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Research Article

Distance from Bloch-Type Functions to the Analytic Space $F(p,q,s)$

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The analytic space $F(p,q,s)$ can be embedded into a Bloch-type space. We establish a distance formula from Bloch-type functions to $F(p,q,s)$, which generalizes the distance formula from Bloch functions to BMOA by Peter Jones, and to $F(p,p-2,s)$ by Zhao.

1. Introduction

Let $\mathbb{D}$ denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$ and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be its boundary. As usual, $H(\mathbb{D})$ denotes the space of all analytic functions on $\mathbb{D}$.

Recall that, for $0 < \alpha < \infty$, the Bloch-type space $\mathscr{B}_\alpha$ is the space of analytic functions on $\mathbb{D}$ satisfying

$$\|f\|_{\mathscr{B}_\alpha} = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty. \tag{1}$$

The little Bloch-type space $\mathscr{B}_\alpha^0$ is the subspace of all $f \in \mathscr{B}_\alpha$ with

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| = 0. \tag{2}$$

It is well known that $\mathscr{B}_\alpha$ is a Banach space under the norm

$$\|f\|_{\mathscr{B}_\alpha}^* = |f(0)| + \|f\|_{\mathscr{B}_\alpha}. \tag{3}$$

In particular, when $\alpha = 1$, $\mathscr{B}_\alpha$ becomes the classic Bloch space $\mathscr{B}$, which is the maximal Möbius invariant Banach space that has a decent linear functional; see [1, 2] for more details on the Bloch spaces.

For $a \in \mathbb{D}$, the involution of the unit disk is denoted by $\sigma_a(z) = (a-z)/(1-\bar{a}z)$. It is well known and easy to check that

$$1 - \left|\sigma_a(z)\right|^2 = \frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{|1-\bar{a}z|^2} = \left(1-|a|^2\right)\left|\sigma_a'(z)\right|. \tag{4}$$

Let $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$, and $-1 < q+s < \infty$. The space $F(p,q,s)$, introduced by Zhao in [3] and known as the *general family of function spaces*, is defined as the set of $f \in H(\mathbb{D})$ for which

$$\|f\|_{F(p,q,s)}^p = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^q \left(1-\left|\sigma_a(z)\right|^2\right)^s \mathrm{d}A(z) < \infty, \tag{5}$$

where $\mathrm{d}A(z)$ is the normalized area measure on $\mathbb{D}$. The space $F_0(p,q,s)$ consists of all $f \in F(p,q,s)$ such that

$$\lim_{|a|\to 1} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^q \left(1-\left|\sigma_a(z)\right|^2\right)^s \mathrm{d}A(z) = 0. \tag{6}$$

For appropriate parameter values p, q, and s, $F(p,q,s)$ coincides with several classical function spaces. For instance, $F(p,q,s) = \mathscr{B}_{(q+2)/p}$ if $1 < s < \infty$. The space $F(p,p,0)$ is the classical Bergman space $L_a^p(\mathbb{D})$, and $F(p,p-2,0)$ is the classical Besov space B_p. The spaces $F(2,0,s)$ are the Q_s spaces, in particular, $F(2,0,1) = \mathrm{BMOA}$, and the function space of *bounded mean oscillation*. See [3–9] for these basic facts.

For $0 < s < \infty$, we say that a nonnegative Borel measure μ defined on $\mathbb{D}$ is an s-Carleson measure if

$$\|\mu\|_{\mathscr{CM}_s} = \sup_{I\subset\mathbb{T}} \frac{\mu(S(I))}{|I|^s} < \infty, \tag{7}$$

where the supremum ranges over all subarcs I of $\mathbb{T}$, $|I|$ denotes the arc length of I, and

$$S(I)=\left\{z=re^{i\theta}\in\mathbb{D}:1-|I|\le r<1,\ e^{i\theta}\in I\right\}\tag{8}$$

is the Carleson square based on a subarc $I\subseteq\mathbb{T}$. We write $\mathscr{CM}_s$ for the class of all s-Carleson measures. Moreover, μ is said to be a vanishing s-Carleson measure if

$$\lim_{|I|\to 0}\frac{\mu(S(I))}{|I|^s}=0.\tag{9}$$

For f an analytic function on $\mathbb{D}$, we define

$$\mathrm{d}\mu_f(z)=\left|f'(z)\right|^p\left(1-|z|^2\right)^{q+s}\mathrm{d}A(z).\tag{10}$$

It was proved in [3] that $f\in F(p,q,s)$ if and only if $\mathrm{d}\mu_f$ is an s-Carleson measure and $f\in F_0(p,q,s)$ if and only if $\mathrm{d}\mu_f$ is a vanishing s-Carleson measure.

