Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 873943, 7 pages
http://dx.doi.org/10.1155/2013/873943

Research Article

Ball-Covering Property in Uniformly Non-$l_3^{(1)}$ Banach Spaces and Application

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Received 7 September 2013; Revised 9 November 2013; Accepted 11 November 2013

Academic Editor: Khalil Ezzinbi

This paper shows the following. (1) X is a uniformly non-$l_3^{(1)}$ space if and only if there exist two constants $\alpha, \beta > 0$ such that, for every 3-dimensional subspace Y of X, there exists a ball-covering $\mathfrak{B}$ of Y with $c(\mathfrak{B}) = 4$ or 5 which is α-off the origin and $r(\mathfrak{B}) \le \beta$. (2) If a separable space X has the Radon-Nikodym property, then X^* has the ball-covering property. Using this general result, we find sufficient conditions in order that an Orlicz function space has the ball-covering property.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space. $S(X)$ and $B(X)$ denote the unit sphere and unit ball, respectively. X^* denote the dual space of X. Let $B(x, r)$ denote the closed ball centered at x and of radius $r > 0$. Let N, R, and R^+ denote the set of natural numbers, reals, and nonnegative reals, respectively.

It is no doubt that the study of geometric and topological properties of unit balls of normed spaces has played an important role in geometry of Banach spaces. Almost all properties of Banach spaces, such as convexity, smoothness, reflexivity, and the Radon-Nikodym property, can be viewed as the corresponding properties of their unit balls. We should mention here that there are many topics studying behavior of ball collections. For example, the Mazur intersection property, the packing sphere problem of unit balls, the measure of noncompactness with respect to topological degree, and the ball topology have also brought great attention of many mathematicians.

Starting with a different viewpoint, a notion of ball-covering property is introduced by Cheng [1].

Definition 1. A Banach space is said to have the ball-covering property if its unit sphere can be contained in the union of countably many balls off the origin. In this case, we also say that the norm has ball-covering property.

In [2], it was established that if X is a locally uniformly convex space and $B(X^*)$ is w^*-separable, then X has the ball-covering property. In [3], Cheng proved that by constructing the equivalent norms on l^∞, there exists a Banach space $(l^\infty, \|\cdot\|^0)$ such that $(l^\infty, \|\cdot\|^0)$ has not ball-covering property. In [4], it was established that for every $\varepsilon > 0$ every Banach space with a w^*-separable dual has an $1 + \varepsilon$-equivalent norm with the ball-covering property. For a ball-covering $\mathfrak{B} = \{B(x_i; r_i)\}_{i\in I}$ of X, we denote by $c(\mathfrak{B})$ its cardinality and by $r(\mathfrak{B})$ the least upper bound of the radius set $\{r_i\}_{i\in I}$, and we call it the radius of $\mathfrak{B}$. We say that a ball covering is minimal if its cardinality is the smallest of all cardinalities of ball coverings. We call a given ball covering $\mathfrak{B}$ α-off the origin if $\inf\{\|x\| : x \in \cup\mathfrak{B}\} \ge \alpha$. Let $\mathfrak{B}_{\min} = \mathfrak{B}_{\min}(X)$ be any minimal ball covering of X. Cheng [1] showed the following results.

Proposition 2. *Suppose that X is an n-dimensional Banach space. Then*

(1) $n + 1 \le c(\mathfrak{B}_{\min}) \le 2n$;

(2) *if X is smooth, then* $c(\mathfrak{B}_{\min}) = n + 1$;

(3) $c(\mathfrak{B}_{\min}) = 2n$ *if and only if X is isometric to* $(R^n, \|\cdot\|_\infty)$.

It is easy to see that $(R^n, \|\cdot\|_1)$ is isometric to $(R^n, \|\cdot\|_\infty)$. Moreover, Cheng [5, 6] showed the following results.

