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Research Article

On the Odd Prime Solutions of the Diophantine Equation $x^y + y^x = z^z$

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Using the elementary method and some properties of the least solution of Pell's equation, we prove that the equation $x^y + y^x = z^z$ has no positive integer solutions (x, y, z) with x and y being odd primes.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of all integers and positive integers, respectively. In recent years, there are many authors who investigated the various properties of exponential diophantine equation with circulating form (see [1–4]). Recently, Zhang et al. [5] are interested in the equation

$$x^y + y^x = z^z, \quad x, y, z \in \mathbb{N}, \quad \min\{x, y, z\} > 1. \tag{1}$$

Using the p-adic lower bound of the log-linear model method, they proved all solutions (x, y, z) of (1) satisfying $z < 2.8 \times 10^9$. Meanwhile, they proposed a conjecture as follows.

Conjecture 1. *Equation* (1) *has no positive integer solution* (x, y, z).

Using the method in [5], it seems to be a very difficult problem to improve the upper bound estimate for z. In this paper, we use the elementary method and some properties of the least solution of Pell's equation to solve the conjecture partly. That is, we will prove the following.

Theorem 2. *Equation* (1) *has no positive integer solution* (x, y, z) *with* x *and* y *being odd primes.*

2. Several Lemmas

Let D be a nonsquare positive integer, and let $h(4D)$ denote the class number of binary quadratic primitive forms with discriminant $4D$. Then we have the following.

Lemma 3. *For the equation*

$$u^2 - Dv^2 = 1, \quad u, v \in \mathbb{Z}, \tag{2}$$

there is a solution (u, v) *with* $uv \neq 0$*, and there is a unique positive integer solution* (u_1, v_1) *satisfying* $0 < u_1 + v_1\sqrt{D} \le u + v\sqrt{D}$*, where* (u, v) *pass through all positive integer solutions of* (2)*. We call* (u_1, v_1) *as the least solution of* (2)*. Every solution* (u, v) *of* (2) *can be expressed as*

$$u + v\sqrt{D} = \pm\left(u_1 + v_1\sqrt{D}\right)^s, \quad s \in \mathbb{Z}. \tag{3}$$

Proof. See Section 10.9 of [6]. □

Lemma 4. *Let D be an odd prime satisfying $D \equiv 1 (\mathrm{mod}\ 4)$ and $D < 10^{11}$, and then every least solution (u_1, v_1) of* (2) *satisfies $v_1 \not\equiv 0 (\mathrm{mod}\ D)$.*

Proof. See [7]. □

Lemma 5. *Consider $h(4D) < D$.*

Proof. According to Table II in Chapter 16 of [6], we know that Lemma 5 holds for $D \leq 25$, and when $D > 25$, by Theorems 12.10.1 and 12.13.3 of [6], we have

$$h(4D) < \frac{2\sqrt{D}(1 + \log 2 + (1/2)\log D)}{\log(u_1 + v_1\sqrt{D})}, \quad (4)$$

where (u_1, v_1) is the least solution of (2). If $h(4D) \geq D$, and $v_1 \geq 1$, $u_1 = \sqrt{Dv_1^2 + 1} \geq \sqrt{D+1}$, then by (4), we have

$$5 < \sqrt{D} < \frac{4 + 4\log 2 + 2\log D}{2\log 2 + \log D} = 2 + \frac{4}{2\log 2 + \log D} < 3, \quad (5)$$

a contradiction. This proves Lemma 5. □

Lemma 6. *Let p be an odd prime with $p \nmid D$. If the equation*

$$X^2 - DY^2 = p^Z, \quad X, Y, Z \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0, \quad (6)$$

has the solution (X, Y, Z), then every solution (X, Y, Z) can be expressed as

$$\begin{gathered} Z = Z_1 t, \quad t \in \mathbb{N}, \\ X + Y\sqrt{D} = (X_1 + \lambda Y_1\sqrt{D})^t (u + v\sqrt{D}), \quad \lambda \in \{\pm 1\}, \end{gathered} \quad (7)$$

where X_1, Y_1, Z_1 are positive integers satisfying

$$X_1^2 - DY_1^2 = p^{Z_1}, \quad \gcd(X_1, Y_1) = 1, \quad Z_1 \mid h(4D), \quad (8)$$

and (u, v) is a solution of (2).