Let $X\subset\mathscr{B}_\alpha$ be an analytic function space. The distance from a Bloch-type function f to X is defined by

$$\operatorname{dist}_{\mathscr{B}_\alpha}(f,X)=\inf_{g\in X}\|f-g\|_{\mathscr{B}_\alpha}.\tag{11}$$

The following result is obtained by Zhao in [9].

Theorem 1. *Suppose $1\le p<\infty$, $0<s\le 1$, and $f\in\mathscr{B}$. The following two quantities are equivalent:*

(1) $\operatorname{dist}_{\mathscr{B}}(f,F(p,p-2,s))$;

(2) $\inf\{\varepsilon:\chi_{\Omega_\varepsilon(f)}(1-|z|^2)^{s-2}\,\mathrm{d}A(z)$ *is a Carleson measure*$\}$,

where $\Omega_\varepsilon(f)=\{z\in\mathbb{D}:|f'(z)|(1-|z|^2)\ge\varepsilon\}$ and χ denotes the characteristic function of a set.

When $p=2$ and $s=1$, the above characterization is Peter Jone's distance formula from a Bloch function to BMOA (Peter Jone never published his result but a proof was provided in [10]). Also, similar type results can be found in [11–13]. Specifically, distance from Bloch function to Q_K-type space is given in [11]; to the little Bloch space is obtained in [12], and to the Q_p space of the ball is characterized in [13]. All these spaces are Möbius invariant.

This paper is dedicated to characterize the distance from $f\in\mathscr{B}_{(q+2)/p}$ to $F(p,q,s)$, which extends Zhao's result. The main result is following.

Theorem 2. *Suppose $1\le p<\infty$, $0<s\le 1$, $-1<q+s<\infty$, and $f\in\mathscr{B}_{(2+q)/p}$. Then*

$$\begin{aligned}&\operatorname{dist}_{\mathscr{B}_{(q+2)/p}}(f,F(p,q,s))\\&\quad\approx\inf\left\{\varepsilon>0:\chi_{\widetilde{\Omega}_\varepsilon(f)}(z)\left(1-|z|^2\right)^{s-2}\mathrm{d}A(z)\in\mathscr{CM}_s\right\},\end{aligned}\tag{12}$$

where

$$\widetilde{\Omega}_\varepsilon(f)=\left\{z\in\mathbb{D}:\left(1-|z|^2\right)^{(q+2)/p}\left|f'(z)\right|\ge\varepsilon\right\}.\tag{13}$$

The strategy in this paper follows from Theorem 3.1.3 in [14]. The distance from a $\mathscr{B}_\alpha$ function to Campanato-Morrey space was given in [15] with similar idea.

Notation. Throughout this paper, we only write $U\lesssim V$ (or $V\gtrsim U$) for $U\le cV$ for a positive constant c, and moreover $U\approx V$ for both $U\lesssim V$ and $V\lesssim U$.

2. Preliminaries

We begin with a lemma quoted from Lemma 3.1.1 in [14].

Lemma 3. *Let $s,\alpha\in(0,\infty)$, and μ be nonnegative Radon measures on $\mathbb{D}$. Then, $\mu\in\mathscr{CM}_s$ if and only if*

$$\|\mu\|_{\mathscr{CM}_s,\alpha}=\sup_{w\in\mathbb{D}}\int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^\alpha}{|1-\overline{w}z|^{\alpha+s}}\mathrm{d}\mu(z)<\infty.\tag{14}$$

According to Lemma 3 and the fact that $f\in F(p,q,s)$ if and only if $\mathrm{d}\mu_f$ is an s-Carleson measure, we can easily get the following corollary.

