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Research Article

Packing Constant in Orlicz Sequence Spaces Equipped with the p-Amemiya Norm

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The problem of packing spheres in Orlicz sequence space $l_{\Phi,p}$ equipped with the p-Amemiya norm is studied, and a geometric characteristic about the reflexivity of $l_{\Phi,p}$ is obtained, which contains the relevant work about l^p $(p > 1)$ and classical Orlicz spaces l_Φ discussed by Rankin, Burlak, and Cleaver. Moreover the packing constant as well as Kottman constant in this kind of spaces is calculated.

1. Introduction and Preliminaries

The packing constant is an important and interesting geometric parameter for studying the geometric structure, isometric embedding, noncompactness, and reflexivity in Banach spaces [1–4]. Let X be a Banach space. We denote by $B(X)$ the unit ball of X and by $S(X)$ the unit sphere of X. The packing constant $P(X)$ of X is the real number such that if $r \le P(X)$, then an infinite number of spheres of radius r can be packed in $B(X)$, and if $r > P(X)$, only a finite number of spheres can be done. It began in the 1950s studying the packing constant of special sequence spaces. Burlak et al. [1] proved that $P(l^1) = P(l^\infty) = 1/2$ and $P(l^p) = 1/(1 + 2^{1-(1/p)})$ for $1 < p < \infty$. Rankin found $P(l^2)$ and $P(l^p)$ $(p > 1)$ in 1955 and 1958, respectively. In 1976, Cleaver discussed Orlicz sequence space l°_Φ equipped with the Orlicz norm under a strong condition, and he found upper and lower bounds of $P(l^\circ_\Phi)$. In 1983, Ye investigated Orlicz sequence space l_Φ equipped with the Luxemburg norm and obtained a formula for $P(l_\Phi)$ [5].

In this paper, an analogue for Orlicz sequence spaces equipped with the p-Amemiya norm is illustrated, and some useful definitions and lemmas are presented.

Definition 1 (see [1]). The packing constant of a Banach space X is defined by

$$P(X) = \sup\Big\{r > 0 : \text{there exists } \{x_i\}_{i=1}^{\infty}, \|x_i\| \le 1 - r, \ \big\|x_i - x_j\big\| \ge 2 \text{ for } i, j \in \mathbb{N}, i \ne j\Big\}. \tag{1}$$

It is obvious that $P(X) = 0$, if $\dim X < \infty$.

Lemma 2 (see [2]). *Let X be an infinite-dimensional Banach space. Define*

$$K(X) = \sup\Big\{\inf\{\|x_n - x_m\| : n \ne m\} : \{x_n\}_{n=1}^{\infty} \subset S(X)\Big\}, \tag{2}$$

which is called the Kottman constant of X. Then

$$P(X) = \frac{K(X)}{K(X) + 2}. \tag{3}$$

It is known that $1 \le K(X) \le 2$. Due to Riesz lemma, it can be summarised that $K(X) \ge 1$ for any infinite-dimensional Banach space X. Finite-dimensional spaces have Kottman constant equal to zero. Furthermore, Elton and Odell in [6] proved that if X is an infinite-dimensional Banach space, then there exists an $\varepsilon > 0$ such that $K(X) \geq 1 + \varepsilon$. Consequently, $1/3 \leq P(X) \leq 1/2$. Hudzik proved that $P(Y) = 1/2$ and $K(Y) = 2$ for every nonreflexive Banach lattice Y [7].

Recall that a Banach space X is said to be P-convex (see [2]) if $P(n, X) < 1/2$, for some $n \in \mathbb{N}$, $n \geq 2$, where

$$P(n, X) = \sup\left\{r > 0 : \text{there exist } \{x_i\}_{i=1}^{n}, \|x_i\| \leq 1 - r,\right. \\ \left.\left\|x_i - x_j\right\| \geq 2r \text{ for } i \neq j\right\}. \tag{4}$$

Kottman [2] has proved that any P-convex Banach space is reflexive.

The packing problem in Orlicz sequence spaces was investigated in [8–11]. The packing constant for Musielak-Orlicz sequence spaces and Cesaro sequence spaces have been calculated in [12, 13].

For any map $\Phi : \mathbb{R} \to [0, \infty]$, define

$$a_\Phi = \max\{u \geq 0 : \Phi(u) = 0\}, \\ b_\Phi = \max\{u \geq 0 : \Phi(u) < \infty\}. \tag{5}$$

A map Φ is said to be an Orlicz function, if $\Phi(0) = 0$; Φ is not identically equal to zero; it is even and convex on the interval $(-b_\Phi, b_\Phi)$ and left-continuous at b_Φ.

For every Orlicz function Φ, we define its complementary function $\Psi : \mathbb{R} \to [0, \infty]$ by the formula

$$\Psi(v) = \sup\{u|v| - \Phi(u) : u \geq 0\}. \tag{6}$$

The complementary function Ψ is also an Orlicz function. The convex modular I_Φ is defined on l^0 (the space of all real sequences) by $I_\Phi(x) = \sum_{i=1}^{\infty} \Phi(x(i))$ for any $x = (x(i))$.

