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Research Article

Global Optimization for the Sum of Certain Nonlinear Functions

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We extend work by Pei-Ping and Gui-Xia, 2007, to a global optimization problem for more general functions. Pei-Ping and Gui-Xia treat the optimization problem for the linear sum of polynomial fractional functions, using a branch and bound approach. We prove that this extension makes possible to solve the following nonconvex optimization problems which Pei-Ping and Gui-Xia, 2007, cannot solve, that the sum of the positive (or negative) first and second derivatives function with the variable defined by sum of polynomial fractional function by using branch and bound algorithm.

1. Introduction

The optimization problem is widely used in sciences, especially in engineering and economy [1–3]. In 2007, Pei-Ping and Gui-Xia considered one global optimization problem in [4]:

$$\begin{aligned} \min \quad & \omega(x) = \sum_{j=1}^{P} c_j \frac{b_j(x)}{a_j(x)} \\ \text{s.t.} \quad & g_k(x) \le 0, \quad x \in X, \end{aligned} \tag{P}$$

where $X := \{x \in \mathbf{R}^N \mid 0 < \underline{x}_i \le x_i \le \overline{x}_i < \infty\ (i = 1, 2, \ldots, N)\}$, $a_j(x)$, $b_j(x)$, $g_k(x)$ are given generalized polynomial. One has

$$\begin{gathered} a_j(x) := \sum_{t=1}^{T_j^a} \beta_{jt}^a \prod_{i=1}^{N} x_i^{\gamma_{jti}^a}, \qquad b_j(x) := \sum_{t=1}^{T_j^b} \beta_{jt}^b \prod_{i=1}^{N} x_i^{\gamma_{jti}^b}, \\ g_k(x) := \sum_{t=1}^{T_k^g} \beta_{kt}^a \prod_{i=1}^{N} x_i^{\gamma_{kti}^g}. \end{gathered} \tag{1}$$

Sum of rations problems like (P) attract a lot of attention, and the reason is that these problems are applied to various economical problems [4].

Pei-Ping and Gui-Xia proposed the method to solve these problems globally by using branch and bound algorithm in [4]. In the above problem, the objective function and constrained function are sums of generalized polynomial fractional functions. We extend these functions to more general functions like below:

$$\begin{aligned} \min \quad & w(x) = \sum_{j=1}^{P} h_j\left(\frac{b_j(x)}{a_j(x)}\right) \\ \text{s.t.} \quad & g_k(x) = \sum_{\acute{j}=1}^{P_k} h_{k\acute{j}}\left(\frac{d_{k\acute{j}}(x)}{c_{k\acute{j}}(x)}\right) \le 0 \\ & \left(\acute{j} = 1, \ldots, P_K,\ k = 1, \ldots, M\right), \\ & x \in X, \end{aligned} \tag{$P0$}$$

where $a_j(x) > 0$, $b_j(x) > 0$, $c_{k\acute{j}}(x) > 0$, $d_{k\acute{j}}(x) > 0$ $x \in X$; that is,

$$\begin{gathered} a_j(x) := \sum_{t=1}^{T_j^a} \beta_{jt}^a \prod_{i=1}^{N} x_i^{\gamma_{jti}^a}, \qquad b_j(x) := \sum_{t=1}^{T_j^b} \beta_{jt}^b \prod_{i=1}^{N} x_i^{\gamma_{jti}^b}, \\ c_{k\acute{j}}(x) := \sum_{t=1}^{T_{k\acute{j}}^c} \beta_{k\acute{j}t}^c \prod_{i=1}^{N} x_i^{\gamma_{k\acute{j}t}^c}, \end{gathered}$$

$$d_{k\acute{j}}(x) := \sum_{t=1}^{T^d_{k\acute{j}}} \beta^d_{k\acute{j}t} \prod_{i=1}^{N} x_i^{\gamma^d_{k\acute{j}t}},$$

$$\left(j = 1, 2, \ldots, P,\ \acute{j} = 1, 2, \ldots, P_k,\ k = 1, 2, \ldots, M\right), \tag{2}$$

where $T^a_j, T^b_j, T^c_{k\acute{j}}, T^d_{k\acute{j}}$ are natural numbers, and $\beta^a_{jt}, \beta^b_{jt}, \beta^c_{k\acute{j}t}, \beta^d_{k\acute{j}t}$ are real constants not zero, and $\gamma^a_{jti}, \gamma^b_{jti}, \gamma^c_{k\acute{j}t}, \gamma^d_{k\acute{j}t}$ are real constants.

We assume that $h_j(y_j), h_{k\acute{j}}(y_{k\acute{j}}) : \mathbf{R} \mapsto \mathbf{R}$ are secondly differentiable functions and monotone increasing or monotone decreasing functions. We divide these functions to monotone increasing or monotone decreasing as follows:

$$h'_j > 0 \quad (j = 1, \ldots, K), \qquad h'_j < 0 \quad (j = K+1, \ldots, P),$$
$$h'_{k\acute{j}} > 0 \quad \left(\acute{j} = 1, \ldots, K_k\right),$$
$$h'_{k\acute{j}} < 0 \quad \left(\acute{j} = K_k + 1, \ldots, P_k\right). \tag{3}$$

Furthermore, we assume the following conditions for the second derivatives:

$$\left\{h_j \circ \exp\left(z_j\right)\right\}'' > 0, \qquad \left\{h_j \circ \exp\left(z_j\right)\right\}'' < 0,$$
$$\left\{h_{k\acute{j}} \circ \exp\left(z_{k\acute{j}}\right)\right\}'' > 0, \qquad \left\{h_{k\acute{j}} \circ \exp\left(z_{k\acute{j}}\right)\right\}'' < 0 \tag{4}$$
$$\left(j = 1, \ldots, P,\ \acute{j} = 1, \ldots, P_k,\ k = 1, \ldots, M\right).$$

To solve the above problem ($P0$), we transform the problem ($P0$) to the equivalent problems ($P1$), ($P2$) and transform ($P2$) into the linear relaxation problem. We prove the equivalency of the problems under above assumption, and we calculate the equivalent problem using branch and bound algorithm corresponding to [4–6].