Corollary 4. *Let f be an analytic function on $\mathbb{D}$. $f\in F(p,q,s)$ if and only if there exists an $\alpha>0$ such that*

$$\begin{aligned}&\|f\|^p_{F(p,q,s),\alpha}\\&\quad=\sup_{w\in\mathbb{D}}\int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^\alpha}{|1-\overline{w}z|^{\alpha+s}}\left|f'(z)\right|^p\left(1-|z|^2\right)^{q+s}\mathrm{d}A(z)<\infty.\end{aligned}\tag{15}$$

We will also need the following standard result from [16].

Lemma 5. *Suppose $t>-1$ and $c>0$. Then,*

$$\int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^t}{|1-\overline{w}z|^{2+t+c}}\mathrm{d}A(w)\approx\frac{1}{\left(1-|z|^2\right)^c}\tag{16}$$

for all $z\in\mathbb{D}$.

The following lemma, quoted from Lemma 1 in [9], is an extension of Lemma 5. See also [17].

Lemma 6. *Suppose $s>-1$ and $r,t>0$. If $t<s+2<r$, then*

$$\int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^s}{|1-\overline{w}z|^r|1-\overline{w}\zeta|^t}\mathrm{d}A(w)\lesssim\frac{1}{\left(1-|z|^2\right)^{r-s-2}\left|1-\overline{\zeta}z\right|^t}.\tag{17}$$

Next, we see that $F(p,q,s)$ is contained in $\mathscr{B}_{(2+q)/p}$. We thank Zhao for pointing out that the following result is firstly proved in [3]. Here, we give another proof with a different approach.

Lemma 7. *For $1\le p<\infty$, $-2<q<\infty$, and $0\le s<\infty$, $F(p,q,s)\subset\mathscr{B}_{(2+q)/p}$. In particular, if $s>1$, then $F(p,q,s)=\mathscr{B}_{(2+q)/p}$.*

Proof. We can use the reproducing formula for f' to get that

$$f'(z) = C \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{b-1} f'(w)}{(1-\overline{w}z)^{b+1}} \mathrm{d}A(w) \tag{18}$$

for some constant C, where b is a real number greater than $1 + (q+s)/p$; see, for example, [14, page 55].

Let $0 < \alpha < 2 + q$. If $p > 1$, denote $p' = p/(p-1)$; it follows from the Hölder's inequality and (15) that

$$\begin{aligned}
&\left|f'(z)\right| \\
&\quad\lesssim \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{(q+s)/p}\left(1-|z|^2\right)^{\alpha/p}\left|f'(w)\right|}{|1-\overline{w}z|^{(s+\alpha)/p}} \\
&\qquad\times \frac{\left(1-|w|^2\right)^{b-1-(q+s)/p}\mathrm{d}A(w)}{\left(1-|z|^2\right)^{\alpha/p}|1-\overline{w}z|^{b+1-(s+\alpha)/p}} \\
&\quad\lesssim \left(\int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^{\alpha}}{|1-\overline{w}z|^{s+\alpha}} |f'(w)|^p \left(1-|w|^2\right)^{q+s} \mathrm{d}A(w)\right)^{1/p} \\
&\qquad\times \left(\int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{p'(b-1-(q+s)/p)}\mathrm{d}A(w)}{\left(1-|z|^2\right)^{p'(\alpha/p)}|1-\overline{w}z|^{p'(b+1-(s+\alpha)/p)}}\right)^{1/p'} \\
&\quad\lesssim \frac{\|f\|_{F(p,q,s),\alpha}}{\left(1-|z|^2\right)^{\alpha/p}} \\
&\qquad\times \left(\int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{p'(b-1-(q+s)/p)}\mathrm{d}A(w)}{|1-\overline{w}z|^{p'(b+1-(s+\alpha)/p)}}\right)^{1/p'} \\
&\quad\lesssim \|f\|_{F(p,q,s),\alpha} \frac{1}{\left(1-|z|^2\right)^{\alpha/p}} \\
&\qquad\times \left(\frac{1}{\left(1-|z|^2\right)^{(2-\alpha+q)/(p-1)}}\right)^{1/p'} \\
&\quad= \|f\|_{F(p,q,s),\alpha} \frac{1}{\left(1-|z|^2\right)^{(2+q)/p}}.
\end{aligned} \tag{19}$$

Apparently, we have used Lemma 5 in the last inequality. This gives that $F(p,q,s) \subset \mathscr{B}_{(q+2)/p}$ when $1 < p < \infty$.