Definition 3 (see [14–16]). The Orlicz sequence space is defined as the set

$$l_\Phi = \{x = (x(i)) : I_\Phi(\lambda x) < \infty, \text{for some } \lambda > 0\}. \tag{7}$$

The Luxemburg norm and the Orlicz norm are expressed as

$$\|x\|_\Phi = \inf\left\{\lambda > 0 : I_\Phi\left(\frac{x}{\lambda}\right) \leq 1\right\}, \\ \|x\|_\Phi^\circ = \inf_{k>0} \frac{1}{k}\left(1 + I_\Phi(kx)\right), \tag{8}$$

respectively. The Orlicz space equipped with the Luxemburg norm and the Orlicz norm are denoted by l_Φ and l_Φ°, respectively.

For any $1 \leq p \leq \infty$ and $u \geq 0$, define

$$s_p(u) = \begin{cases} (1 + u^p)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \max\{1, u\}, & \text{for } p = \infty \end{cases} \tag{9}$$

and define $s_{\Phi,p}(x) = s_p \circ I_\Phi(x)$ for all $1 \leq p \leq \infty$. Note that the functions s_p and $s_{\Phi,p}$ are convex. Moreover, the function s_p is increasing on $\mathbb{R}_+$, for $1 \leq p < \infty$, but the function s_∞ is increasing on the interval $[1, \infty)$ only.

Definition 4 (see [17, 18]). Let $1 \leq p \leq \infty$. For any $x = (x(i))$, define the p-Amemiya norm by the formula

$$\|x\|_{\Phi,p} = \inf_{k>0} \frac{1}{k} s_{\Phi,p}(kx). \tag{10}$$

The Orlicz space equipped with the p-Amemiya norm will be denoted by $l_{\Phi,p}$.

It is known that $\|x\|_{\Phi,1} = \|x\|_\Phi^\circ$ and $\|x\|_{\Phi,\infty} = \|x\|_\Phi$. If $1 \leq p < \infty$, $x \neq 0$, then

$$\frac{1}{2}\|x\|_\Phi^\circ \leq \|x\|_\Phi \leq \|x\|_{\Phi,p} \leq 2^{1/p}\|x\|_\Phi < 2^{1/p}\|x\|_\Phi^\circ. \tag{11}$$

(See [17].)

Let p_+ be the right-hand side derivative of Φ on $[0, b_\Phi)$ and put $p_+(b_\Phi) = \lim_{u \to b_\Phi^-} p_+(u)$. Define the function $\alpha_p : l_{\Phi,p} \to [-1, \infty]$ by

$$\alpha_p(x) = \begin{cases} I_\Phi^{p-1}(x) I_\Psi(p_+(|x|)) - 1, & 1 \leq p < \infty, \\ -1, & p = \infty, \ I_\Phi(x) \leq 1, \\ I_\Psi(p_+(|x|)), & p = \infty, \ I_\Phi(x) > 1 \end{cases} \tag{12}$$

and the functions $k_p^* : l_{\Phi,p} \to [0, \infty)$ and $k_p^{**} : l_{\Phi,p} \to [0, \infty)$ by

$$k_p^*(x) = \inf\left\{k \geq 0 : \alpha_p(kx) \geq 0\right\}, \quad (\text{with } \inf \phi = \infty), \\ k_p^{**}(x) = \inf\left\{k \geq 0 : \alpha_p(kx) \leq 0\right\}. \tag{13}$$

It is obvious that $k_p^*(x) \leq k_p^{**}(x)$ for every $1 \leq p \leq \infty$ and $x \in l_{\Phi,p}$.

Set $K_p(x) = \{0 < k < \infty : k_p^*(x) \leq k \leq k_p^{**}(x)\}$.

Definition 5 (see [14]). We say an Orlicz function Φ satisfies the $\Delta_2(0)$-condition ($\Phi \in \Delta_2(0)$, for short) if there exist constants $K \geq 2$ and $u_0 > 0$ such that $\Phi(u_0) > 0$ and

$$\Phi(2u) \leq K\Phi(u) \quad \text{for every } |u| \leq u_0. \tag{14}$$

For more details about Orlicz spaces, we refer the reader to [15, 16, 18, 19].

Lemma 6 (see [20]). *Assume that $\Phi \in \Delta_2(0)$, $1 \leq p < \infty$. Then, for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in l^0$ there holds the implication*

$$(I_\Phi(x) \leq L) \wedge (I_\Phi(y) \leq \delta) \Longrightarrow \left|I_\Phi^p(x + y) - I_\Phi^p(x)\right| < \varepsilon. \tag{15}$$

2. Main Results

Assume that $\Phi \in \Delta_2(0)$, $1 \leq p < \infty$. Then, for any $x \in S(l_{\Phi,p})$ and $k > 1$, there exists a unique $d_{x,k} > 0$ such that

$$I_\Phi^p\left(\frac{kx}{d_{x,k}}\right) = \frac{k^p - 1}{2^p}. \tag{16}$$

Set

$$d_x = \inf\{d_{x,k} : k > 1\},$$
$$d = \sup\{d_x : x \in S(l_{\Phi,p})\}. \tag{17}$$

Then $d_x > 1$ and $1 < d \le 2$. Denote

$$k' = \inf\{k : k \in K_p(x), \|x\|_{\Phi,p} = 1\},$$
$$k'' = \sup\{k : k \in K_p(x), \|x\|_{\Phi,p} = 1\}. \tag{18}$$

In the sequel, the packing constant $l_{\Phi,p}$ is calculated, and the main results of this paper are proposed.