Proposition 3. *Suppose that X is a Banach space. Then X is a uniformly nonsquare space if and only if there exist two constants $\alpha, \beta > 0$ such that, for every 2-dimensional subspace Y of X, there exists a ball-covering $\mathfrak{B}$ of Y with $c(\mathfrak{B}) = 3$ which is α-off the origin and $r(\mathfrak{B}) \leq \beta$.*

Definition 4. A Banach space X is said to be non-$l_n^{(1)}$ space, if, for all $x_1, x_2, \ldots, x_n \in S(X)$,

$$\min \{\|\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n\| : \xi_i = \pm 1,\ i \in \{1, 2, \ldots, n\}\} < n. \tag{1}$$

Definition 5. A Banach space X is said to be uniformly non-$l_n^{(1)}$ space, if there exists $\delta > 0$ such that, for all $x_1, x_2, \ldots, x_n \in S(X)$,

$$\min \{\|\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n\| : \xi_i = \pm 1,\ i \in \{1, 2, \ldots, n\}\} < n - \delta. \tag{2}$$

Relationships between various kinds of convexity of Banach spaces and reflexivity have been developed by many authors. Giesy [7] and James [8] raised the question whether Banach spaces which are uniformly non-$l_n^{(1)}$ with some positive integer $n \geq 2$ are reflexive. James [8] settled the question affirmatively for $n = 2$ and gave a partial result for $n = 3$. Afterwards, the same author presented in [9] an example of a nonreflexive uniformly non-$l_3^{(1)}$ Banach space.

Definition 6. A Banach space X is said to have the Radon-Nikodym property whenever if (T, Σ, μ) is a nonatomic measure space and ν is a vector measure on Σ with values in X which is absolutely continuous with respect to μ and has bounded variation; then there exists $f \in L_1(X)$ such that, for any $A \in \Sigma$,

$$\nu(A) = \int_A f(t)\, dt. \tag{3}$$

Let us recall some geometrical notions that will be used in the further part of this paper. A point $x \in C$ is said to be a strongly exposed point of C if there exists $x^* \in X^*$ such that $x_n \to x$ whenever $x^*(x_n) \to x^*(x) = \sup\{x^*(x) : x \in C\}$. It is well known that Banach spaces have the Radon-Nikodym property if and only if every bounded closed convex subset of X is the closed convex hull of its strongly exposed points. A point $x \in S(X)$ is said to be a smooth point if it has a unique supporting functional f_x. If every $x \in S(X)$ is a smooth point, then X is called smooth. Let D be a nonempty open convex subset of X and let f be a real-valued continuous convex function on D. Recall that f is said to be Gateaux differentiable at the point x in D if the limit

$$f'(x)(y) = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} \tag{$*$}$$

exists for all $y \in X$. When this is the case, the limit is a continuous linear function of y, denoted by $f'(x)$.

In this paper, firstly, we prove that X is a uniformly non-$l_3^{(1)}$ nonsquare if and only if there exist two constants $\alpha, \beta > 0$ such that, for every 3-dimensional subspace Y of X, there exists a ball-covering $\mathfrak{B}$ of Y with $c(\mathfrak{B}) = 4$ or 5 which is α-off the origin and $r(\mathfrak{B}) \leq \beta$. Secondly, we will also prove that if a separable Banach space X has the Radon-Nikodym property, then X^* has the ball-covering property. Using this general result, we find sufficient conditions for an Orlicz function space to have ball-covering property. The topic of this paper is related to the topic of [1–6, 10–12].

2. Main Results

Theorem 7. *Suppose that X is a Banach space. Then, X is a uniformly non-$l_3^{(1)}$ space if and only if there exist two constants $\alpha, \beta > 0$ such that, for every 3-dimensional subspace Y of X, there exists a ball-covering $\mathfrak{B}$ of Y with $c(\mathfrak{B}) = 4$ or 5 which is α-off the origin and $r(\mathfrak{B}) \leq \beta$.*

In order to prove the theorem, we give some lemmas.

Lemma 8. *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in X. If $\alpha_n \geq 0$, $\beta_n \geq 0$, and $\lim_{n\to\infty}\|x_n + y_n\| = \lim_{n\to\infty}\|2x_n\| = \lim_{n\to\infty}\|2y_n\| = 2$, then $\lim_{n\to\infty}\|\alpha_n x_n + \beta_n y_n\| = 1$ if and only if $\lim_{n\to\infty}(\alpha_n + \beta_n) = 1$.*

Proof

Sufficiency. Let $\lim_{n\to\infty}(\alpha_n + \beta_n) = 1$. By $\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|y_n\| = 1$, we obtain that

$$\begin{aligned} \limsup_{n\to\infty} \|\alpha_n x_n + \beta_n y_n\| &\leq \limsup_{n\to\infty} (\alpha_n \|x_n\| + \beta_n \|y_n\|) \\ &= \limsup_{n\to\infty} (\alpha_n + \beta_n) = 1. \end{aligned} \tag{4}$$