For example, according to this extension at approach, we can calculate the following global optimization problem:

$$\begin{aligned} \min \quad & \sin\left(\frac{x_1^2 + 3x_2 - 2x_2^2 + 1}{x_1^2 + x_2 + 2}\right) \\ & + \cos\left(\frac{-x_2^2 + 2x_1 + 2x_2}{x_1 + 2.5}\right) \\ \text{s.t.} \quad & x_1^2 - \frac{x_1}{x_2} - 1 \le 0 \\ & x_1 + 3\frac{x_2}{x_1} - 5 \le 0 \\ & X = \left\{x : 1 \le x_1 \le 3, 1 \le x_2 \le 3\right\}. \end{aligned} \tag{5}$$

In this paper, we explain how to make equivalent relaxation linear problem from original problem in Section 2. In Section 3, we present the branch and bound algorithm and its convergence. In Section 4, we introduce numerical experiments result.

2. Equivalence Transformation and Linear Relaxation

In this section we firstly transform the problem ($P0$) to the equivalent problems ($P1$) and secondly transform ($P1$) to ($P2$). Thirdly we linearize the problem ($P2$) corresponding to [4].

2.1. Translation of the Problem ($P0$) *into* ($P1$). For the problem ($P0$), we put new variables $m_j, l_j, t_{k\acute{j}}$ and $s_{k\acute{j}}$, and the function $\rho(l,m)$ and $\xi_k(s,t)$ depending on h_j, $h_{k\acute{j}}$ in the original problem ($P0$):

$$\rho(l,m) := \sum_{j=1}^{P} h_j\left(\frac{m_j}{l_j}\right) \quad (j = 1, 2, \ldots, P),$$

$$\xi_k(s,t) := \sum_{\acute{j}=1}^{P_k} h_{k\acute{j}}\left(\frac{t_{k\acute{j}}}{s_{k\acute{j}}}\right) \quad \left(\acute{j} = 1, \ldots, P_k,\ k = 1, \ldots, M\right). \tag{6}$$

Since $a_j(x)$, $b_j(x)$, $c_{k\acute{j}}(x)$, $d_{k\acute{j}}(x)$ are polynomials on closed interval X, it is easy to calculate the minimums and maximums of the functions on X; we denote them by $\underline{a}_j, \overline{a}_j, \underline{b}_j, \overline{b}_j, \underline{c}_{k\acute{j}}, \overline{c}_{k\acute{j}}, \underline{d}_{k\acute{j}}, \overline{d}_{k\acute{j}}$.

Let H be the closed interval:

$$\begin{aligned} H := \Big\{ & (l,m,s,t) \in \mathbf{R}^{P_{\text{sum}}} \mid \\ & \underline{a}_j \le l_j \le \overline{a}_j,\ \underline{b}_j \le m_j \le \overline{b}_j, \\ & \underline{c}_{k\acute{j}} \le s_{k\acute{j}} \le \overline{c}_{k\acute{j}},\ \underline{d}_{k\acute{j}} \le t_{k\acute{j}} \le \overline{d}_{k\acute{j}} \Big\}, \end{aligned} \tag{7}$$

where $P_{\text{sum}} = 2(P + \sum_{k=1}^{M} P_k)$.

Let Z_H be the following closed domain in $X \times H$; that is,

$$\begin{aligned} Z_H := \Big\{ & (x,l,m,s,t) \in X \times H \mid \\ & \xi_k(s,t) \le 0 \quad (k = 1, \ldots, M), \\ & l_j - a_j(x) \le 0, \quad b_j(x) - m_j \le 0 \\ & \qquad (j = 1, \ldots, K), \\ & a_j(x) - l_j \le 0, \quad m_j - b_j(x) \le 0 \\ & \qquad (j = K+1, \ldots, P), \\ & s_{k\acute{j}} - c_{k\acute{j}}(x) \le 0, \quad d_{k\acute{j}}(x) - t_{k\acute{j}} \le 0 \\ & \qquad \left(\acute{j} = 1, \ldots, K_k, k = 1, \ldots, M\right), \\ & c_{k\acute{j}}(x) - s_{k\acute{j}} \le 0, \quad t_{k\acute{j}} - d_{k\acute{j}}(x) \le 0 \\ & \qquad \left(\acute{j} = K_k + 1, \ldots, P_k, k = 1, \ldots, M\right) \Big\}. \end{aligned} \tag{8}$$

We give the problem $(P1)$ on Z_H. Consider

$$\begin{aligned} \min \quad & \rho(l, m) \\ \text{s.t.} \quad & \xi_k(s, t) \leq 0, \quad (k = 1, \ldots, M), \\ & (x, l, m, s, t) \in Z_H. \end{aligned} \tag{P1}$$

Now we obtain Theorem 1 that proves the equivalence of $(P0)$ and $(P1)$.

Theorem 1. *The problem* $(P0)$ *on* X *is equivalent to the problem* $(P1)$ *on* Z_H.

Proof. Let x^* be the optimal solution for $(P0)$; we denote

$$\begin{aligned} l_j^* &:= a_j(x^*), \qquad m_j^* := b_j(x^*), \\ s_{k\acute{j}}^* &:= c_{k\acute{j}}(x^*), \qquad t_{k\acute{j}}^* := d_{k\acute{j}}(x^*), \end{aligned} \tag{9}$$

and then

$$\begin{aligned} \sum_{j=1}^{P} h_j\left(\frac{b_j(x^*)}{a_j(x^*)}\right) &= \sum_{j=1}^{P} h_j\left(\frac{m_j^*}{l_j^*}\right), \\ \sum_{\acute{j}=1}^{P_k} h_{k\acute{j}}\left(\frac{d_{k\acute{j}}(x^*)}{c_{k\acute{j}}(x^*)}\right) &= \sum_{\acute{j}=1}^{P_k} h_{k\acute{j}}\left(\frac{t_{k\acute{j}}^*}{s_{k\acute{j}}^*}\right). \end{aligned} \tag{10}$$