Theorem 7. *If $\Phi \in \Delta_2(0)$, $1 \le p < \infty$, then $K(l_{\Phi,p}) = d$ and $P(l_{\Phi,p}) = d/(d+2)$.*

Proof. For any $\varepsilon > 0$, there exists $x \in S(l_{\Phi,p})$ such that $d_x > d - \varepsilon$, so $d_{x,k} > d - \varepsilon$ for all $k > 1$. Define

$$x^n = \sum_{i=1}^{\infty} x(i)\, e_{2^{n-1}(2i-1)}, \quad \forall n \in \mathbb{N}. \tag{19}$$

Then $\{x^n\}$ have pairwise disjoint supports and $\|x^n\|_{\Phi,p} = \|x\|_{\Phi,p} = 1$ $(n \in \mathbb{N})$. For all $n \neq m$ and all $k > 1$,

$$\begin{aligned}
&\frac{1}{k}\left(1 + I_\Phi^p\left(k\frac{x^n - x^m}{d-\varepsilon}\right)\right)^{1/p} \\
&= \frac{1}{k}\left(1 + 2^p I_\Phi^p\left(\frac{kx}{d-\varepsilon}\right)\right)^{1/p} \\
&> \frac{1}{k}\left(1 + 2^p I_\Phi^p\left(\frac{kx}{d_{x,k}}\right)\right)^{1/p} \\
&= \frac{1}{k}\left(1 + 2^p \cdot \frac{k^p - 1}{2^p}\right)^{1/p} = 1.
\end{aligned} \tag{20}$$

Then $\|x^n - x^m\|_{\Phi,p} = \inf_{k>0}(1/k)(1 + I_\Phi^p(k(x^n - x^m)))^{1/p} \ge d - \varepsilon$, so we have $K(l_{\Phi,p}) \ge d$, since ε is arbitrary.

In the following, $K(l_{\Phi,p}) \le d$ will be illustrated as an important part of our results.

For any sequence $\{x_n\} \subset S(l_{\Phi,p})$, which means that $x_n = (x_n(i))_i$, $\|x_n\|_{\Phi,p} = \|\sum_{i=1}^{\infty} x_n(i)e_i\|_{\Phi,p} = 1$, for any $n \in \mathbb{N}$, then $\{\|x_n(i)e_i\|_{\Phi,p}\}_n$ is bounded for all $i \in \mathbb{N}$.

Since $\{\|x_n(1)e_1\|_{\Phi,p}\}_n$ is bounded, there exists a subsequence $\{x_{1_n}\} \subset \{x_n\}$ such that $\{\|x_{1_n}(1)e_1\|_{\Phi,p}\}_n$ is convergent, but $\{\|x_{1_n}(2)e_2\|_{\Phi,p}\}_n$ is bounded, so there exists a subsequence $\{x_{2_n}\} \subset \{x_{1_n}\}$ such that $\{\|x_{2_n}(2)e_2\|_{\Phi,p}\}_n$ is convergent. In a similar way, using the diagonal method, we can find a subsequence $\{x_{n_n}\} \subset \{x_n\}$ such that, for any $i \in \mathbb{N}$, $\{\|x_{n_n}(i)e_i\|_{\Phi,p}\}_{n\ge i}$ is convergent. Denoting $\|e_i\|_{\Phi,p} = s_i$ and setting $\|x_{n_n}(i)e_i\|_{\Phi,p} \to b_i$ as $n \to \infty$, then $|x_{n_n}(i)| \to b_i/s_i$ as $n \to \infty$ for all $i \in \mathbb{N}$.

Let $x = (b_i/s_i)_i$, $|x_{n_n}| = (|x_{n_n}(i)|)_i$, and $z_n = |x_{n_n}| - x$. Then

$$z_n(i) \longrightarrow 0 \text{ as } n \longrightarrow \infty \quad \forall i \in \mathbb{N},$$
$$\operatorname{sep}(z_n) = \operatorname{sep}(|x_{n_n}|) \ge \operatorname{sep}(x_n). \tag{21}$$

Since $\Phi \in \Delta_2$, then $x \in S(l_{\Phi,p})$. For any $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$, such that $\|\sum_{i=i_0+1}^{\infty} x(i)e_i\|_{\Phi,p} < \varepsilon$. Moreover, $|x_{n_n}(i)| \to x(i)$ as $n \to \infty$ for $i = 1, \ldots, i_0$. So we have

$$\begin{aligned}
\|z_n\|_{\Phi,p} &= \left\|\sum_{i=1}^{\infty}\left(|x_{n_n}(i)| - x(i)\right)e_i\right\|_{\Phi,p} \\
&\le \left\|\sum_{i=1}^{i_0}\left(|x_{n_n}(i)| - x(i)\right)e_i\right\|_{\Phi,p} \\
&\quad + \left\|\sum_{i=i_0+1}^{\infty}|x_{n_n}(i)|\, e_i\right\|_{\Phi,p} + \varepsilon,
\end{aligned} \tag{22}$$

and, consequently, $\limsup_n \|z_n\|_{\Phi,p} \le 1 + \varepsilon$.