Moreover, we may assume without loss of generality that $\alpha_n \geq \beta_n$. Noticing that $\lim_{n\to\infty}\|x_n + y_n\| = \lim_{n\to\infty}\|2x_n\| = \lim_{n\to\infty}\|2y_n\| = 2$, we have

$$\begin{aligned} &\liminf_{n\to\infty} \|\alpha_n x_n + \beta_n y_n\| \\ &\quad = \liminf_{n\to\infty} \|\alpha_n (x_n + y_n) - (\alpha_n - \beta_n) y_n\| \\ &\quad \geq \liminf_{n\to\infty} (\alpha_n \|x_n + y_n\| - (\alpha_n - \beta_n) \|y_n\|) \\ &\quad = \liminf_{n\to\infty} (\alpha_n + \beta_n) \\ &\quad = 1. \end{aligned} \tag{5}$$

Hence, we have $\lim_{n\to\infty}\|\alpha_n x_n + \beta_n y_n\| = 1$.

Necessity. Let $\lim_{n\to\infty}\|\alpha_n x_n + \beta_n y_n\| = 1$. By $(\alpha_n + \beta_n)^{-1}(\alpha_n + \beta_n) = 1$ for any $n \in N$ and the sufficiently part of the proof that has been just finished, we obtain that

$$\lim_{n\to\infty} \left\| \frac{\alpha_n}{\alpha_n + \beta_n} x_n + \frac{\beta_n}{\alpha_n + \beta_n} y_n \right\| = 1. \tag{6}$$

This implies that

$$\begin{aligned}1 &= \lim_{n\to\infty}\left\|\alpha_n x_n + \beta_n y_n\right\| \\ &= \lim_{n\to\infty}\left(\alpha_n+\beta_n\right)\left\|\frac{\alpha_n}{\alpha_n+\beta_n}x_n + \frac{\beta_n}{\alpha_n+\beta_n}y_n\right\| \\ &= \lim_{n\to\infty}\left(\alpha_n+\beta_n\right),\end{aligned} \quad (7)$$

which completes the proof. □

Lemma 9. *If $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ are three sequences and $\lim_{n\to\infty}\min\{\|\xi_1 x_n+\xi_2 y_n+\xi_3 z_n\| : \xi_i=\pm 1\} = \lim_{n\to\infty}\|3x_n\| = \lim_{n\to\infty}\|3y_n\| = \lim_{n\to\infty}\|3z_n\| = 3$, then*

$$\lim_{n\to\infty}\left\|z_n + \frac{\alpha_n}{\alpha_n+\beta_n}x_n + \frac{\beta_n}{\alpha_n+\beta_n}y_n\right\| = 2. \quad (8)$$

Proof. Since $\lim_{n\to\infty}\min\{\|\xi_1 x_n+\xi_2 y_n+\xi_3 z_n\| : \xi_i=\pm 1\} = \lim_{n\to\infty}\|3x_n\| = \lim_{n\to\infty}\|3y_n\| = \lim_{n\to\infty}\|3z_n\| = 3$, we obtain that $\liminf_{n\to\infty}\|x_n+y_n+z_n\| = 3$ and $\liminf_{n\to\infty}\|x_n-y_n+z_n\| = 3$. Therefore, by $\lim_{n\to\infty}\|x_n\|=1$ and $\lim_{n\to\infty}\|y_n\|=1$, we have

$$\begin{aligned}2 &= \lim_{n\to\infty}\|x_n\| + \lim_{n\to\infty}\|y_n\| \\ &\geq \liminf_{n\to\infty}\|x_n+y_n\| \\ &\geq \liminf_{n\to\infty}\left(\|x_n+y_n+z_n\|-\|z_n\|\right) \\ &\geq \liminf_{n\to\infty}\|x_n+y_n+z_n\| - \limsup_{n\to\infty}\|z_n\| \\ &= 2.\end{aligned} \quad (9)$$