If $p = 1$, then

$$\begin{aligned}
&\left(1-|z|^2\right)^{2+q}\left|f'(z)\right| \\
&\quad\lesssim \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{q+s}\left(1-|z|^2\right)^{\alpha}\left|f'(w)\right|}{|1-\overline{w}z|^{\alpha+s}} \\
&\qquad\times \frac{\left(1-|w|^2\right)^{b-1-q-s}\mathrm{d}A(w)}{\left(1-|z|^2\right)^{\alpha-2-q}|1-\overline{w}z|^{b+1-s-\alpha}} \\
&\quad\lesssim \int_{\mathbb{D}} \left|f'(w)\right|\left(1-|w|^2\right)^{q+s} \frac{\left(1-|z|^2\right)^{\alpha}}{|1-\overline{w}z|^{\alpha+s}} \mathrm{d}A(w) \\
&\qquad\times \sup_{w\in\mathbb{D}} \frac{\left(1-|w|^2\right)^{b-1-q-s}\left(1-|z|^2\right)^{2+q-\alpha}}{|1-\overline{w}z|^{b+1-\alpha-s}} \\
&\quad\lesssim \|f\|_{F(p,q,s),\alpha} \sup_{w\in\mathbb{D}} \frac{\left(1-|w|^2\right)^{b-1-q-s}\left(1-|z|^2\right)^{2+q-\alpha}}{|1-\overline{w}z|^{b+1-\alpha-s}}.
\end{aligned} \tag{20}$$

Recall that $b > 1 + q + s$ and $0 < \alpha < 2 + q$. We can easily use (4) to check that

$$\sup_{z,w\in\mathbb{D}} \frac{\left(1-|w|^2\right)^{b-1-q-s}\left(1-|z|^2\right)^{2+q-\alpha}}{|1-\overline{w}z|^{b+1-\alpha-s}} \lesssim 1. \tag{21}$$

Thus, $F(p,q,s) \subset \mathscr{B}_{(q+2)/p}$ when $p = 1$.

Now, suppose $s > 1$ and let $f \in \mathscr{B}_{(q+2)/p}$, then

$$\left|f'(z)\right|\left(1-|z|^2\right)^{(q+2)/p} \leq \|f\|_{(q+2)/p} < \infty \tag{22}$$

for all $z \in \mathbb{D}$. It follows that

$$\begin{aligned}
&\|f\|_{F(p,q,s)}^p \\
&\quad= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^{q+s} \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^s \mathrm{d}A(z) \\
&\quad= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^{q+2} \\
&\qquad\times \left(1-|z|^2\right)^{s-2} \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^s \mathrm{d}A(z) \\
&\quad\lesssim \|f\|_{\mathscr{B}_{(q+2)/p}}^p \sup_{a\in\mathbb{D}} \left(1-|a|^2\right)^s \int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^{s-2}}{|1-\overline{a}z|^{2s}} \mathrm{d}A(z) \\
&\quad\approx \|f\|_{\mathscr{B}_{(q+2)/p}}^p.
\end{aligned} \tag{23}$$

Again, the above inequality follows from Lemma 5. This completes the proof. □

Our strategy relies on an integral operator preserving the s-Carleson measures. For $a, b > 0$, we define the integral operator $T_{a,b}$ as

$$T_{a,b} f(z) = \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{b-1}}{|1-\overline{w}z|^{a+b}} f(w)\, \mathrm{d}A(w) \quad \forall z \in \mathbb{D}. \tag{24}$$

The following lemma is similar to Theorem 2.5 in [18]. Indeed, Qiu and Wu proved the case $1 < p < \infty$. Specially, the $p = 2$ case is just Lemma 3.1.2 in [14].