For the above $\varepsilon > 0$, since $l_{\Phi,p}$ is order continuous, there exists $i_1 \in \mathbb{N}$ such that $\|\sum_{i=i_1+1}^{\infty} z_{n_1}(i)e_i\|_{\Phi,p} < \varepsilon$ for $n_1 = 1$. Take $n_2 > n_1$ such that $\|\sum_{i=1}^{i_1} z_{n_2}(i)e_i\|_{\Phi,p} < \varepsilon$. And for n_2, there exists $i_2 > i_1$ such that $\|\sum_{i=i_2+1}^{\infty} z_{n_2}(i)e_i\|_{\Phi,p} < \varepsilon$. Then

$$\begin{aligned}
\operatorname{sep}(z_n) &\le \|z_{n_1} - z_{n_2}\|_{\Phi,p} \\
&\le \left\|\sum_{i=1}^{i_1} z_{n_1}(i)e_i - \sum_{i=i_1+1}^{i_2} z_{n_2}(i)e_i\right\|_{\Phi,p} \\
&\quad + \left\|\sum_{i=i_1+1}^{\infty} z_{n_1}(i)e_i\right\|_{\Phi,p} \\
&\quad + \left\|\sum_{i=1}^{i_1} z_{n_2}(i)e_i\right\|_{\Phi,p} + \left\|\sum_{i=i_2+1}^{\infty} z_{n_2}(i)e_i\right\|_{\Phi,p} \\
&\le \left\|\sum_{i=1}^{i_1} z_{n_1}(i)e_i - \sum_{i=i_1+1}^{i_2} z_{n_2}(i)e_i\right\|_{\Phi,p} + 3\varepsilon.
\end{aligned} \tag{23}$$

Take $n_3 > n_2$ such that $\|\sum_{i=1}^{i_2} z_{n_3}(i)e_i\|_{\Phi,p} < \varepsilon$, and for n_3, there exists $i_3 > i_2$ such that $\|\sum_{i=i_3+1}^{\infty} z_{n_3}(i)e_i\|_{\Phi,p} < \varepsilon$. Then

$$\begin{aligned}
&\|z_{n_1} - z_{n_3}\|_{\Phi,p} \\
&\le \left\|\sum_{i=1}^{i_1} z_{n_1}(i)e_i - \sum_{i=i_2+1}^{i_3} z_{n_3}(i)e_i\right\|_{\Phi,p} \\
&\quad + \left\|\sum_{i=i_1+1}^{\infty} z_{n_1}(i)e_i\right\|_{\Phi,p} + \left\|\sum_{i=1}^{i_2} z_{n_3}(i)e_i\right\|_{\Phi,p} \\
&\quad + \left\|\sum_{i=i_3+1}^{\infty} z_{n_3}(i)e_i\right\|_{\Phi,p}
\end{aligned}$$

$$\leq \left\| \sum_{i=1}^{i_1} z_{n_1}(i)e_i - \sum_{i=i_2+1}^{i_3} z_{n_3}(i)e_i \right\|_{\Phi,p} + 3\varepsilon,$$

$$\begin{aligned}
\|z_{n_2} - z_{n_3}\|_{\Phi,p} &\leq \left\| \sum_{i=i_1+1}^{i_2} z_{n_2}(i)e_i - \sum_{i=i_2+1}^{i_3} z_{n_3}(i)e_i \right\|_{\Phi,p} \\
&\quad + \left\| \sum_{i=1}^{i_1} z_{n_2}(i)e_i \right\|_{\Phi,p} + \left\| \sum_{i=i_2+1}^{\infty} z_{n_2}(i)e_i \right\|_{\Phi,p} \\
&\quad + \left\| \sum_{i=1}^{i_2} z_{n_3}(i)e_i \right\|_{\Phi,p} + \left\| \sum_{i=i_3+1}^{\infty} z_{n_3}(i)e_i \right\|_{\Phi,p} \\
&\leq \left\| \sum_{i=i_1+1}^{i_2} z_{n_2}(i)e_i - \sum_{i=i_2+1}^{i_3} z_{n_3}(i)e_i \right\|_{\Phi,p} + 4\varepsilon.
\end{aligned} \tag{24}$$