Noticing that $\limsup_{n\to\infty}\|x_n+y_n\| \leq 2$, we have $\lim_{n\to\infty}\|x_n+y_n\| = 2$. Similarly, we have $\lim_{n\to\infty}\|x_n-y_n\| = 2$. This implies that $\lim_{n\to\infty}\min\{\|x_n-y_n\|, \|x_n+y_n\|\} = 2$. Moreover, we may assume without loss of generality that $\alpha_n \geq \beta_n$. Then

$$\begin{aligned}&\liminf_{n\to\infty}\left\|z_n + \frac{\alpha_n}{\alpha_n+\beta_n}x_n + \frac{\beta_n}{\alpha_n+\beta_n}y_n\right\| \\ &= \liminf_{n\to\infty}\left\|\left(x_n+y_n+z_n\right) - \frac{\beta_n}{\alpha_n+\beta_n}\left(x_n-y_n\right) - \frac{\beta_n-\alpha_n}{\alpha_n+\beta_n}y_n\right\| \\ &\geq \liminf_{n\to\infty}\left(\|x_n+y_n+z_n\| - \frac{\beta_n}{\alpha_n+\beta_n}\|x_n-y_n\| - \frac{\alpha_n-\beta_n}{\alpha_n+\beta_n}\|y_n\|\right) \\ &= \liminf_{n\to\infty}\left(3 - \frac{2\beta_n}{\alpha_n+\beta_n} - \frac{\alpha_n-\beta_n}{\alpha_n+\beta_n}\right) \\ &= 2.\end{aligned} \quad (10)$$

Moreover, it is easy to see that

$$\limsup_{n\to\infty}\left\|z_n + \frac{\alpha_n}{\alpha_n+\beta_n}x_n + \frac{\beta_n}{\alpha_n+\beta_n}y_n\right\| \leq 2. \quad (11)$$

This implies that

$$\lim_{n\to\infty}\left\|z_n + \frac{\alpha_n}{\alpha_n+\beta_n}x_n + \frac{\beta_n}{\alpha_n+\beta_n}y_n\right\| = 2, \quad (12)$$

which completes the proof. □

Lemma 10. *Suppose that X is a Banach space. Then, X is a uniformly non-$l_3^{(1)}$ space if and only if there exists $\varepsilon > 0$ such that, for every 3-dimensional subspace X_3 of X, if $T : X_3 \to (R^3, \|\cdot\|_1)$ is a linear isomorphism, then $\|T\|\cdot\|T^{-1}\| \geq 1+\varepsilon$.*

Proof

Necessity. Suppose that, for any natural number k, there exist a 3-dimensional subspace $X_{3,k}$ of X and a linear operator T_k such that $T_k : X_{3,k} \to (R^3, \|\cdot\|_1)$ is a linear isomorphism and $\|T_k\|\cdot\|T_k^{-1}\| < 1+1/k$. We may assume without loss of generality that $\|T_k\| = 1$. Moreover, it is easy to see that there exist $y_1, y_2, y_3 \in S((R^3, \|\cdot\|_1))$ such that

$$\min\left\{\|\xi_1 y_1 + \xi_2 y_2 + \xi_n y_3\| : \xi_i = \pm 1\right\} = 3. \quad (13)$$

By $\|T_k\|\cdot\|T_k^{-1}\| < 1+1/k$ and $\|T_k\| = 1$, we have

$$\begin{gathered}1 \leq \left\|T_k^{-1}y_1\right\| \leq 1+\frac{1}{k}, \qquad 1 \leq \left\|T_k^{-1}y_2\right\| \leq 1+\frac{1}{k}, \\ 1 \leq \left\|T_k^{-1}y_3\right\| \leq 1+\frac{1}{k}.\end{gathered} \quad (14)$$

Let

$$x_{1,k} = \frac{T_k^{-1}y_1}{\|T_k^{-1}y_1\|}, \qquad x_{2,k} = \frac{T_k^{-1}y_2}{\|T_k^{-1}y_2\|}, \qquad x_{3,k} = \frac{T_k^{-1}y_3}{\|T_k^{-1}y_3\|}. \quad (15)$$

Then

$$\begin{aligned}3 &\geq \min\left\{\|\xi_1 x_{1,k} + \xi_2 x_{2,k} + \xi_3 x_{3,k}\| : \xi_i = \pm 1\right\} \\ &= \min\left\{\left\|\xi_1\frac{T_k^{-1}y_1}{\|T_k^{-1}y_1\|} + \xi_2\frac{T_k^{-1}y_2}{\|T_k^{-1}y_2\|} + \xi_3\frac{T_k^{-1}y_3}{\|T_k^{-1}y_3\|}\right\| : \xi_i = \pm 1\right\} \\ &= \|T_k\|\min\left\{\left\|\xi_1\frac{T_k^{-1}y_1}{\|T_k^{-1}y_1\|} + \xi_2\frac{T_k^{-1}y_2}{\|T_k^{-1}y_2\|} + \xi_3\frac{T_k^{-1}y_3}{\|T_k^{-1}y_3\|}\right\| : \xi_i = \pm 1\right\} \\ &\geq \min\left\{\left\|\xi_1\frac{T_kT_k^{-1}y_1}{\|T_k^{-1}y_1\|} + \xi_2\frac{T_kT_k^{-1}y_2}{\|T_k^{-1}y_2\|} + \xi_3\frac{T_kT_k^{-1}y_3}{\|T_k^{-1}y_3\|}\right\| : \xi_i = \pm 1\right\} \\ &= \min\left\{\left\|\xi_1\frac{y_1}{\|T_k^{-1}y_1\|} + \xi_2\frac{y_2}{\|T_k^{-1}y_2\|} + \xi_3\frac{y_3}{\|T_k^{-1}y_3\|}\right\| : \xi_i = \pm 1\right\}.\end{aligned} \quad (16)$$

