gphQ}, (\hat{x}, -q(\hat{x}))) \\ &= N(\mathbb{R}^n, \hat{x}) \times N\left(C^l, -q(\hat{x})\right) \\ &= \{0_{\mathbb{R}^n}\} \times N(C, (-G_1(\hat{x}), -H_1(\hat{x}))) \\ &\quad \times \cdots \times N(C, (-G_l(\hat{x}), -H_l(\hat{x}))). \end{aligned} \tag{85}$$

Together with Lemma 21, we have for every $(x^*, y^*) \in \mathbb{R}^{n+2l}$ with $x^* \in D^*Q(\hat{x}, -q(\hat{x}))(y^*)$,

$$\begin{gathered} x^* = 0_{\mathbb{R}^n}, \\ -y^* \in \left\{ \begin{pmatrix} \eta_1 \\ \theta_1 \\ \vdots \\ \eta_l \\ \theta_l \end{pmatrix} \begin{array}{ll} \mid \eta_i \in \mathbb{R}, \theta_i = 0 & \text{if } i \in I_{0+} \\ \ \eta_i = 0, \theta_i \in \mathbb{R} & \text{if } i \in I_{+0} \\ \ \text{either } \eta_i > 0, & \\ \ \theta_i > 0 & \\ \ \text{or } \eta_i\theta_i = 0 & \text{if } i \in I_{00}. \end{array} \right. \end{gathered} \tag{86}$$

Moreover, we get

$$\nabla q(\hat{x}) = \begin{pmatrix} \nabla G_1(\hat{x})^T \\ \nabla H_1(\hat{x})^T \\ \vdots \\ \nabla G_l(\hat{x})^T \\ \nabla H_l(\hat{x})^T \end{pmatrix}. \tag{87}$$

Taking $\lambda^* = (\lambda^f, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+r+s+2l}$ with $\lambda^f = \lambda$, $\lambda^g = \beta$, $\lambda^h = \gamma$, $\lambda_i^G = \tau\eta_i$, and $\lambda_i^H = \tau\theta_i$, for all $i \in \{1, 2, \ldots, l\}$, and substituting (86) and (87) into (84), then we can conclude that $\hat{x} \in \hat{S}$ is a M-stationary point of (MOPCC) with respect to the Lagrange multiplier λ^*. □

4.2. Applications to (MOPWVVI). Consider the following multiobjective optimization problem with weak vector variational inequality constraints:

(MOPWVVI)

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \in -\mathbb{R}^r_+, \\ & h(x) = 0_{\mathbb{R}^s}, \\ & x \in \widetilde{\Theta}, \end{aligned} \tag{88}$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$ is locally Lipschitz, $g : \mathbb{R}^n \to \mathbb{R}^r$ and $h : \mathbb{R}^n \to \mathbb{R}^s$ are continuously Fréchet differentiable. $\widetilde{\Theta}$ is the solution set of the weak vector variational inequality (in short, (WVVI)): find a vector $\hat{x} \in \Theta$ such that

$$\begin{gathered} (\langle F_1(\hat{x}), w - \hat{x}\rangle, \langle F_2(\hat{x}), w - \hat{x}\rangle, \ldots, \\ \langle F_m(\hat{x}), w - \hat{x}\rangle)^T \notin -\operatorname{int} \mathbb{R}^m_+, \quad \forall w \in \Theta, \end{gathered} \tag{89}$$

where $F_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \ldots, m$ are continuously Fréchet differentiable and Θ is a nonempty, closed, and convex subset of $\mathbb{R}^n$. In the sequel, we denote $\widetilde{S} := \{x \in \mathbb{R}^n \mid g(x) \in -\mathbb{R}^r_+, h(x) = 0_{\mathbb{R}^s}, x \in \widetilde{\Theta}\}$ by the feasible set of (MOPWVVI). Then it is clear that $\widetilde{S}$ is closed.

Take $e_0 := (1, 1, \ldots, 1) \in \mathbb{R}^m$. Then it follows from Lemma 6 that $\hat{x} \in \mathbb{R}^n$ is a solution of (WVVI) if and only if

$$\begin{gathered} \hat{x} \in \Theta, \quad \inf_{w\in\Theta} \xi_{e_0}\left((\langle F_1(\hat{x}), w - \hat{x}\rangle, \langle F_2(\hat{x}), w - \hat{x}\rangle, \ldots, \right. \\ \left. \langle F_m(\hat{x}), w - \hat{x}\rangle)^T \right) = 0. \end{gathered} \tag{90}$$

Moreover, since Θ is nonempty, closed, and convex, and ξ_{e_0} is a convex function, we can conclude from Theorem 8.15 in [7] and Proposition 5 that $\hat{x}$ is a solution of (WVVI) if and only if

$$\begin{aligned} 0_{\mathbb{R}^n} &\in \partial\xi_{e_0}\left((\langle F_1(\hat{x}), \bullet - \hat{x}\rangle, \langle F_2(\hat{x}), \bullet - \hat{x}\rangle, \ldots, \right. \\ &\qquad \left. \langle F_m(\hat{x}), \bullet - \hat{x}\rangle)^T \right)(\hat{x}) + N(\Theta, \hat{x}) \\ &\subset (F_1(\hat{x}), F_2(\hat{x}), \ldots, F_m(\hat{x}))\, \partial\xi_{e_0}(0_{\mathbb{R}^m}) + N(\Theta, \hat{x}). \end{aligned} \tag{91}$$