Lemma 8. *Assume $0 < s \leq 1$, $1 \leq p < \infty$, and $\alpha > -1$. Let $b > (\alpha+1)/p$, let $a > 1 - (\alpha+1)/p$, and let f be Lebesgue measurable on $\mathbb{D}$. If $|f(z)|^p(1-|z|^2)^{\alpha}\, \mathrm{d}A(z)$ belongs to $\mathscr{CM}_s$, then $|T_{a,b}f(z)|^p(1-|z|^2)^{p(a-1)+\alpha}\, \mathrm{d}A(z)$ also belongs to $\mathscr{CM}_s$.*

Proof. We firstly prove the case $p = 1$ and then sketch the outline argument of the case $1 < p < \infty$ modified from [18] for the completeness.

When $p = 1$, according to Lemma 3, it is sufficient to show that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^x}{|1-\overline{a}z|^{x+s}} \left|T_{a,b} f(z)\right| \left(1-|z|^2\right)^{a-1+\alpha} dA(z) < \infty \tag{25}$$

for some $x > 0$. That is to show

$$\begin{aligned} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^x}{|1-\overline{a}z|^{x+s}} &\left| \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{b-1} f(w)}{|1-\overline{w}z|^{a+b}} dA(w) \right| \\ &\times \left(1-|z|^2\right)^{a-1+\alpha} dA(z) \end{aligned} \tag{26}$$

is finite. By Fubini's theorem, it is enough to verify that

$$\begin{aligned} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(1-|a|^2\right)^x \int_{\mathbb{D}} &\frac{\left(1-|z|^2\right)^{a-1+\alpha} dA(z)}{|1-\overline{w}z|^{a+b}|1-\overline{a}z|^{x+s}} \\ &\times |f(w)| \left(1-|w|^2\right)^{b-1} dA(w) \end{aligned} \tag{27}$$

is finite.

Choosing x such that $x+s < a+1+\alpha$, we can use Lemma 6 to control the last integral by

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^x}{|1-\overline{a}w|^{x+s}} |f(w)| \left(1-|w|^2\right)^{\alpha} dA(w). \tag{28}$$

Since $|f(z)|(1-|z|^2)^{\alpha} dA(z)$ is an s-Carleson measure, we can complete the proof by using Lemma 3 again.

When $1 < p < \infty$, we need to verify that

$$\frac{1}{|I|^s} \int_{S(I)} \left|T_{a,b} f(z)\right|^p \left(1-|z|^2\right)^{p(a-1)+\alpha} dA(z) \lesssim 1 \tag{29}$$

holds for any arc $I \subset \mathbb{T}$. In order to make this estimate, let N_I, be the biggest integer satisfying $N_I \leq -\log_2|I|$, and let I_n, $n = 0, 1, 2, \ldots, N_I$, denotes the arcs on $\mathbb{T}$ with the same center as I and length $2^n|I|$, and I_{N_I+1} is just $\mathbb{T}$. We can control and decompose the integral as

$$\begin{aligned} &\int_{S(I)} \left|T_{a,b} f(z)\right|^p \left(1-|z|^2\right)^{p(a-1)+\alpha} dA(z) \\ &\lesssim \int_{S(I)} \left(\int_{S(I_1)} \frac{\left(1-|w|^2\right)^{b-1} \left(1-|z|^2\right)^{(a-1)+\alpha/p}}{|1-\overline{w}z|^{a+b}} \times |f(w)| \, dA(w) \right)^p dA(z) \\ &\quad + \int_{S(I)} \left(\int_{\mathbb{D}\setminus S(I_1)} \frac{\left(1-|w|^2\right)^{b-1} \left(1-|z|^2\right)^{(a-1)+\alpha/p}}{|1-\overline{w}z|^{a+b}} \times |f(w)| \, dA(w) \right)^p dA(z) \\ &= \mathrm{Int}_1 + \mathrm{Int}_2. \end{aligned} \tag{30}$$

In order to estimate Int_1, we define the linear operator $B : L^p(\mathbb{D}) \to L^p(\mathbb{D})$ as

$$B(f)(z) = \int_{\mathbb{D}} K(z, w) f(w) \, dA(w), \tag{31}$$

where

$$K(z, w) = \frac{\left(1-|w|^2\right)^{b-1} \left(1-|z|^2\right)^{(a-1)+\alpha/p}}{|1-\overline{w}z|^{a+b}}. \tag{32}$$