Therefore, by (13) and (14), we have

$$\lim_{k\to\infty}\min\left\{\|\xi_1 x_{1,k} + \xi_2 x_{2,k} + \xi_3 x_{3,k}\| : \xi_i = \pm 1\right\} = 3, \quad (17)$$

a contradiction. This implies that if there exists $\varepsilon > 0$ such that, for every 3-dimensional subspace X_3 of X, if $T : X_3 \to (R^3, \|\cdot\|_1)$ is a linear isomorphism, then $\|T\|\cdot\|T^{-1}\| \geq 1+\varepsilon$.

Sufficiency. Suppose that X is not a uniformly non-$l_3^{(1)}$ space. Then, for any natural number k, there exist $x_{1,k}, x_{2,k}, x_{3,k} \in S(X)$ such that

$$\lim_{k\to\infty}\min\left\{\|\xi_1 x_{1,k} + \xi_2 x_{2,k} + \xi_3 x_{3,k}\| : \xi_i = \pm 1\right\} = 3. \quad (18)$$

We define the subspace $Y_k = \{t_1x_{1,k}+t_2x_{2,k}+t_3x_{3,k} : \{t_i\}_{i=1}^3 \subset R\}$ of X. We claim that $\dim Y_k = 3$. In fact, suppose that $\dim Y_k < 3$. By (18), we obtain that $\lim_{k\to\infty}\min\{\|x_{1,k}+x_{2,k}\|, \|x_{1,k}-x_{2,k}\|\} = 2$. Hence, for any natural number k, we may assume without loss of generality that $x_{1,k}$ and $x_{2,k}$ are linearly independent. Notice that $\dim Y_k < 3$, so we obtain that $\dim Y_k = 2$. This implies that there exist $t_k \in R$ and $h_k \in R$ such that $x_{3,k} = t_kx_{1,k}+h_kx_{2,k}$. By (18) and $\lim_{k\to\infty}\min\{\|x_{1,k}+x_{2,k}\|, \|x_{1,k}-x_{2,k}\|\} = 2$, we may assume without loss of generality that $t_k \ge h_k \ge 0$. By $1 = \|x_{3,k}\| = \|t_kx_{1,k}+h_kx_{2,k}\|$ and Lemma 8, we have $\lim_{k\to\infty}(t_k+h_k) = 1$. Hence, we have

$$\begin{aligned} 3 &= \lim_{k\to\infty}\|x_{1,k}+x_{2,k}-x_{3,k}\| \\ &= \lim_{k\to\infty}\|x_{1,k}+x_{2,k}-(t_kx_{1,k}+h_kx_{2,k})\| \\ &= \lim_{k\to\infty}\|(1-t_k)x_{1,k}+(1-h_k)x_{2,k}\| \\ &\le \limsup_{k\to\infty}(1-t_k)\|x_{1,k}\| + \limsup_{k\to\infty}(1-h_k)\|x_{2,k}\| \\ &\le 2, \end{aligned} \tag{19}$$

a contradiction. This implies that $x_{1,k}$, $x_{2,k}$, and $x_{3,k}$ are linearly independent. Then, $\dim Y_k = 3$. We define the linear operator $T_k : Y_k \to (R^3, \|\cdot\|_1)$ by the formula

$$\begin{aligned} &T_k(t_{1,k}x_{1,k}+t_{2,k}x_{2,k}+t_{3,k}x_{3,k}) \\ &\quad = t_{1,k}\begin{pmatrix}1\\0\\0\end{pmatrix} + t_{2,k}\begin{pmatrix}0\\1\\0\end{pmatrix} + t_{3,k}\begin{pmatrix}0\\0\\1\end{pmatrix}. \end{aligned} \tag{20}$$

It is easy to see that T_k is one-one mapping.