Furthermore let $(x^\sharp, l^\sharp, m^\sharp, s^\sharp, t^\sharp)$ be the optimal solution for $(P1)$. Then by the restricted condition we have the following:

for $j = 1, \ldots, K$, $0 < l_j^\sharp \leq a_j(x^\sharp)$ and $0 < b_j(x^\sharp) \leq m_j^\sharp$; that is, $0 < b_j(x^\sharp)/a_j(x^\sharp) \leq m_j^\sharp/l_j^\sharp$;

for $j = K + 1, \ldots, P$, $0 < a_j(x^\sharp) \leq l_j^\sharp$ and $0 < m_j^\sharp \leq b_j(x^\sharp)$; that is, $0 < m_j^\sharp/l_j^\sharp \leq b_j(x^\sharp)/a_j(x^\sharp)$;

for $\acute{j} = 1, \ldots, K_k$ and $k = 1, \ldots, M$, $0 < s_{k\acute{j}}^\sharp \leq c_{k\acute{j}}(x^\sharp)$ and $0 < d_{k\acute{j}}(x^\sharp) \leq t_{k\acute{j}}^\sharp$; that is, $0 < d_{k\acute{j}}(x^\sharp)/c_{k\acute{j}}(x^\sharp) \leq t_{k\acute{j}}^\sharp/s_{k\acute{j}}^\sharp$;

for $\acute{j} = K_k + 1, \ldots, P_k$ and $k = 1, \ldots, M$, $0 < c_{k\acute{j}}(x^\sharp) \leq s_{k\acute{j}}^\sharp$ and $0 < t_{k\acute{j}}^\sharp \leq d_{k\acute{j}}(x^\sharp)$; that is, $0 < t_{k\acute{j}}^\sharp/s_{k\acute{j}}^\sharp \leq d_{k\acute{j}}(x^\sharp)/c_{k\acute{j}}(x^\sharp)$.

The conditions $h_j' > 0$ $(j = 1, \ldots, K)$, or $h_j' < 0$ $(j = K + 1, \ldots, P)$ and $h_{k\acute{j}}' > 0$ $(\acute{j} = 1, \ldots, K_k)$, or $h_{k\acute{j}}' < 0$ $(\acute{j} = K_k + 1, \ldots, P_k)$ lead to

$$\begin{aligned} & h_j\left(\frac{b_j(x^\sharp)}{a_j(x^\sharp)}\right) \leq h_j\left(\frac{m_j^\sharp}{l_j^\sharp}\right) \quad (j = 1, \ldots, P), \\ & h_{k\acute{j}}\left(\frac{d_{k\acute{j}}(x^\sharp)}{c_{k\acute{j}}(x^\sharp)}\right) \leq h_{k\acute{j}}\left(\frac{t_{k\acute{j}}^\sharp}{s_{k\acute{j}}^\sharp}\right) \\ & \left(\acute{j} = 1, \ldots, P_k,\ k = 1, \ldots, M\right). \end{aligned} \tag{11}$$

Therefore we obtain

$$\begin{aligned} \sum_{j=1}^{P} h_j\left(\frac{b_j(x^\sharp)}{a_j(x^\sharp)}\right) &\leq \sum_{j=1}^{P} h_j\left(\frac{m_j^\sharp}{l_j^\sharp}\right), \\ \sum_{\acute{j}=1}^{P_k} h_{k\acute{j}}\left(\frac{d_{k\acute{j}}(x^\sharp)}{c_{k\acute{j}}(x^\sharp)}\right) &\leq \sum_{\acute{j}=1}^{P_k} h_{k\acute{j}}\left(\frac{t_{k\acute{j}}^\sharp}{s_{k\acute{j}}^\sharp}\right). \end{aligned} \tag{12}$$

Now,

$$\sum_{\acute{j}=1}^{P_k} h_{k\acute{j}}\left(\frac{d_{k\acute{j}}(x^\sharp)}{c_{k\acute{j}}(x^\sharp)}\right) \leq \sum_{\acute{j}=1}^{P_k} h_{k\acute{j}}\left(\frac{t_{k\acute{j}}^\sharp}{s_{k\acute{j}}^\sharp}\right) \leq 0, \tag{13}$$

that is, $x^\sharp$ satisfied constant for $(P0)$.

Since the optimal solution for $(P0)$ is x^*, we obtain

$$h_j\left(\frac{b_j(x^*)}{a_j(x^*)}\right) \leq h_j\left(\frac{b_j(x^\sharp)}{a_j(x^\sharp)}\right), \tag{14}$$

so

$$\begin{aligned} \sum_{j=1}^{P} h_j\left(\frac{m_j^*}{l_j^*}\right) = \sum_{j=1}^{P} h_j\left(\frac{b_j(x^*)}{a_j(x^*)}\right) &\leq \sum_{j=1}^{P} h_j\left(\frac{m_j^\sharp}{l_j^\sharp}\right), \\ \sum_{j=1}^{P} h_j\left(\frac{m_j^*}{l_j^*}\right) &\leq \sum_{j=1}^{P} h_j\left(\frac{m_j^\sharp}{l_j^\sharp}\right). \end{aligned} \tag{15}$$

For the optimal solution x^* of $(P0)$, we denote

$$\begin{aligned} l_j^* &:= a_j(x^*), \qquad m_j^* := b_j(x^*), \\ s_{k\acute{j}}^* &:= c_{k\acute{j}}(x^*), \qquad t_{k\acute{j}}^* := d_{k\acute{j}}(x^*); \end{aligned} \tag{16}$$

then

$$\rho(l^*, m^*) = w(x^*), \qquad \xi_k(s^*, t^*) = g(x^*). \tag{17}$$

The element $(x^*, l^*, m^*, s^*, t^*)$ satisfies the conditions for Z_H. Since $(x^\sharp, l^\sharp, m^\sharp, s^\sharp, t^\sharp)$ is the optimal solution for $(P1)$, it satisfies $\sum_{j=1}^{P} h_j(m_j^\sharp/l_j^\sharp) \leq \sum_{j=1}^{P} h_j(m_j^*/l_j^*)$.

Hence $\sum_{j=1}^{P} h_j(m_j^\sharp/l_j^\sharp) = \sum_{j=1}^{P} h_j(m_j^*/l_j^*)$; that is, the two problems are equivalent. □

2.2. Translation of the Problem $(P1)$ *into* $(P2)$. We change the variables by the logarithmic function log. Since x_i, l_j, m_j, $s_{k\acute{j}}$, $t_{k\acute{j}}$ are positive, we can write x_i, l_j, m_j, $s_{k\acute{j}}$, $t_{k\acute{j}}$ as $\exp(y_n)$ are using new variables y_n $(n = 1, \ldots, N + P_{\text{sum}})$; that is, $y_i := \ln x_i$, $y_{N+j} := \ln l_j$, $y_{N+P+j} := \ln m_j$, $y_{N+2P+(k-1)P_k+\acute{j}} := \ln s_{k\acute{j}}$, and $y_{N+2P+(M+k-1)P_k+\acute{j}} := \ln t_{k\acute{j}}$.