If we choose a test function $g(z) = (1-|z|^2)^{-1/pp'}$, then Schur's lemma combines with Lemma 5 implying that

$$\begin{aligned} &\int_{\mathbb{D}} K(w, z) g^p(w) \, dA(w) \lesssim g^p(z), \\ &\int_{\mathbb{D}} K(w, z) g^{p'}(z) \, dA(z) \lesssim g^{p'}(w). \end{aligned} \tag{33}$$

Hence, B is a bounded operator. Letting $h(w) = |f(w)|(1-|w|^2)^{\alpha/p} \chi_{S(I_1)}(w)$, then $h \in L^p(\mathbb{D})$ with

$$\|h\|_{L^p}^p = \int_{S(I_1)} |f(w)|^p \left(1-|w|^2\right)^{\alpha} dA(w) \lesssim |I|^s. \tag{34}$$

Thus,

$$\mathrm{Int}_1 \lesssim \int_{\mathbb{D}} |B(h)(z)|^p dA(z) = \|B(h)\|_{L^p}^p \lesssim \|h\|_{L^p}^p \lesssim |I|^s. \tag{35}$$

To handle Int_2, first note that, for $n = 0, 1, \ldots, N_I$, if $z \in S(I)$ and $w \in S(I_{n+1}) \setminus S(I_n)$, then $|1-\overline{w}z| \gtrsim 2^n|I|$. Further, it is easy to check that, for any fixed $\beta > -1$,

$$\int_{S(I_n)} \left(1-|w|^2\right)^{\beta} dA(w) \lesssim \left(2^n |I|\right)^{\beta+2}, \quad n = 0, 1, \ldots, N_I. \tag{36}$$

Now, splitting $\mathbb{D} \setminus S(I_1)$ as

$$\bigcup_{n=1}^{N_I} S(I_{n+1}) \setminus S(I_n) = \bigcup_{n=1}^{N_I} \widetilde{S}_{n+1}, \tag{37}$$

we have

$$\begin{aligned}\mathrm{Int}_2 &\lesssim \int_{S(I)}\left|\sum_{n=1}^{N_I}\int_{\widetilde{S}_{n+1}}\frac{\left(1-|w|^2\right)^{b-1}|f(w)|}{|1-\overline{w}z|^{a+b}}\mathrm{d}A(w)\right|^p \\ &\quad\times\left(1-|z|^2\right)^{p(a-1)+\alpha}\mathrm{d}A(z) \\ &\lesssim |I|^{p(a-1)+\alpha+2} \\ &\quad\times\left(\sum_{n=1}^{N_I}\frac{1}{\left(2^n|I|\right)^{a+b}}\right. \\ &\quad\left.\times\int_{S(I_{n+1})}\left(1-|w|^2\right)^{b-1}|f(w)|\,\mathrm{d}A(w)\right)^p.\end{aligned} \tag{38}$$

Recall that $|f(z)|^p(1-|z|^2)^\alpha \mathrm{d}A(z) \in \mathscr{CM}_s$. It follows from Hölder's inequality that

$$\begin{aligned}&\int_{S(I_{n+1})}\left(1-|w|^2\right)^{b-1}|f(w)|\,\mathrm{d}A(w) \\ &\quad\lesssim \left|I_{n+1}\right|^{s/p}\cdot\left(2^{n+1}|I|\right)^{b-1-\alpha/p+2/p'}.\end{aligned} \tag{39}$$

Now, an easy computation gives that

$$\mathrm{Int}_2 \lesssim \left(\sum_{n=1}^{N_I}2^{-n(a-1+(\alpha+2-s)/p)}\right)^p|I|^s \lesssim |I|^s, \tag{40}$$

since $a > 1-(\alpha+1)/p$ and $0 < s \le 1$. This completes the proof. □

3. Proof of the Main Result

Proof of Theorem 2. For $f \in \mathscr{B}_{(q+2)/p}$, it is easy to establish the following formula (see, e.g., [19, (1.1)] or [14, page 55]. Notice that it is a special case of the α-order derivative of f, as $\alpha = 0$ in [14], which holds for all holomorphic f on $\mathbb{D}$). Consider

$$f(z) = f(0) + \int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^{(q+2)/p}f'(w)}{\overline{w}(1-\overline{w}z)^{1+(q+2)/p}}\mathrm{d}A(w) \quad \forall z \in \mathbb{D}. \tag{41}$$