Next, we will prove that $\lim_{k\to\infty}\|T_k\| = 1$. In fact, it is easy to see that, for any natural number k, we have $\|T_k\| \ge 1$. Suppose that $\lim_{k\to\infty}\|T_k\| \ne 1$. Then, we may assume without loss of generality that there exists $r > 0$ such that $\lim_{k\to\infty}\|T_k\| > 1+r$. This implies that there exists a sequence $\{t_{1,k}x_{1,k}+t_{2,k}x_{2,k}+t_{3,k}x_{3,k}\}_{k=1}^\infty \subset S(Y_k)$ such that

$$\lim_{k\to\infty}\|T_k(t_{1,k}x_{1,k}+t_{2,k}x_{2,k}+t_{3,k}x_{3,k})\| > 1+\frac{1}{2}r. \tag{21}$$

By (18), we may assume without loss of generality that $t_{1,k} \ge 0$, $t_{2,k} \ge 0$, and $t_{3,k} \ge 0$. By Lemma 8, we have

$$\lim_{k\to\infty}\left\|\frac{t_{2,k}}{t_{2,k}+t_{3,k}}x_{2,k}+\frac{t_{3,k}}{t_{2,k}+t_{3,k}}x_{3,k}\right\| = 1. \tag{22}$$

Moreover, by Lemma 9, we have

$$\lim_{k\to\infty}\left\|x_{1,k}+\frac{t_{2,k}}{t_{2,k}+t_{3,k}}x_{2,k}+\frac{t_{3,k}}{t_{2,k}+t_{3,k}}x_{3,k}\right\| = 2. \tag{23}$$

By $\|t_{1,k}x_{1,k}+t_{2,k}x_{2,k}+t_{3,k}x_{3,k}\| = 1$, we obtain

$$\begin{aligned} 1 &= \|t_{1,k}x_{1,k}+t_{2,k}x_{2,k}+t_{3,k}x_{3,k}\| \\ &= \Bigg\|t_{1,k}x_{1,k} \\ &\quad + (t_{2,k}+t_{3,k})\left(\frac{t_{2,k}}{t_{2,k}+t_{3,k}}x_{2,k}+\frac{t_{3,k}}{t_{2,k}+t_{3,k}}x_{3,k}\right)\Bigg\|. \end{aligned} \tag{24}$$

Therefore, by (22)–(24) and Lemma 8, we obtain $\lim_{k\to\infty}(t_{1,k}+t_{2,k}+t_{3,k}) = 1$. Noticing that $T_k(t_{1,k}x_{1,k}+t_{2,k}x_{2,k}+t_{3,k}x_{3,k}) = (t_{1,k}, t_{2,k}, t_{3,k})$, we have

$$\begin{aligned} &\lim_{k\to\infty}\|T_k(t_{1,k}x_{1,k}+t_{2,k}x_{2,k}+t_{3,k}x_{3,k})\| \\ &\quad = \lim_{k\to\infty}(t_{1,k}+t_{2,k}+t_{3,k}) = 1, \end{aligned} \tag{25}$$

a contradiction. This implies that $\lim_{k\to\infty}\|T_k\| = 1$.

Moreover, we claim that $\lim_{k\to\infty}\|T_k^{-1}\| = 1$. In fact, it is easy to see that, for any natural number k, we have $\|T_k^{-1}\| \ge 1$. Suppose that $\lim_{k\to\infty}\|T_k^{-1}\| \ne 1$. Then, we may assume without loss of generality that there exists $d > 0$ such that $\lim_{k\to\infty}\|T_k^{-1}\| > 1+d$. This implies that there exists a sequence $\{(t_{1,k}, t_{2,k}, t_{3,k})\}_{k=1}^\infty \subset S((R^3, \|\cdot\|_1))$ such that

$$\begin{aligned} &\lim_{k\to\infty}\|T_k^{-1}(t_{1,k}, t_{2,k}, t_{3,k})\| \\ &\quad = \lim_{k\to\infty}\|t_{1,k}x_{1,k}+t_{2,k}x_{2,k}+t_{3,k}x_{3,k}\| > 1+\frac{1}{2}d. \end{aligned} \tag{26}$$