The closed domain Z_H corresponds to the following S^0, where

$$S^0 := \{y \in \mathbf{R}^{N+P_{\text{sum}}} \mid \ln \underline{x}_i \le y_i \le \ln \overline{x}_i, \ \ln \underline{a}_j \le y_{N+j} \le \ln \overline{a}_j, \ \ln \underline{b}_j \le y_{N+P+j} \le \ln \overline{b}_j, \ \ln \underline{c}_{k\acute{j}} \le y_{N+2P+(k-1)P_k+\acute{j}} \le \ln \overline{c}_{k\acute{j}}, \ \ln \underline{d}_{k\acute{j}} \le y_{N+2P+(M+k-1)P_{k\acute{j}}} \le \ln \overline{d}_{k\acute{j}}\}. \tag{18}$$

Using such transformation of variables, the objective function and the restricted functions of $(P1)$ are changed to the following:

$$\begin{aligned}
&\sum_{j=1}^{P} h_j \left(\frac{m_j}{l_j}\right) = \sum_{j=1}^{P} h_j \circ \exp\left(y_{N+P+j} - y_{N+j}\right), \\
&\sum_{\acute{j}=1}^{P_k} h_{k\acute{j}} \left(\frac{t_{k\acute{j}}}{s_{k\acute{j}}}\right) = \sum_{\acute{j}=1}^{P_k} h_{k\acute{j}} \circ \exp\left(y_{N+2P+(M+k-1)P_k+\acute{j}} - y_{N+2P+(k-1)P_k-\acute{j}}\right), \\
&l_j - a_j(x) = \exp\left(y_{N+j}\right) - \sum_{t=1}^{T_j^a} \beta_{jt}^a \exp\left(\sum_{i=1}^{N} \gamma_{jti}^a y_i\right), \\
&b_j(x) - m_j = \sum_{t=1}^{T_j^b} \beta_{jt}^b \exp\left(\sum_{i=1}^{N} \gamma_{jti}^b y_i\right) - \exp\left(y_{N+P+j}\right), \\
&s_{k\acute{j}} - c_{k\acute{j}}(x) = \exp\left(y_{N+2P+(k-1)P_k+\acute{j}}\right) - \sum_{t=1}^{T_j^c} \beta_{jt}^c \exp\left(\sum_{i=1}^{N} \gamma_{jti}^c y_i\right), \\
&d_{k\acute{j}}(x) - t_{k\acute{j}} = \sum_{t=1}^{T_j^d} \beta_{jt}^d \exp\left(\sum_{i=1}^{N} \gamma_{jti}^d y_i\right) - \exp\left(y_{N+2P+(M+k-1)P_K+\acute{j}}\right).
\end{aligned} \tag{19}$$

Now $\rho(l,m)$, $\xi_k(s,t)$, $l_j - a_j(x)$, $b_j(x) - m_j$, $s_{k\acute{j}} - c_{k\acute{j}}(x)$, $d_{k\acute{j}}(x) - t_{k\acute{j}}$, are represented as

$$\sum_{t=1}^{T_m} \Psi_{mt} \circ \exp\left(\sum_{i=1}^{N+P_{\text{sum}}} \lambda_{mti} y_i\right), \tag{20}$$

where λ_{mti} is real number and Ψ_{mt} satisfies $\Psi'_{mt}(x) > 0$ or $\Psi'_{mt}(x) < 0$, and $\{\Psi_{mt} \circ \exp(y)\}'' > 0$ or $\{\Psi_{mt} \circ \exp(y)\}'' < 0$.

Let $f_{mt}(y)$ be $\Psi_{mt} \circ \exp(y)$ and let $\mu_m(y)$ be $\sum_{t=1}^{T_m} f_{mt}(\sum_{i=1}^{N+P_{\text{sum}}} \lambda_{mti} y_i)$.

Then the objective function $\rho(l,m)$ and the restricted functions are changed functions which are changed to $\mu_0(y)$ and $\mu_m(y)$ $(m = 1, \ldots, M + 2(P + \sum_{k=1}^{M} P_k))$.

Now we put

$$S_\mu^0 := \left\{ y \in S^0 \mid \mu_m(y) \le 0 \left(m = 1, 2, \ldots, M + 2\left(P + \sum_{k=1}^{M} P_k\right)\right)\right\}. \tag{21}$$

Then the problem $(P1)$ is transformed naturally to the following problem $(P2)$:

$$\begin{aligned} \min \quad & \mu_0(y) \\ \text{s.t.} \quad & y \in S_\mu^0. \end{aligned} \tag{P2}$$

2.3. Linearization of the Problem $(P2)$. The objective and restricted function for $(P2)$ are nonlinear. On S_μ^0, we approximate $\mu_m(y)$ to lower bounded linear functions, and we can transform $(P2)$ into the linear optimization problem. The solution of it is lower bound of the optimal value on $(P2)$. We denote $\underline{y}_i$, $\overline{y}_i$, $\underline{y}_{N+j}$, $\overline{y}_{N+j}$, $\underline{y}_{N+P+j}$, $\overline{y}_{N+P+j}$, $\underline{y}_{N+2P+(k-1)P_k+\acute{j}}$, $\overline{y}_{N+2P+(k-1)P_k+\acute{j}}$, $\underline{y}_{N+2P+(M+k-1)P_k+\acute{j}}$, $\overline{y}_{N+2P+(M+k-1)P_k+\acute{j}}$ as minimums and maximums for $\ln \underline{x}_i$, $\ln \overline{x}_i$, $\ln \underline{a}_j$, $\ln \overline{a}_j$, $\ln \underline{b}_j$, $\ln \overline{b}_j$, $\ln \underline{c}_{k\acute{j}}$, $\ln \overline{c}_{k\acute{j}}$, $\ln \underline{d}_{k\acute{j}}$, $\ln \overline{d}_{k\acute{j}}$ $(j = 1, \ldots, P,\ \acute{j} = 1, \ldots, P_k,\ k = 1, \ldots, M)$.