Define, for each $\varepsilon > 0$,

$$\begin{aligned}f_1(z) &= f(0) + \int_{\widetilde{\Omega}_\varepsilon(f)}\frac{\left(1-|w|^2\right)^{(q+2)/p}f'(w)}{\overline{w}(1-\overline{w}z)^{1+(q+2)/p}}\mathrm{d}A(w), \\ f_2(z) &= \int_{\mathbb{D}\backslash\widetilde{\Omega}_\varepsilon(f)}\frac{\left(1-|w|^2\right)^{(q+2)/p}f'(w)}{\overline{w}(1-\overline{w}z)^{1+(q+2)/p}}\mathrm{d}A(w).\end{aligned} \tag{42}$$

Then,

$$\begin{aligned}&\left|f_1'(z)\right| \\ &\quad\lesssim \|f\|_{\mathscr{B}_{(q+2)/p}}\int_{\mathbb{D}}\frac{\chi_{\widetilde{\Omega}_\varepsilon(f)}(w)}{|1-\overline{w}z|^{2+(q+2)/p}}\mathrm{d}A(w) \\ &\quad= \|f\|_{\mathscr{B}_{(q+2)/p}}\int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^{2/p}}{|1-\overline{w}z|^{2+(q+2)/p}} \\ &\qquad\times\frac{\chi_{\widetilde{\Omega}_\varepsilon(f)}(w)}{\left(1-|w|^2\right)^{2/p}}\mathrm{d}A(w).\end{aligned} \tag{43}$$

Write

$$g(w) = \frac{\chi_{\widetilde{\Omega}_\varepsilon(f)}(w)}{\left(1-|w|^2\right)^{2/p}}. \tag{44}$$

Then,

$$|g(w)|^p\left(1-|w|^2\right)^s\mathrm{d}A(w) = \chi_{\widetilde{\Omega}_\varepsilon(f)}(w)\left(1-|w|^2\right)^{s-2}\mathrm{d}A(w). \tag{45}$$

So, if

$$\chi_{\widetilde{\Omega}_\varepsilon(f)}(z)\left(1-|z|^2\right)^{s-2}\mathrm{d}A(z) \tag{46}$$

is in $\mathscr{CM}_s$, Lemma 8 implies that

$$\left|f_1'(z)\right|^p\left(1-|z|^2\right)^{q+s}\mathrm{d}A(z) \in \mathscr{CM}_s. \tag{47}$$

By Corollary 4, $f_1 \in F(p,q,s)$. Meanwhile, recall that, for $w \in \mathbb{D}\backslash\widetilde{\Omega}_\varepsilon(f)$ and $(1-|w|^2)^{(q+2)/p}|f'(w)| < \varepsilon$, we can use Lemma 5 to obtain

$$\begin{aligned}\left|f_2'(z)\right| &\le \int_{\mathbb{D}\backslash\widetilde{\Omega}_\varepsilon(f)}\frac{\left(1-|w|^2\right)^{(q+2)/p}\left|f'(w)\right|}{|1-\overline{w}z|^{2+(q+2)/p}}\mathrm{d}A(w) \\ &< \varepsilon\int_{\mathbb{D}}\frac{1}{|1-\overline{w}z|^{2+(q+2)/p}}\mathrm{d}A(w) \\ &\approx \frac{\varepsilon}{\left(1-|z|^2\right)^{(2+q)/p}}.\end{aligned} \tag{48}$$

This means that

$$\left(1-|z|^2\right)^{(2+q)/p}\left|f_2'(z)\right| \lesssim \varepsilon. \tag{49}$$

To summarize the above argument, we have $f = f_1 + f_2$, $f_1 \in F(p,q,s)$ (by (47)), and $f_2 \in \mathscr{B}_{(2+q)/p}$ (by (49)), and $\chi_{\widetilde{\Omega}_\varepsilon(f)}(z)(1-|z|^2)^{s-2}\mathrm{d}A(z)$ is an s-Carleson measure for each $\varepsilon > 0$. Thus,

$$\begin{aligned}&\mathrm{dist}_{\mathscr{B}_{(2+q)/p}}(f, F(p,q,s)) \\ &\quad\lesssim \inf\left\{\varepsilon > 0 : \chi_{\widetilde{\Omega}_\varepsilon(f)}(z)\left(1-|z|^2\right)^{s-2}\mathrm{d}A(z) \in \mathscr{CM}_s\right\}.\end{aligned} \tag{50}$$