By (18), we may assume without loss of generality that $t_{1,k} \ge 0$, $t_{2,k} \ge 0$, and $t_{3,k} \ge 0$. Noticing that $\{(t_{1,k}, t_{2,k}, t_{3,k})\}_{k=1}^\infty \subset S((R^3, \|\cdot\|_1))$, we have $t_{1,k}+t_{2,k}+t_{3,k} = 1$. Therefore, by (22)–(24) and Lemma 8, we have

$$\begin{aligned} 1 &= \lim_{k\to\infty}\Bigg\|t_{1,k}x_{1,k} \\ &\quad + (t_{2,k}+t_{3,k})\left(\frac{t_{2,k}}{t_{2,k}+t_{3,k}}x_{2,k}+\frac{t_{3,k}}{t_{2,k}+t_{3,k}}x_{3,k}\right)\Bigg\| \\ &= \lim_{k\to\infty}\|t_{1,k}x_{1,k}+t_{2,k}x_{2,k}+t_{3,k}x_{3,k}\|, \end{aligned} \tag{27}$$

a contradiction. This implies that $\lim_{k\to\infty}\|T_k^{-1}\| = 1$. Thus $\lim_{k\to\infty}\|T_k\|\cdot\|T_k^{-1}\| = 1$, a contradiction. Hence, we obtain that X is a uniformly non-$l_3^{(1)}$ space, which completes the proof. □

Lemma 11. *Suppose that (1) there exists a ball-covering $\mathfrak{B}_n$ of $(R^3, \|\cdot\|_n)$ and $c(\mathfrak{B}_n) = 5$, (2) $\|\cdot\|_n$ is uniformly convergent to $\|\cdot\|_\infty$ in $B(R^3, \|\cdot\|_\infty)$ and (3) $\mathfrak{B}_n$ is α-off the origin for any n. Then, $r(\mathfrak{B}_n) \to \infty$ as $n \to \infty$.*

Proof. Let $S(R^2, \|\cdot\|_n) \subset \cup_{i=1}^5 B(y_n^i, r_n^i)$, $\|y_n^i\| \ge r_n^i > 0$, and $\inf\{\|y_n^i\|-r_n^i : 1 \le i \le 5, n \in N\} \ge \alpha$. Suppose that there exists $\delta > 0$ such that $\max_{1\le i\le 5}\{r_n^i\} = r(\mathfrak{B}_n) \le \delta$. Then, $\{y_n^i\}_{n=1}^\infty$ and $\{r_n^i\}_{n=1}^\infty$ are bounded sequences. Hence, we may assume without loss of generality that $y_n^i \to y^i$ and $r_n^i \to r^i$ in $\|\cdot\|_\infty$ for any $\{1,\dots,5\}$. Then, $\alpha \le \|y_n^i\|-r_i^n \to \|y^i\|-r$ as $n \to \infty$. We claim that

$$S(R^3, \|\cdot\|_\infty) \subset \bigcup_{i=1}^5 B(y^i, r^i). \tag{28}$$

In fact, for any $A, B \subset R^3$ and $n \in N \cup +\infty$, let

$$d_n = \max\left\{\sup_{a\in A}\left\{\inf_{b\in B}\|a-b\|_n\right\}, \sup_{b\in B}\left\{\inf_{a\in A}\|a-b\|_n\right\}\right\}. \quad (29)$$

Since $\|\cdot\|_n$ is uniformly convergent to $\|\cdot\|_\infty$ in $B(R^3, \|\cdot\|_\infty)$, we obtain that, for any bounded set, $\|\cdot\|_n$ is uniformly convergent to $\|\cdot\|_\infty$. Moreover, it is easy to see that

$$\lim_{n\to\infty} d_\infty\left(B\left(R^3, \|\cdot\|_n\right), B\left(R^3, \|\cdot\|_\infty\right)\right) = 0,$$
$$\lim_{n\to\infty} d_\infty\left(B\left(y_n^i, r_n^i\right), B\left(y^i, r^i\right)\right) = 0. \quad (30)$$

This implies that

$$\lim_{n\to\infty} d_\infty\left(S\left(R^3, \|\cdot\|_n\right), S\left(R^3, \|\cdot\|_\infty\right)\right) = 0,$$
$$\lim_{n\to\infty} d_\infty\left(\bigcup_{i=1}^5 B\left(y_n^i, r_n^i\right), \bigcup_{i=1}^5 B\left(y^i, r^i\right)\right) = 0. \quad (31)$$