Analogously, we can find by induction a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and $\{i_k\} \subset \mathbb{N}$ such that $n_1 < n_2 < \cdots < n_k < \cdots$, $i_1 < i_2 < \cdots < i_k < \cdots$ such that, for any $k \in \mathbb{N}$,

$$\begin{gathered}
\left\| \sum_{i=1}^{i_{k-1}} z_{n_k}(i)e_i \right\|_{\Phi,p} < \varepsilon, \\
\left\| \sum_{i=i_k+1}^{\infty} z_{n_k}(i)e_i \right\|_{\Phi,p} < \varepsilon, \\
\|z_{n_1} - z_{n_k}\|_{\Phi,p} \leq \left\| \sum_{i=1}^{i_1} z_{n_1}(i)e_i - \sum_{i=i_{k-1}+1}^{i_k} z_{n_k}(i)e_i \right\|_{\Phi,p} + 3\varepsilon, \\
\|z_{n_l} - z_{n_k}\|_{\Phi,p} \leq \left\| \sum_{i=i_{l-1}+1}^{i_l} z_{n_l}(i)e_i - \sum_{i=i_{k-1}+1}^{i_k} z_{n_k}(i)e_i \right\|_{\Phi,p} + 4\varepsilon, \\
\forall 1 < l < k.
\end{gathered} \tag{25}$$

Since $\| \sum_{i=1}^{\infty} z_n(i)e_i \|_{\Phi,p} \leq 1+\varepsilon$, $\| \sum_{i=i_{k-1}+1}^{i_k} z_{n_k}(i)e_i \|_{\Phi,p}/(1+\varepsilon) \leq 1$ for all $k \in \mathbb{N}$. Therefore, for any $l, k \in \mathbb{N}$,

$$\begin{aligned}
&\|z_{n_l} - z_{n_k}\|_{\Phi,p} \\
&\quad \leq (1+\varepsilon) \\
&\quad \times \left\| \frac{\sum_{i=i_{l-1}+1}^{i_l} z_{n_l}(i)e_i}{\| \sum_{i=i_{l-1}+1}^{i_l} z_{n_l}(i)e_i \|_{\Phi,p}} - \frac{\sum_{i=i_{k-1}+1}^{i_k} z_{n_k}(i)e_i}{\| \sum_{i=i_{k-1}+1}^{i_k} z_{n_k}(i)e_i \|_{\Phi,p}} \right\|_{\Phi,p} \\
&\quad + 4\varepsilon.
\end{aligned} \tag{26}$$

Setting $y_k = \sum_{i=i_{k-1}+1}^{i_k} z_{n_k}(i)e_i / \| \sum_{i=i_{k-1}+1}^{i_k} z_{n_k}(i)e_i \|_{\Phi,p}$ (for all $k \in \mathbb{N}$), then

$$\{y_m\} \subset S(l_{\Phi,p}), \qquad \operatorname{supp}(y_l) \cap \operatorname{supp}(y_m) = \phi. \tag{27}$$

In this way, we get

$$\begin{aligned}
K(l_{\Phi,p}) &\leq \operatorname{sep}(x_n) \leq \operatorname{sep}(z_n) \leq \operatorname{sep}(z_{n_i}) \\
&\leq (1+\varepsilon) \|y_m - y_l\|_{\Phi,p} + 4\varepsilon.
\end{aligned} \tag{28}$$

For any $\varepsilon > 0$, by the definition of d, there exists $k_m > 1$ such that $d_{y_m,k_m} < d + \varepsilon$, where d_{y_m,k_m} satisfies the equality $I_\Phi^p(k_m y_m / d_{y_m,k_m}) = (k_m^p - 1)/2^p \quad (m \in \mathbb{N})$.

Setting $\|y_m - y_l\| = \lambda_{ml}$ and taking $k_{ml} \in K_p((y_m - y_l)/\lambda)$, we have

$$\begin{aligned}
1 &= \left\| \frac{y_m - y_l}{\lambda_{ml}} \right\|_{\Phi,p} \\
&= \frac{1}{k_{ml}} \left(1 + I_\Phi^p \left(k_{ml} \left(\frac{y_m - y_l}{\lambda_{ml}} \right) \right) \right)^{1/p} \\
&= \frac{1}{k_{ml}} \left(1 + \left(I_\Phi \left(k_{ml} \left(\frac{y_m}{\lambda_{ml}} \right) \right) + I_\Phi \left(k_{ml} \left(\frac{y_l}{\lambda_{ml}} \right) \right) \right)^p \right)^{1/p}.
\end{aligned} \tag{29}$$

Then

$$\left(k_{ml}^p - 1 \right)^{1/p} = I_\Phi \left(k_{ml} \left(\frac{y_m}{\lambda_{ml}} \right) \right) + I_\Phi \left(k_{ml} \left(\frac{y_l}{\lambda_{ml}} \right) \right). \tag{30}$$