Proof. See Lemma 4 of [8]. □

3. Proof of Theorem 2

Let (x, y, z) be one of the solutions of (1). Without loss of generality we may assume that $x \geq y$, since x and y are symmetrical in (1). By [5], we know that x, y, z are coprime, and $x > z > y > 1$. When x and y are odd primes, z must be even. Note that $y \geq 3$; by (1) we have $z^z > y^x$; then $z\log z > x\log y \geq x\log 3 > x$, so by the result $z < 2.8 \times 10^9$ in [5], we get

$$3 \leq y < z < x < z\log z < 6.2 \times 10^{10}. \quad (9)$$

By (9), we know that $z > 2$; then from (1) we get $0 \equiv z^z \equiv x^y + y^x \equiv x + y \pmod 4$. Therefore,

$$x \equiv \varepsilon \pmod 4, \quad y \equiv -\varepsilon \pmod 4, \quad \varepsilon \in \{\pm 1\}. \quad (10)$$

If $\varepsilon = 1$, by (10), x is an odd prime with $x \equiv 1 \pmod 4$. We see from (1) that the equation

$$X^2 - xY^2 = y^Z, \quad X, Y, Z \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0, \quad (11)$$

has the solution

$$(X, Y, Z) = (z^{z/2}, x^{(y-1)/2}, x). \quad (12)$$

Since y is an odd prime with $y \nmid x$, applying Lemma 6 to (11) and (12), we have

$$x = Z_1 t, \quad t \in \mathbb{N}, \quad (13)$$

$$z^{z/2} + x^{(y-1)/2}\sqrt{x} = (X_1 + \lambda Y_1\sqrt{x})^t (u + v\sqrt{x}), \quad \lambda \in \{\pm 1\}, \quad (14)$$

where X_1, Y_1, Z_1 are positive integers satisfying

$$X_1^2 - xY_1^2 = y^{Z_1}, \quad \gcd(X_1, Y_1) = 1, \quad (15)$$

$$Z_1 \mid h(4x), \quad (16)$$

(u, v) is a solution of the equation

$$u^2 - xv^2 = 1, \quad u, v \in \mathbb{Z}, \quad (17)$$

and $h(4x)$ denotes the class number of binary quadratic primitive forms with discriminant $4x$.

Since x is an odd prime, we know from (13) that $t = 1$ or x. If $t = 1$, $Z_1 = x$ by (13), so from (16) we have $x \mid h(4x)$ and $x \leq h(4x)$. But, by Lemma 5, this is impossible; thus $t = x$.

Since $t = x$, by (13) we know that $Z_1 = 1$, so that (14) and (15) read

$$\begin{aligned} z^{z/2} + x^{(y-1)/2}\sqrt{x} &= (X_1 + \lambda Y_1\sqrt{x})^x \\ &\quad \times (u + v\sqrt{x}), \quad \lambda \in \{-1, 1\}, \end{aligned} \quad (18)$$

$$X_1^2 - xY_1^2 = y, \quad \gcd(X_1, Y_1) = 1. \quad (19)$$

Further, $z^{z/2} + x^{(y-1)/2}\sqrt{x} > 0$, and from (19) we know $X_1 + \lambda Y_1\sqrt{x} > 0$, since $u + v\sqrt{x} > 0$ by Lemma 3. Thus, according to Lemma 3 there is $s \in Z$ such that

$$u + v\sqrt{x} = (u_1 + v_1\sqrt{x})^s, \quad s \in \mathbb{Z}, \quad (20)$$

where (u_1, v_1) is the least solution of (17). For the integer s, there exist integers q and r satisfying

$$s = xq + r, \quad 0 \leq r < x. \quad (21)$$

Let

$$X_2 + Y_2\sqrt{x} = (X_1 + \lambda Y_1\sqrt{x})(u_1 + v_1\sqrt{x})^q. \quad (22)$$

From (17), (19), and (22), we know that X_2 and Y_2 are integers satisfying

$$X_2^2 - xY_2^2 = y, \quad \gcd(X_2, Y_2) = 1. \quad (23)$$

And from (18), (20), (21), and (22), we have

$$z^{z/2} + x^{(y-1)/2}\sqrt{x} = (X_2 + Y_2\sqrt{x})^x (u_1 + v_1\sqrt{x})^r. \quad (24)$$

If $r = 0$ in (21), then, from (24), we have

$$x^{(y-1)/2} = Y_2 \sum_{i=0}^{(x-1)/2} \binom{x}{2i+1} X_2^{x-2i-1} (xY_2^2)^i. \quad (25)$$

However, since $x > y$ from (9), and by (23), we have $X_2^2 > x$. According to (25), we get $x^{(y-1)/2} > xX_2^{x-1} > x^{(x+1)/2} > x^{(y+1)/2}$, which is impossible. Thus, from (21), we have $0 < r < x$ and

$$x \nmid r. \tag{26}$$

Let

$$X' + Y'\sqrt{x} = (X_2 + Y_2\sqrt{x})^x, \qquad u' + v'\sqrt{x} = (u_1 + v_1\sqrt{x})^r. \tag{27}$$