And we denote $S^q \subset S_\mu^0$; that is,

$$\begin{aligned}
&S^q := \{y \in \mathbf{R}^{N+P_{\text{sum}}} \mid \underline{y}_i \le \underline{y}_i^q \le y_i \le \overline{y}_i^q \le \overline{y}_i \ (i = 1, \ldots, N + P_{\text{sum}})\}, \\
&Y_{mt}^{S^q} := \sum_{i=1}^{N+P_{\text{sum}}} \lambda_{mti} y_i, \\
&\underline{Y}_{mt}^{S^q} := \sum_{i=1}^{N+P_{\text{sum}}} \min\{\lambda_{mti} \underline{y}_i^q, \lambda_{mti} \overline{y}_i^q\}, \\
&\overline{Y}_{mt}^{S^q} := \sum_{i=1}^{N+P_{\text{sum}}} \max\{\lambda_{mti} \underline{y}_i^q, \lambda_{mti} \overline{y}_i^q\}, \\
&\left(m = 0, 1, 2, \ldots, M + 2\left(P + \sum_{k=1}^{M} P_k\right),\ t = 1, \ldots, T_m\right).
\end{aligned} \tag{22}$$

Now, $f'_{mt}(y) > 0$ or $f'_{mt}(y) < 0$, and $f''_{mt}(y) > 0$ or $f''_{mt}(y) < 0$, and $f_{mt}(Y_{mt}^{S^q})$ is monotonic convex function on $[\underline{Y}_{mt}^{S^q}, \overline{Y}_{mt}^{S^q}]$. And there exist the upper and lower bounded linear functions ($F_{mt}^{S^q}(Y_{mt}^{S^q})$ and $G_{mt}^{S^q}(Y_{mt}^{S^q})$) of $f_{mt}(Y_{mt}^{S^q})$.

We denote

$$F_{mt}^{S^q}\left(Y_{mt}^{S^q}\right) := \frac{f_{mt}\left(\overline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)}{\overline{Y}_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}}\left(Y_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}\right) + f_{mt}\left(\underline{Y}_{mt}^{S^q}\right). \tag{23}$$

As $f_{mt}(Y_{mt}^{S^q})$ is continuous on $[\underline{Y}_{mt}^{S^q}, \overline{Y}_{mt}^{S^q}]$ and differentiable on $(\underline{Y}_{mt}^{S^q}, \overline{Y}_{mt}^{S^q})$, there exists $c_{mt}^{S^q} \in (\underline{Y}_{mt}^{S^q}, \overline{Y}_{mt}^{S^q})$, such that

$$f'_{mt}\left(c_{mt}^{S^q}\right) = \frac{f_{mt}\left(\overline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)}{\overline{Y}_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}}, \tag{24}$$

by the mean value theorem.

Since $f'_{mt}(y) > 0$ or $f'_{mt}(y) < 0$, $f_{mt}(Y_{mt}^{S^q})$ is monotonic function on $[\underline{Y}_{mt}^{S^q}, \overline{Y}_{mt}^{S^q}]$, there exists the inverse function of $f_{mt}(\underline{Y}_{mt}^{S^q})$. Hence $c_{mt}^{S^q}$ is uniquely decided, $f_{mt}^{-1}(Y_{mt}^{S^q})$, such that

$$c_{mt}^{S^q} = f_{mt}^{-1}\left(\frac{f_{mt}\left(\overline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)}{\overline{Y}_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}}\right), \tag{25}$$

and we define

$$G_{mt}^{S^q}\left(Y_{mt}^{S^q}\right) := \frac{f_{mt}\left(\overline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)}{\overline{Y}_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}}\left(Y_{mt}^{S^q} - c_{mt}^{S^q}\right) + f_{mt}\left(c_{mt}^{S^q}\right), \tag{26}$$

$$L_{mt}^{S^q}\left(Y_{mt}^{S^q}\right) := \begin{cases} G_{mt}^{S^q}\left(Y_{mt}^{S^q}\right) & \left(f''_{mt}(y) > 0\right) \\ F_{mt}^{S^q}\left(Y_{mt}^{S^q}\right) & \left(f''_{mt}(y) < 0\right). \end{cases}$$

By the definition, $f_{mt}(Y_{mt}^{S^q}) \geq L_{mt}(Y_{mt}^{S^q})$.

For all $y \in S^q$, $\mu_m(y) := \sum_{t=1}^{T_m} f_{mt}(y) \geq \sum_{t=1}^{T_m} L_{mt}^{S^q}(y)$.

Let $L_m^{S^q}(y) := \sum_{t=1}^{T_m} L_{mt}^{S^q}(y)$ $(0 \leq m \leq M + 2(P + \sum_{k=1}^{M} P_k))$. Then $L_0(y)$ is a linear function which is lower function for the convex envelope of $\mu_0(y)$ on the rectangle.

$(\mathrm{LRP}(S^q))$ is the linear problem of $(P2)$ by the lower bounded function of $\mu_m(y)$:

$$\begin{aligned} \min\quad & L_0^{S^q}(y) \\ \text{s.t.}\quad & L_m^{S^q}(y) \leq 0 \\ & \left(m = 1, 2, \ldots, M + 2\left(P + \sum_{k=1}^{M} P_k\right)\right). \end{aligned} \tag{LRP (S^q)}$$

By the definition $(\mathrm{LRP}(S^q))$, any y in S^q satisfying the restricted condition of $(P2)$ satisfy the restricted condition of $(\mathrm{LRP}(S^q))$.

Lemma 2. *The value of* $(\mathrm{LRP}(S^q))$ is less than the optimal value for the problem $(P2)$ on S^q.

Proof. The definition of $(\mathrm{LRP}(S^q))$ implies the statement naturally. □

Lemma 3. *Assume that* $S^{q+1} \subset S^q \subset \cdots \subset S^0 \subset R^{N+P_{sum}}$, *and* $\bigcap_{q=0}^{\infty} S^q = \{y^*\}$. *For each* $m = 0, 1, 2, \ldots, M + 2(P + \sum_{k=1}^{M} P_k)$ *and* $t = 1, 2, \ldots, T_m$, $\lim_{q\to\infty} \max_{y\in S^q} |F_{mt}^{S^q}(y) - f_{mt}(y)| = 0$ *and* $\lim_{q\to\infty} \max_{y\in S^q} |G_{mt}^{S^q}(y) - f_{mt}(y)| = 0$.

Proof. Let $\underline{y}^q := \{y \in S^q \mid \min(\lambda_{mti}\underline{y}_i^q, \lambda_{mti}\overline{y}_i^q)$ $(i = 1, \ldots, N + P_{\mathrm{sum}})\}$ and $\overline{y}^q := \{y \in S^q \mid \max(\lambda_{mti}\underline{y}_i^q, \lambda_{mti}\overline{y}_i^q)$ $(i = 1, \ldots, N + P_{\mathrm{sum}})\}$.