Now we obtain that $\lambda_{ml} \leq \max\{d_{y_m,k_{ml}}, d_{y_l,k_{ml}}\}$. If not, $\lambda_{ml} > \max\{d_{y_m,k_{ml}}, d_{y_l,k_{ml}}\}$, we have

$$\begin{aligned}
I_\Phi^p \left(k_{ml} \left(\frac{y_m}{\lambda_{ml}} \right) \right) &< \frac{k_{ml}^p - 1}{2^p}, \\
I_\Phi^p \left(k_{ml} \left(\frac{y_l}{\lambda_{ml}} \right) \right) &< \frac{k_{ml}^p - 1}{2^p},
\end{aligned} \tag{31}$$

whence

$$\begin{aligned}
&\left(I_\Phi \left(k_{ml} \left(\frac{y_m}{\lambda_{ml}} \right) \right) + I_\Phi \left(k_{ml} \left(\frac{y_l}{\lambda_{ml}} \right) \right) \right)^p \\
&\quad < \left(\left(\frac{k_{ml}^p - 1}{2^p} \right)^{1/p} + \left(\frac{k_{ml}^p - 1}{2^p} \right)^{1/p} \right)^p \\
&\quad = k_{ml}^p - 1.
\end{aligned} \tag{32}$$

This is a contradiction. Hence,

$$\|y_m - y_l\|_{\Phi,p} = \lambda_{ml} \leq \max\left\{ d_{y_m,k_{ml}}, d_{y_l,k_{ml}} \right\} \leq d. \tag{33}$$

So $K(l_{\Phi,p}) \leq (1+\varepsilon)d + 4\varepsilon$; we get $K(l_{\Phi,p}) \leq d$ due to the arbitrariness of ε. □

Theorem 8. *If $\Phi \notin \Delta_2(0)$, $1 \leq p < \infty$, then $K(l_{\Phi,p}) = 2$.*

Proof. Denote

$$\begin{aligned}
l_\alpha = \Big\{ x \in l_{\Phi,p} : \lim_{n \to \infty} &\|(0, \ldots, 0, x(n+1), \\
&x(n+2), \ldots)\|_{\Phi,p} = 0 \Big\}.
\end{aligned} \tag{34}$$

Since $\Phi \notin \Delta_2(0)$, then $l_\alpha \neq l_{\Phi,p}$; so for $\varepsilon > 0$, according to Riesz lemma, there exists $x_\varepsilon \in S(l_{\Phi,p})$ satisfying $\operatorname{dist}(x_\varepsilon, l_\alpha) > 1-\varepsilon$. Then we have

$$\left\|(0,\ldots,0,x_\varepsilon(n+1),x_\varepsilon(n+2),\ldots)\right\|_{\Phi,p} > 1-\varepsilon. \quad (35)$$

Since

$$\lim_{m\to\infty}\left\|(0,\ldots,0,x_\varepsilon(n+1),\ldots,x_\varepsilon(m),0,\ldots)\right\|_{\Phi,p} > 1-\varepsilon, \quad (36)$$

there exists a subsequence $\{n_i\} \subset \mathbb{N}$ such that $n_1 < n_2 < \cdots < n_k < \cdots$ and

$$\left\|(0,\ldots,0,x_\varepsilon(n_i+1),\ldots,x_\varepsilon(n_{i+1}),0,\ldots)\right\|_{\Phi,p} > 1-\varepsilon. \quad (37)$$

Let

$$\begin{aligned} x_1 &= (-x_\varepsilon(1),\ldots,-x_\varepsilon(n_1),x_\varepsilon(n_1+1),\ldots,\\ &\qquad x_\varepsilon(n_2),x_\varepsilon(n_2+1),\ldots),\\ x_2 &= (x_\varepsilon(1),\ldots,x_\varepsilon(n_1),-x_\varepsilon(n_1+1),\ldots,\\ &\qquad -x_\varepsilon(n_2),x_\varepsilon(n_2+1),\ldots). \end{aligned} \quad (38)$$

Then for any $m, l \in \mathbb{N}$,

$$\begin{aligned} &\|x_m - x_l\|_{\Phi,p}\\ &\quad = 2\left\|(\ldots,0,x_\varepsilon(n_{m-1}+1),\ldots,x_\varepsilon(n_m),0,\ldots,\right.\\ &\qquad\quad \left.0,x_\varepsilon(n_{l-1}+1),\ldots,x_\varepsilon(n_l),0,\ldots)\right\|_{\Phi,p}\\ &\quad \geq 2\left\|(0,\ldots,0,x_\varepsilon(n_{m-1}+1),\ldots,x_\varepsilon(n_m),0,\ldots)\right\|_{\Phi,p}\\ &\quad \geq 2(1-\varepsilon). \end{aligned} \quad (39)$$

Due to the arbitrariness of $\varepsilon > 0$, we have $K(l_{\Phi,p}) = 2$. □

Lemma 9. *If $\Phi \in \Delta_2(0) \cap \nabla_2(0)$, $1 \leq p < \infty$, then*

$$1 < k' \leq k'' < \infty. \quad (40)$$

Proof. (1) Since $\Phi \in \Delta_2(0)$ and the norm convergence and the modular convergence are equivalent, there exists $c > 0$ such that

$$\inf_{\|x\|_{\Phi,p}=1} I_\Phi(x) = c > 0. \quad (41)$$

For any $x \in S(l_{\Phi,p})$ and $k \in K_p(x)$, we have

$$1 = \|x\|_{\Phi,p} = \frac{1}{k}\left(1 + I_\Phi^p(kx)\right)^{1/p}, \quad (42)$$

so $k = (1 + I_\Phi^p(kx))^{1/p} \geq 1$; then

$$\begin{aligned} k' &= \inf_{\|x\|_{\Phi,p}=1} k = \inf_{\|x\|_{\Phi,p}=1}\left(1 + I_\Phi^p(kx)\right)^{1/p}\\ &\geq \inf_{\|x\|_{\Phi,p}=1}\left(1 + I_\Phi^p(x)\right)^{1/p} \geq \left(1 + c^p\right)^{1/p} > 1. \end{aligned} \quad (43)$$