Then X', Y', u', v' are integers with $\gcd(X', Y') = \gcd(u', v') = 1$, and

$$Y' = Y_2 \sum_{i=0}^{(x-1)/2} \binom{x}{2i+1} X_2^{x-2i-1} (xY_2^2)^i, \tag{28}$$

$$v' = v_1 \sum_{j=0}^{[(r-1)/2]} \binom{r}{2j+1} u_1^{r-2j-1} (xv_1^2)^j, \tag{29}$$

where $[(r-1)/2]$ is the integral part of $(r-1)/2$. From (29), we have

$$Y' \equiv 0 \pmod{x}, \qquad v' \equiv ru_1^{r-1}v_1 \pmod{x}. \tag{30}$$

Applying (27) to (24), we get

$$x^{(y-1)/2} = X'v' + Y'u'. \tag{31}$$

From (17) and (26), $\gcd(ru_1^{r-1}, x) = 1$, by (30), $v_1 \equiv 0 \pmod{x}$. However, we get from (9) that x is an odd prime satisfying $x \equiv 1 \pmod 4$ and $x < 6.2 \times 10^{10}$; then from Lemma 4, we know it is impossible. Thus, the theorem holds for $\varepsilon = 1$.

Similarly, if $\varepsilon = -1$, by (10) y is an odd prime with $y \equiv 1 \pmod 4$. We see from (1) that the equation

$$X^2 - yY^2 = x^Z, \quad X, Y, Z \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0, \tag{32}$$

has the solution

$$(X, Y, Z) = \left(z^{z/2}, y^{(x-1)/2}, y\right). \tag{33}$$

Applying Lemmas 5 and 6 to (11) and (12), we have

$$z^{z/2} + y^{(x-1)/2}\sqrt{y} = (X_1 + \lambda Y_1\sqrt{y})^y (u + v\sqrt{y}), \quad \lambda \in \{\pm 1\}, \tag{34}$$

where X_1, Y_1, Z_1 are positive integers satisfying

$$X_1^2 - yY_1^2 = x, \qquad \gcd(X_1, Y_1) = 1, \tag{35}$$

and (u, v) is a solution of the equation

$$u^2 - yv^2 = 1, \quad u, v \in \mathbb{Z}. \tag{36}$$

Applying Lemma 3 to (34) and (35), we have

$$u + v\sqrt{y} = (u_1 + v_1\sqrt{y})^s, \quad s \in \mathbb{Z}, \tag{37}$$

where (u_1, v_1) is the least solution of (36). In addition, the integer s can be expressed as

$$s = yq + r, \quad q, r \in \mathbb{Z}, \ 0 \le r < y. \tag{38}$$

Let

$$X_2 + Y_2\sqrt{y} = (X_1 + \lambda Y_1\sqrt{y})(u_1 + v_1\sqrt{y})^q. \tag{39}$$

From (35) and (36), we know that X_2 and Y_2 are integers satisfying

$$X_2^2 - yY_2^2 = x, \qquad \gcd(X_2, Y_2) = 1. \tag{40}$$

And from (34), (37), (38), and (39), we get

$$z^{z/2} + y^{(x-1)/2}\sqrt{y} = (X_2 + Y_2\sqrt{y})^y (u_1 + v_1\sqrt{y})^r. \tag{41}$$

If $r = 0$ in (38), then, from (41), we have

$$y^{(x-1)/2} = Y_2 \sum_{i=0}^{(y-1)/2} \binom{y}{2i+1} X_2^{y-2i-1} (yY_2^2)^i. \tag{42}$$

Since y is an odd prime, $x > y \ge 5$, by (42), we know

$$0 \equiv y^{(x-1)/2} \equiv yX_2^{y-1}Y_2 \pmod{y^2}. \tag{43}$$

From (40), we know $\gcd(X_2, y) = 1$; then from (43) we get $y \mid Y_2$. Let $y^\alpha \parallel Y_2$, and $y \ge 5$, so

$$y^{\alpha+1} \parallel Y_2 \binom{y}{1} X_2^{y-1}, \tag{44}$$

$$Y_2 \binom{y}{2i+1} X_2^{y-2i-1} (yY_2^2)^i \equiv yY_2 \binom{y-1}{2i} \frac{X_2^{y-2i-1}(yY_2^2)^i}{2i+1} \equiv 0 \pmod{y^{\alpha+2}}, \quad i \ge 1. \tag{45}$$

By (45),

$$y^{\alpha+1} \parallel Y_2 \sum_{i=0}^{(y-1)/2} \binom{y}{2i+1} X_2^{y-2i-1} (yY_2^2)^i. \tag{46}$$

Combining (42) and (46) we may immediately get

$$\alpha + 1 = \frac{x-1}{2}. \tag{47}$$

However, from (42) and (47), we get $|yY_2| \ge y^{\alpha+1} = y^{(x-1)/2} > |yX_2^{y-1}Y_2| > |yY_2|$, but it is impossible. Therefore, we have $0 < r < y$ and

$$y \nmid r. \tag{48}$$

Now, using the similarly proof with $\varepsilon = 1$, from (41) and (48) can obtain contradiction.

This completes the proof of our theorem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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