Since $\bigcap_{q=0}^{\infty} S^q = \{y^*\}$, the values $\underline{y}^q$ and $\overline{y}^q$ satisfy $\lim_{q\to\infty}\underline{y}^q = \lim_{q\to\infty}\overline{y}^q = y^*$.

Hence, $\overline{Y}_{mt}^{S^q} - \underline{Y}_{mt}^{S^q} = \sum_{i=1}^{N+P_{\mathrm{sum}}} |\lambda_{mti}|(\overline{y}_i^q - \underline{y}_i^q) \xrightarrow{q\to\infty} 0$.

Now,

$$\begin{aligned} &\left|F_{mt}^{S^q}(y) - f_{mt}(y)\right| \\ &\quad = \left|F_{mt}^{S^q}\left(Y_{mt}^{S^q}\right) - f_{mt}\left(Y_{mt}^{S^q}\right)\right| \\ &\quad = \left|\frac{f_{mt}\left(\overline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)}{\overline{Y}_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}}\left(Y_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}\right) + f_{mt}\left(\underline{Y}_{mt}^{S^q}\right) - f_{mt}\left(Y_{mt}^{S^q}\right)\right|. \end{aligned} \tag{27}$$

The function $|F_{mt}^{S^q}(y) - f_{mt}(y)|$ is concave on $[\underline{Y}_{mt}^{S^q}, \overline{Y}_{mt}^{S^q}]$; therefore $c_{mt}^{S^q}$ attains the maximum value of $|F_{mt}^{S^q}(y) - f_{mt}(y)|$: $\max_{y\in S^q}|F_{mt}^{S^q}(y) - f_{mt}(y)|$

$$\begin{aligned} &\max_{y\in S^q}\left|F_{mt}^{S^q}(y) - f_{mt}(y)\right| \\ &\quad = \left|F_{mt}^{S^q}\left(c_{mt}^{S^q}\right) - f_{mt}\left(c_{mt}^{S^q}\right)\right| \\ &\quad = \left|\frac{f_{mt}\left(\overline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)}{\overline{Y}_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}}\left(c_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}\right) + f_{mt}\left(\underline{Y}_{mt}^{S^q}\right) - f_{mt}\left(c_{mt}^{S^q}\right)\right|. \end{aligned} \tag{28}$$

We denote

$$I_{mt}^{S^q} = \overline{Y}_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}, \qquad c_{mt}^{S^q} = \underline{Y}_{mt}^{S^q} + \theta_{mt}^{S^q} I_{mt}^{S^q} \quad \left(0 < \theta_{mt}^{S^q} < 1\right). \tag{29}$$

Since $I_{mt}^{S^q} \to 0$ for $q \to \infty$,

$$\begin{aligned}
&\left| \frac{f_{mt}\left(\overline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)}{\overline{Y}_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}} \left(c_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}\right) \right. \\
&\qquad \left. + f_{mt}\left(\underline{Y}_{mt}^{S^q}\right) - f_{mt}\left(c_{mt}^{S^q}\right) \right| \\
&= \left| \frac{f_{mt}\left(\underline{Y}_{mt}^{S^q} + I_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)}{\underline{Y}_{mt}^{S^q} + I_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}} \left(\underline{Y}_{mt}^{S^q} + \theta_{mt}^{S^q} I_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}\right) \right. \\
&\qquad \left. + f_{mt}\left(\underline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q} + \theta_{mt}^{S^q} I_{mt}^{S^q}\right) \right| \\
&= \left| \frac{f_{mt}\left(\underline{Y}_{mt}^{S^q} + I_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)}{\underline{Y}_{mt}^{S^q} + I_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}} \left(\theta_{mt}^{S^q} I_{mt}^{S^q}\right) \right. \\
&\qquad \left. + f_{mt}\left(\underline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q} + \theta_{mt}^{S^q} I_{mt}^{S^q}\right) \right| \\
&= \left| f_{mt}\left(\underline{Y}_{mt}^{S^q} + I_{mt}^{S^q}\right)\theta_{mt}^{S^q} - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)\theta_{mt}^{S^q} + f_{mt}\left(\underline{Y}_{mt}^{S^q}\right) \right. \\
&\qquad \left. - f_{mt}\left(\underline{Y}_{mt}^{S^q} + \theta_{mt}^{S^q} I_{mt}^{S^q}\right) \right| \\
&\xrightarrow{q \to \infty} \left| f_{mt}\underline{Y}_{mt}^{S^q}\theta_{mt}^{S^q} - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)\theta_{mt}^{S^q} \right. \\
&\qquad \left. + f_{mt}\left(\underline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right) \right| = 0.
\end{aligned} \tag{30}$$

Thus

$$\begin{aligned}
&\left| G_{mt}^{S^q}(y) - f_{mt}(y) \right| \\
&\quad = \left| G_{mt}^{S^q}\left(Y_{mt}^{S^q}\right) - f_{mt}\left(Y_{mt}^{S^q}\right) \right| \\
&\quad = \left| \frac{f_{mt}\left(\overline{Y}_{mt}^{S^q}\right) - f_{mt}\left(\underline{Y}_{mt}^{S^q}\right)}{\overline{Y}_{mt}^{S^q} - \underline{Y}_{mt}^{S^q}} \left(Y_{mt}^{S^q} - c_{mt}^{S^q}\right) \right. \\
&\qquad \left. + f_{mt}\left(c_{mt}^{S^q}\right) - f_{mt}\left(Y_{mt}^{S^q}\right) \right|.
\end{aligned} \tag{31}$$

On the other hand $|G_{mt}^{S^q}(y) - f_{mt}(y)|$ is a convex function by the same argument, and we obtain the following $\max_{y \in S^q} |G_{mt}^{S^q}(y) - f_{mt}(y)| \to 0$ for $q \to \infty$. □

Lemma 4. *Under the same assumption of Lemma 3* $\max_{y \in S^q} |L_m^{S^q}(y) - \mu_m(y)| \to 0$ *for* $q \to \infty$ *and each* $m = 0, 1, 2, \ldots, M + 2(P + \sum_{k=1}^{M} P_k)$.

Proof. Lemma 3 and the definitions $L_m^{S^q}(y)$, $\mu_m(y)$ imply Lemma 4, standardly. □

3. Branch and Bound Algorithm and Its Convergence

In Section 2, we transformed the initial problem ($P0$) into the equivalent problem ($P2$), and we make the linear relaxation problem (LRP) of ($P2$) to find the approximate value of ($P2$) easily. Now we get it by using branch and bound algorithm.