(2) If $\Phi \in \nabla_2(0)$, then there exists $\alpha > 1$ such that

$$up_+(u) \geq \alpha\Phi(u), \qquad \left(|u| \leq q_+\left(\Psi^{-1}\left(\frac{1}{c^{p-1}}\right)\right)\right). \quad (44)$$

For any $x \in S(l_{\Phi,p})$ and $k \in K_p(x)$, we have $1 < k' \leq k \leq k_p^{**}(x)$; then for any $\varepsilon \in (0, k'-1)$, we get

$$\begin{aligned} 1 &\geq I_\Phi^{p-1}((k-\varepsilon)x)\, I_\Psi\left(p_+(|(k-\varepsilon)x|)\right)\\ &\geq I_\Phi^{p-1}(x)\, I_\Psi\left(p_+(|(k-\varepsilon)x|)\right)\\ &\geq c^{p-1}\sum_{i=1}^{\infty}\Psi\left(p_+(|(k-\varepsilon)x(i)|)\right)\\ &\geq c^{p-1}\Psi\left(p_+(|(k-\varepsilon)x(i)|)\right), \quad (\forall i = 1,2,\ldots); \end{aligned} \quad (45)$$

whence $|(k-\varepsilon)x(i)| \leq q_+(\Psi^{-1}(1/c^{p-1}))$. Moreover, according to the Young inequality

$$\begin{aligned} 1 &\geq I_\Phi^{p-1}((k-\varepsilon)x)\, I_\Psi\left(p_+(|(k-\varepsilon)x|)\right)\\ &\geq c^{p-1}\sum_{i=1}^{\infty}\Psi\left(p_+(|(k-\varepsilon)x(i)|)\right)\\ &\geq c^{p-1}\sum_{i=1}^{\infty}\left\{|(k-\varepsilon)x(i)|\, p_+(|(k-\varepsilon)x(i)|)\right.\\ &\qquad\qquad \left.-\Phi((k-\varepsilon)x(i))\right\}\\ &\geq c^{p-1}(\alpha-1)\, I_\Phi((k-\varepsilon)x)\\ &\geq c^{p-1}(\alpha-1)(k-\varepsilon)\, I_\Phi(x)\\ &\geq c^p(\alpha-1)(k-\varepsilon), \end{aligned} \quad (46)$$

since $\varepsilon > 0$ is arbitrary, we deduce that $k \leq 1/(\alpha-1)c^p < \infty$. □

Ye et al. [21] have proved that Orlicz function space as well as Orlicz sequence space equipped with the Luxemburg norm is P-convex if and only if it is reflexive; that is, Φ satisfies the suitable Δ_2-condition and ∇_2-condition (i.e., the Δ_2-condition at zero in the sequence case). We will prove now an analogous result for $l_{\Phi,p}$ in terms of $P(l_{\Phi,p})$.

Theorem 10. *If $\Phi \in \Delta_2(0) \cap \nabla_2(0)$, $1 \leq p < \infty$, then $P(l_{\Phi,p}) < 1/2$.*

Proof. If $\Phi \in \Delta_2(0)$, then due to Lemma 6, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} &(I_\Phi(x) \leq 1) \wedge \left(I_\Phi(y) < \frac{\delta k''}{2(2-\delta)}\right)\\ &\quad \Longrightarrow I_\Phi^p(x+y) \leq I_\Phi^p(x) + \varepsilon\left(\left(k'\right)^p - 1\right), \end{aligned} \quad (47)$$

where $k'' = \sup\{k : k \in K_p(x), \|x\|_{\phi,p} = 1\}$. According to Lemma 9, $k'' < \infty$. Set $\inf_{\|x\|_{\Phi,p}=1} I_\Phi(x) = c > 0$. If $K(l_{\Phi,p}) = 2$, then there exists $x \in S(l_{\Phi,p})$ such that $d_x > 2 - \delta$, so $d_{x,k} > 2 - \delta$ for all $k \in K_p(x)$.