This shows that there exists some $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m) \in \mathbb{R}^m_+$ with $\sum_{i=1}^m \zeta_i = 1$ such that

$$0_{\mathbb{R}^n} \in \sum_{i=1}^m \zeta_i F_i(\hat{x}) + N(\Theta, \hat{x}). \tag{92}$$

Given $\hat{x} \in \widetilde{S}$ being a local efficient (resp. local weak efficient) solution for (MOPWVVI), then $\hat{x}$ is a solution of (MOPWVVI). Next, we define the concept of (MOPWVVI-) calmness with order $\sigma > 0$ at $\hat{x}$ for (MOPWVVI) with respect to the corresponding $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m) \in \mathbb{R}^m_+$ with $\sum_{i=1}^m \zeta_i = 1$ satisfying (92).

Definition 23. Given $\sigma > 0$, $\hat{x} \in \widetilde{S}$ being a local efficient (resp. local weak efficient) solution for (MOPWVVI) and $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m) \in \mathbb{R}^m_+$ with $\sum_{i=1}^m \zeta_i = 1$ satisfying (92), then (MOPWVVI) is said to be (MOPWVVI-) calm with order σ at $\hat{x}$ if and only if there exists $M > 0$ such that, for every sequence $\{(u_k, v_k, y_k, z_k)\} \subset \mathbb{R}^{r+s+n+m}$ with $(u_k, v_k, y_k, z_k) \to 0_{\mathbb{R}^{r+s+n+m}}$ and every sequence $\{x_k\} \subset \Theta$ satisfying $g(x_k) + u_k \in \mathbb{R}^r_+$, $h(x_k) + v_k = 0_{\mathbb{R}^s}$, $z_k \in \sum_{i=1}^m \zeta_i F_i(x_k) + N(\Theta, x_k + y_k)$, and $x_k \to \hat{x}$, it holds that

$$f(x_k) + M\|(u_k, v_k, y_k, z_k)\|^\sigma e \notin f(\hat{x}) - \operatorname{int} \mathbb{R}^p_+. \tag{93}$$

Obviously, we can define a similar local error bound condition at a local weak efficient solution $\hat{x} \in \widetilde{S}$ for (MOPWVVI) with respect to $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m) \in \mathbb{R}^m_+$ with $\sum_{i=1}^m \zeta_i = 1$ satisfying (92). Moreover, we can obtain a corresponding relationship between the (MOPWVVI-) calmness condition and the local error bound condition. However, we omit the details here for simplicity.

Next, we establish a M-stationary condition for (MOPWVVI) under the (MOPWVVI-) calmness with order 1 assumption.

Theorem 24. *Suppose that $\hat{x} \in \widetilde{S}$ is a local weak efficient solution for (MOPWVVI) and $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m) \in \mathbb{R}^m_+$ with $\sum_{i=1}^m \zeta_i = 1$ satisfy (92). If in addition (MOPWVVI) is (MOPWVVI-) calm with order 1 at $\hat{x}$, then there exist $\lambda \in \mathbb{R}^p_+$ with $\sum_{i=1}^p \lambda_i = 1$, $\beta \in \mathbb{R}^r_+$, $\gamma \in \mathbb{R}^s$, $\tau > 0$, and $(x^*, y^*) \in \mathbb{R}^{2n}$ with $x^* \in D^* N_\Theta(\hat{x}, -\sum_{i=1}^m \zeta_i F_i(\hat{x}))(y^*)$ such that*

$$\begin{aligned} 0_{\mathbb{R}^n} \in \partial \langle \lambda, f \rangle (\hat{x}) &+ \sum_{i=1}^r \beta_i \nabla g_i(\hat{x}) + \sum_{i=1}^s \gamma_i \nabla h_i(\hat{x}) \\ &+ \tau \left[x^* + \left(\sum_{i=1}^m \zeta_i \nabla F_i(\hat{x}) \right)^* (y^*) \right] + N(\Theta, \hat{x}), \\ &\beta_i g_i(\hat{x}) = 0, \quad \forall i = 1, 2, \ldots, r, \end{aligned} \tag{94}$$

where the set-valued map $N_\Theta : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined by $N_\Theta(x) = N(\Theta, x)$ for all $x \in \mathbb{R}^n$.

Proof. Consider the problem (MOPEC) with $q(x) = \sum_{i=1}^m \zeta_i F_i(x)$ and $Q(x) = N(\Theta, x)$ for all $x \in \mathbb{R}^n$. Obviously, $\hat{x}$ is a feasible point of (MOPEC) and the feasible set of (MOPEC) is contained in $\widetilde{S}$. By assumption, $\hat{x}$ is a local weak efficient solution for (MOPEC). Moreover, it is easy to verify that the (MOPWVVI-) calmness of (MOPWVVI) with order 1 at $\hat{x}$ implies the (MOPEC-) calmness of (MOPEC) with order 1 at $\hat{x}$. Thus, together with F_i, $i = 1, 2, \ldots, m$ being continuously Fréchet differentiable and $\nabla q(\hat{x}) = \sum_{i=1}^m \zeta_i \nabla F_i(\hat{x})$, we immediately complete the proof by Theorem 15. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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