3.1. Branch and Bound Algorithm. We solve the linear relaxation problem on initial domain S^0 to get the linear optimal value as lower bound of ($P2$) and upper bound of ($P2$). For preparing to separate the active domains, we let the active domains set be $\mathcal{Q}_q$ and active domain $S^{q(k)} \subset S^0$. q presents the times of the cutting domains and stage number and k presents the number of active domains on stage q. If $S^{q(k)}$ is active domain, we divide $S^{q(k)}$ into half domains $S^{q(k)\cdot 1}$, $S^{q(k)\cdot 2}$ and linearize ($P2$) on each domain and solve the linear problems. After the above calculations, we get the lower and upper bound value of ($P2$). After the repeat calculations, we get the convergence for the sequences of the lower and upper bound values, and we get the optimal value and solution.

3.1.1. Branching Rule. We denote that $S^{q(k)} = \{y \mid \underline{y}_n^{q(k)} \le y_n^{q(k)} \le \overline{y}_n^{q(k)},\ n = 1, \ldots, N + P_{\text{sum}}\} \subseteq S^0$. We select the branching variable i such that $i = n \max\{\overline{y}_n^{q(k)} - \underline{y}_n^{q(k)}, n = 1, 2, \ldots, N + P_{\text{sum}}\}$, and we divide the interval $[\underline{y}_i^{q(k)}, \overline{y}_i^{q(k)}]$ into half intervals: $[\underline{y}_i^{q(k)}, (\underline{y}_i^{q(k)} + \overline{y}_i^{q(k)})/2]$ and $[(\underline{y}_i^{q(k)} + \overline{y}_i^{q(k)})/2, \overline{y}_i^{q(k)}]$.

3.1.2. Algorithm Statement

Step 0. Firstly, we let q be 0 and let k be 1. And we set an appropriate ϵ-value as a convergence tolerance, the initial upper bound $V^* = \infty$, and $\mathcal{Q}_0 = S^{0(1)}$. We solve LRP($S^{0(1)}$), and we denote the linear optimal solution and optimal value by $\hat{y}(S^{0(1)})$ and $\mathrm{LB}_{0(1)}$. If $\hat{y}(S^{0(1)})$ is feasible for ($P2$), then update $V^* = \mu_0(\hat{y}(S^{0(1)}))$ and we set the initial lower bound $\mathrm{LB} = \mathrm{LB}_{0(1)}$. If $V^* - \mathrm{LB} \le \epsilon$, then we get the ϵ-approximate optimal value $\mu_0(\hat{y}(S^{0(1)}))$ and optimal solution $\hat{y}(S^{0(1)})$ of ($P2$), so we stop this algorithm. Otherwise, we proceed to Step 1.

Step 1. For all k, we divide $S^{q(k)}$ to get two half domains, $S^{q(k)\cdot 1}$ and $S^{q(k)\cdot 2}$, according to the above branching rule.

Step 2. For all k and each domain $S^{q(k)\cdot v}$ $(v = 1, 2)$, we calculate

$$\underline{\mu}_m(v) = \sum_{t=1, c_{mt}>0}^{\Gamma_m} c_{mt} \exp\left(\underline{Y}_{mt}^{S^{q(k)\cdot v}}\right) + \sum_{t=1, c_{mt}<0}^{\Gamma_m} c_{mt} \exp\left(\overline{Y}_{mt}^{S^{q(k)\cdot v}}\right)$$
$$\left(m = 1, \ldots, M + 2\left(P + \sum_{k=1}^{M} P_k\right)\right), \tag{32}$$

where c_{mt}, $\underline{Y}_{mt}^{S^{q(k)\cdot v}}$, and $\overline{Y}_{mt}^{S^{q(k)\cdot v}}$ are defined in Section 2.3.

If there is the $\underline{\mu}_m(v)$ that satisfy $\underline{\mu}_m(v) > 0$ for some $m \in \{1, 2, \ldots, M + 2(P + \sum_{k=1}^{M} P_k)\}$, $S^{q(k)v}$ is infeasible domain for $(P2)$, then we delete the domain from $\mathcal{Q}_q$. If $S^{q(k)\cdot v}$ $(v = 1, 2)$ are all deleted for all k, then the problem has no feasible solutions.

Step 3. For left domains, we compute $A_{mt}^{S^{q(k)\cdot v}}$, $B_{mt}^{S^{q(k)\cdot v}}$, $\underline{Y}_{mt}^{S^{q(k)\cdot v}}$, and $\overline{Y}_{mt}^{S^{q(k)\cdot v}}$ as defined in Sections 2.2 and 2.3. We solve the $\text{LRP}(S^{q(k)\cdot v})$ by simplex algorithm, and we denote the obtained linear optimal solutions and values by $(\hat{y}(S^{q(k)\cdot v}), \text{LB}_{q(k)\cdot v})$. Then if $\hat{y}(S^{q(k)\cdot v})$ is feasible for $(P2)$, we update $V^* = \min\{V^*, \mu_0(\hat{y}(S^{q(k)\cdot v}))\}$. If $\text{LB}_{q(k)\cdot v} > V^*$, then delete the corresponding domains from $\mathcal{Q}_q$. If $V^* - \text{LB}_{q(k)\cdot v} \leq \epsilon$, then we get the ϵ-approximate optimal value $\mu_0(\hat{y}(S^{q(k)\cdot v}))$ and optimal solution $\hat{y}(S^{q(k)\cdot v})$ of $(P2)$, so we stop this algorithm. Otherwise, we proceed to Step 4.

Step 4. We update the index of left domains $S^{q(k)\cdot v}$ to $S^{q+1(k)}$; then we initialize k. And we settle that $\mathcal{Q}_{q+1}$ is a set of $S^{q+1(k)}$, and go to Step 1.

3.2. The Convergence of the Algorithm. Corresponding to [4], we obtain the convergence of the algorithm (cf. [4]).

Theorem 5. *Suppose that problem $(P2)$ has a global optimal solution, and let μ_0^* be the global optimal value of $(P2)$. Then one has the following:*

(i) *for the case $\epsilon > 0$, the algorithm always terminates after finitely many iterations yielding a global ϵ-optimal solution y^* and a global ϵ-optimal value V^* for problem $(P2)$ in the sense that*

$$y^* \in S, \qquad V^* - \epsilon \leq \mu_0^* \text{ with } V^* = \mu_0(y^*); \tag{33}$$

(ii) *for the case $\epsilon \to 0$, we assume the sequence ϵ_n is convergence tolerance, such that $\epsilon_1 > \epsilon_2 >, \ldots, > \epsilon_n > \epsilon_{n+1} >, \ldots, > 0$; that is, $\lim_{n\to\infty}\epsilon_n = 0$. And we assume the sequence y_n^* is optimal solution of $(P2)$ corresponding to ϵ_n. Then the accumulation point of y_n^* is global optimal solution of $(P2)$.*

Proof. (i) It is obvious by algorithm statement.