Since $\in \nabla_2(0)$, we can find $\theta > 1$ such that

$$\Phi\left(\frac{u}{2}\right) \leq \frac{1}{2\theta}\Phi(u), \qquad \left(|u| \leq q_+\left(\Psi^{-1}\left(\frac{1}{c^{p-1}}\right)\right)\right). \tag{48}$$

Let us notice that

$$I_\Phi\left(\frac{\delta k}{2(2-\delta)}x\right) \leq \left\|\frac{\delta k}{2(2-\delta)}x\right\|_{\Phi,p} = \frac{\delta k}{2(2-\delta)} \leq \frac{\delta k''}{2(2-\delta)}. \tag{49}$$

Thus,

$$\begin{aligned}
\frac{k^p-1}{2^p} &= I_\Phi^p\left(\frac{kx}{d_{x,k}}\right) < I_\Phi^p\left(\frac{kx}{2-\delta}\right) \\
&= I_\Phi^p\left(\frac{kx}{2} + \frac{\delta kx}{2(2-\delta)}\right) \\
&\leq I_\Phi^p\left(\frac{kx}{2}\right) + \varepsilon\left(\left(k'\right)^p - 1\right) \\
&\leq \frac{1}{(2\theta)^p} I_\Phi^p(kx) + \varepsilon\left(k^p - 1\right) \\
&= \frac{1}{(2\theta)^p}\left(k^p - 1\right) + \varepsilon\left(k^p - 1\right) \\
&= \left(\frac{1}{(2\theta)^p} + \varepsilon\right)\left(k^p - 1\right);
\end{aligned} \tag{50}$$

we have $1/2^p \leq 1/(2\theta)^p + \varepsilon$. Since ε is arbitrary, we obtain $\theta < 1$; this is a contradiction. Therefore, $K(l_{\Phi,p}) < 2$ and $P(l_{\Phi,p}) < 1/2$. ☐

Corollary 11. *If* $X = l^{p_1}$ $(1 < p_1 < \infty)$, *then*

$$K\left(l^{p_1}\right) = 2^{1/p_1}, \qquad P\left(l^{p_1}\right) = \frac{1}{1 + 2^{1-(1/p_1)}}. \tag{51}$$

Proof. For any $x \in l^{p_1}$,

$$\begin{aligned}
\|x\|_{\Phi,p} &= \left(p_1 - 1\right)^{-1/pq_1} p_1^{1/p-1/p_1} \|x\|_{l^{p_1}} \\
&= \left(p_1 - 1\right)^{-1/pq_1} p_1^{1/p-1/p_1} \Phi^{-1}\left(I_\Phi(x)\right),
\end{aligned} \tag{52}$$

where $\Phi(u) = |u|^{p_1}/p_1$ and $1/p + 1/q = 1$, $1/p_1 + 1/q_1 = 1$.

In fact, since $\Phi(u) = |u|^{p_1}/p_1$, then $\Phi(\|x\|_{l^{p_1}}) = I_\Phi(x)$ and $\Phi^{-1}(u) = (p_1 u)^{1/p_1}$. Set

$$f(k) = \frac{1}{k^p}\left(1 + I_\Phi^p(kx)\right) = \frac{1}{k^p}\left(1 + \left(\frac{k^{p_1}\|x\|_{l^{p_1}}^{p_1}}{p_1}\right)^p\right). \tag{53}$$

By $f'(k) = 0$, we get $k_0 = (p_1 - 1)^{-1/pp_1} p_1^{1/p_1} 1/\|x\|_{l^{p_1}}$. Since $f''(k_0) < 0$, we have

$$\begin{aligned}
\|x\|_{\Phi,p} &= \inf_{k>0}\frac{1}{k}\left(1 + I_\Phi^p(kx)\right)^{1/p} = \left(f\left(k_0\right)\right)^{1/p} \\
&= \left(p_1 - 1\right)^{-1/pq_1} p_1^{1/p-1/p_1}\|x\|_{l^{p_1}}.
\end{aligned} \tag{54}$$

Set $\alpha = (p_1 - 1)^{-1/pq_1} p_1^{1/p-1/p_1}$. From the equation

$$\frac{k^p - 1}{2^p} = I_\Phi^p\left(\frac{kx}{d_{x,k}}\right) = \Phi^p\left(\frac{1}{\alpha}\left\|\frac{kx}{d_{x,k}}\right\|_{\Phi,p}\right) = \Phi^p\left(\frac{k}{\alpha d_{x,k}}\right), \tag{55}$$

we deduce that $d_{x,k} = (k/\alpha)(\Phi^{-1}(((k^p - 1)/2^p)^{1/p}))^{-1}$. Therefore,

$$\begin{aligned}
K\left(l^{p_1}\right) = d &= \sup_{\|x\|_{\Phi,p}=1}\inf_{k>1} d_{x,k} \\
&= \frac{1}{\alpha}\inf_{k>1}\left\{k\left(\Phi^{-1}\left(\left(\frac{k^p-1}{2^p}\right)^{1/p}\right)\right)^{-1}\right\} \\
&= \left(p_1 - 1\right)^{1/pq_1} p_1^{-1/p} 2^{1/p_1} \inf_{k>1}\frac{k}{\left(k^p - 1\right)^{1/pp_1}} \\
&= 2^{1/p_1}.
\end{aligned} \tag{56}$$

We have $K(l^{p_1}) = 2^{1/p_1}$. So $P(l^{p_1}) = 1/(1 + 2)^{1-(1/p_1)}$ for $1 < p_1 < \infty$. ☐

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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