In order to prove the other direction of the inequality, we assume that ε_0 equals the right-hand quantity of the last inequality and

$$\operatorname{dist}_{\mathscr{B}_{(2+q)/p}}(f, F(p,q,s)) < \varepsilon_0. \tag{51}$$

We only consider the case $\varepsilon_0 > 0$. Then, there exists an ε_1 such that

$$0 < \varepsilon_1 < \varepsilon_0, \quad \operatorname{dist}_{\mathscr{B}_{(2+q)/p}}(f, F(p,q,s)) < \varepsilon_1. \tag{52}$$

Hence, by definition, we can find a function $h \in F(p,q,s)$ such that

$$\|f - h\|_{\mathscr{B}_{(2+q)/p}} < \varepsilon_1. \tag{53}$$

Now, for any $\varepsilon \in (\varepsilon_1, \varepsilon_0)$, we have that

$$\chi_{\widetilde{\Omega}_\varepsilon(f)}(z)\left(1-|z|^2\right)^{s-2} dA(z) \tag{54}$$

is not in $\mathscr{CM}_s$. But, according to (53), we get

$$\left(1-|z|^2\right)^{(2+q)/p}\left|h'(z)\right| > \left(1-|z|^2\right)^{(2+q)/p}\left|f'(z)\right| - \varepsilon_1 \quad \forall z \in \mathbb{D}, \tag{55}$$

and so

$$\chi_{\widetilde{\Omega}_\varepsilon(f)}(z) \le \chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z) \quad \forall z \in \mathbb{D}. \tag{56}$$

This implies that

$$\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)\left(1-|z|^2\right)^{s-2} dA(z) \tag{57}$$

does not belong to $\mathscr{CM}_s$. But, it follows from (13) that $\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h) = \{z \in \mathbb{D} : (1-|z|^2)^{(q+2)/p}|h'(z)| \ge \varepsilon - \varepsilon_1\}$. Therefore,

$$\begin{aligned}
&\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)\left(1-|z|^2\right)^{s-2} dA(z) \\
&\quad = \chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)\frac{\left(1-|z|^2\right)^{q+s}}{\left(1-|z|^2\right)^{q+2}} dA(z) \\
&\quad \le \frac{\left|h'(z)\right|^p}{(\varepsilon-\varepsilon_1)^p}\left(1-|z|^2\right)^{q+s}\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)\, dA(z) \\
&\quad \le \frac{1}{(\varepsilon-\varepsilon_1)^p}\left|h'(z)\right|^p\left(1-|z|^2\right)^{q+s} dA(z).
\end{aligned} \tag{58}$$

Since $h \in F(p,q,s)$, Corollary 4 implies that

$$\left|h'(z)\right|^p\left(1-|z|^2\right)^{q+s} dA(z) \tag{59}$$

is in $\mathscr{CM}_s$. This means that

$$(\varepsilon-\varepsilon_1)^p\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)\left(1-|z|^2\right)^{s-2} dA(z) \tag{60}$$

is in $\mathscr{CM}_s$, and so is $\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)(1-|z|^2)^{s-2}dA(z)$. This contradicts (57). Thus, we must have

$$\operatorname{dist}_{\mathscr{B}_{(2+q)/p}}(f, F(p,q,s)) \ge \varepsilon_0 \tag{61}$$

as required. □

Remark 9. Theorem 2 characterizes the closure of $F(p,q,s)$ in the $\mathscr{B}_{(q+2)/p}$ norm. That is, for $f \in \mathscr{B}_{(q+2)/p}$, f is in the closure of $F(p,q,s)$ in the $\mathscr{B}_{(q+2)/p}$ norm if and only if, for every $\varepsilon > 0$,

$$\int_{\widetilde{\Omega}_\varepsilon(f)\cap S(I)}\left(1-|z|^2\right)^{s-2} dA(z) \lesssim |I|^s \tag{62}$$

for any Carleson square $S(I)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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