(ii) We assume that the upper bound corresponding to ϵ_n is V_n^*:

$$\mu_0(y_n^*) \in [V_n^* - \epsilon, V_n^*]; \tag{34}$$

y_n^* is the point sequence on bounded closed set, so y_n^* has a converge subsequence y_{ni}^*. We assume that $\lim_{i\to\infty} y_{ni}^* = y^*$; then

$$\begin{gathered} V_{ni}^* - \epsilon_{ni} \leq \mu_0(V_{ni}^*) \leq V_{ni}^* \quad i \longrightarrow \infty, \\ \text{then } ni \longrightarrow \infty, \text{ so } \lim_{ni\to\infty} \epsilon_{ni} = 0. \end{gathered} \tag{35}$$

V_n^* is monotone decreasing sequence, so it converges. We assume that $\lim_{n\to\infty} V_n^* = \mu_0^*$:

$$\lim_{i\to\infty}(V_{ni}^* - \epsilon_{ni}) \leq \lim_{i\to\infty}\mu_0(y_{ni}^*) \leq \lim_{i\to\infty} V_{ni}^*; \tag{36}$$

$\mu_0(y)$ is continuous function, so $\lim_{i\to\infty}\mu_0(y_{ni}) = \mu_0(y^*)$. And $\mu_0^* \leq \mu_0^{y^*} \leq \mu_0^*$; that is, $\mu_0(y^*) = \mu_0^*$. For $\forall m$, $\mu_m(y_n^*) \leq 0$. As $\mu_m(y^*)$ is continuous, $\lim_{n\to\infty}\mu_m(y_n^*) = \mu_m(y^*) \leq 0$. □

4. Numerical Experiment

In this chapter, we show the numerical experiments of these optimization problems according to the former rules. We make the algorithms coded with MATLAB. In these codes, we use MATLAB's unique function code "linprog" to solve the linear optimization problems.

Example 1. Consider

$$\begin{aligned} \min \quad & h(x) = \sin\left(\frac{x_1^2 + 3x_2 - 2x_2^2 + 1}{x_1^2 + x_2 + 2}\right) \\ & \qquad + \cos\left(\frac{-x_2^2 + 2x_1 + 2x_2}{x_1 + 2.5}\right) \\ \text{s.t.} \quad & x_1^2 - \frac{x_1}{x_2} - 1 \leq 0 \\ & x_1 + 3\frac{x_2}{x_1} - 5 \leq 0 \\ & X = \{x : 1 \leq x_1 \leq 3, 1 \leq x_2 \leq 3\}. \end{aligned} \tag{37}$$

We set $\epsilon = 0.0001$. After the algorithm, we found a global ϵ-optimal value $V^* = 1.0748$ when the global ϵ-optimal solution is $(x_1, x_2)^T = (1.34977, 1.64232)$.

Example 2. Consider

$$\begin{aligned} \min \quad & \exp\left(\frac{-x_1^2 + 3x_1 + 2x_2^2 + 3x_2 + 3.5}{x_1 + 1}\right) \\ & - \exp\left(\frac{x_2}{x_1^2 - 2x_1 + x_2^2 - 8x_2 + 20}\right) \\ \text{s.t.} \quad & x_1 - \frac{x_2}{x_1} \leq 1 \\ & 2\frac{x_1}{x_2} + x_2 + \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & X = \{x : 1 \leq x_1 \leq 3, 1 \leq x_2 \leq 3\}. \end{aligned} \tag{38}$$

We set $\epsilon = 0.0001$. After the algorithm, we found a global ϵ-optimal value $V^* = 58.2723$ when the global ϵ-optimal solution is $(x_1, x_2)^T = (1, 1.6180)$.

Example 3. Consider

$$\begin{aligned} \min \quad & \sin\left(\frac{x_1^2 + 2x_2 - 2x_1 + x_2^2 + 1}{x_1 + x_2^2 + 2}\right) \\ & + \cos\left(\frac{3x_1^2 - 3x_2 + 2x_1 + x_2^2 + 5}{x_1^2 + 2x_2^2 + 10}\right) \end{aligned}$$

$$\text{s.t.} \quad \sin\left(\frac{x_1^2 + 3x_2 - 2x_2^2 + 2}{x_1^2 + x_2 + 2}\right) + \cos\left(\frac{-x_2^2 + 2x_1 + 2x_2}{x_1 + 2.5}\right) \leq 2 \qquad (39)$$
$$X = \{x : 1 \leq x_1 \leq 2, 1 \leq x_2 \leq 2\}.$$

We set $\epsilon = 0.0001$. After the algorithm, we found a global ϵ-optimal value $V^* = 1.09133$ when the global ϵ-optimal solution is $(x_1, x_2)^T = (2, 1)$.

Example 4. Consider

$$\begin{aligned} \min \quad & \exp\left(\frac{x_1^2 - 2x_2^2 + 8}{2x_1^2 + x_2 + 1}\right) + \exp\left(\frac{3x_1 - x_2^2 + 5}{x_1^2 - x_1 + x_2^2 - 3x_2 + 10}\right) \\ \text{s.t.} \quad & x_1^2 - 2x_2 \leq 1 \\ & x_1 - \frac{x_2}{x_1} \leq 1 \\ & 2x_1 + x_2^2 \leq 6 \\ & X = \{x : 1.5 \leq x_1 \leq 2, 1.5 \leq x_2 \leq 2\}. \end{aligned} \qquad (40)$$

We set $\epsilon = 0.0001$. After the algorithm, we found a global ϵ-optimal value $V^* = 3.9378$ when the global ϵ-optimal solution is $(x_1, x_2)^T = (1.5, 1.7321)$.

5. Concluding Remarks

In this paper, we proved that we can solve the nonconvex optimization problems which [4] cannot solve that the sum of the positive (or negative) first and second derivatives function with the variable defined by sum of polynomial fractional function by using branch and bound algorithm.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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