Since $\lim_{n\to\infty} d_\infty(S(R^3, \|\cdot\|_n), S(R^3, \|\cdot\|_\infty)) = 0$, then, for any $x \in S(R^3, \|\cdot\|_\infty)$, there exists a sequence $\{x_n\}_{n=1}^\infty \subset S(R^3, \|\cdot\|_n)$ such that $x_n \to x$ as $n \to \infty$. Since $S(R^3, \|\cdot\|_n) \subset \cup_{i=1}^5 B(y_n^i, r_n^i)$, then there exists $i \in \{1, \ldots, 5\}$ such that $x_n \in B(y_n^i, r_n^i)$ for any $n \in N$. Noticing that

$$\lim_{n\to\infty} d_\infty\left(B\left(y_n^i, r_n^i\right), B\left(y^i, r^i\right)\right) = 0, \quad (32)$$

we have $x \in B(y^i, r^i)$. This implies that $S(R^3, \|\cdot\|_\infty) \subset \cup_{i=1}^5 B(y^i, r^i)$. By Proposition 2, we have $c(\mathfrak{B}_{\min}(R^3, \|\cdot\|_\infty)) = 6$, a contradiction. This implies that $r(\mathfrak{B}_n) \to \infty$ as $n \to \infty$, which completes the proof. □

Proof of Theorem 7

Sufficiency. It is easy to see that there exist two constants $\alpha, \beta > 0$ such that, for every 3-dimensional subspace Y of X, there exists a ball-covering $\mathfrak{B}$ of Y with $c(\mathfrak{B}) = 5$ which is α-off the origin and $r(\mathfrak{B}) \le \beta$. Suppose that X is not a uniformly non-$l_3^{(1)}$ space. By Lemma 10, for any natural number k, there exist a 3-dimensional subspace $X_{3,k}$ of X and a linear operator T_k such that $T_k : X_{3,k} \to (R^3, \|\cdot\|_1)$ is a linear isomorphism and $\|T_k\|\cdot\|T_k^{-1}\| < 1+1/k$. Since $(R^3, \|\cdot\|_1)$ and $(R^3, \|\cdot\|_\infty)$ are isomorphism, there exists a linear operator G_k such that $G_k : X_{3,k} \to (R^3, \|\cdot\|_\infty)$ is a linear isomorphism and

$$\left\|G_k\right\|\cdot\left\|G_k^{-1}\right\| < 1 + \frac{1}{k}. \quad (33)$$

Moreover, we may assume without loss of generality that $\|G_k\| = 1$. Let

$$\|x\|_k = \left\|G_k^{-1}x\right\|, \quad \forall x \in R^3, \quad (34)$$

and let $\mathfrak{B}_k$ be a ball covering of $X_{3,k}$, where $c(\mathfrak{B}_k) = 5$ which is α-off the origin and $r(\mathfrak{B}_k) \le \beta$. It is easy to see that $\|\cdot\|_k$ is uniformly convergent to $\|\cdot\|_\infty$ in $B(R^3, \|\cdot\|_\infty)$. By Lemma 11, we have that $r(\mathfrak{B}_k) \to \infty$ as $k \to \infty$, a contradiction. Hence, we obtain that X is a uniformly non-$l_3^{(1)}$ space.

Necessity. By Lemma 10, we have $c(\mathfrak{B}) = 4$ or 5. Using the method of Theorem 3.5 in [6] and Lemma 10, similarly, we obtain that $\mathfrak{B}$ is α-off the origin and $r(\mathfrak{B}) \le \beta$, which completes the proof. □

Theorem 12. *Suppose that X is a uniformly non-$l_3^{(1)}$ space and smooth space. Then, there exist two constants $\alpha, \beta > 0$ such that for every 3-dimensional subspace Y of X, there exists a minimal ball-covering $\mathfrak{B}$ of Y with $c(\mathfrak{B}) = 4$ which is α-off the origin and $r(\mathfrak{B}) \le \beta$.*

Proof. By Proposition 2, we obtain that there exist two constants $\alpha, \beta > 0$ such that, for every 3-dimensional subspace Y of X, there exists a minimal ball-covering $\mathfrak{B}$ of Y with $c(\mathfrak{B}) = 4$. Using the method of Theorem 3.5 in [6] and Lemma 10, similarly, we obtain that $\mathfrak{B}$ is α-off the origin and $r(\mathfrak{B}) \le \beta$, which completes the proof. □

Theorem 13. *Suppose that there exist two constants $\alpha, \beta > 0$ such that, for every 3-dimensional subspace Y of X, there exists a minimal ball-covering $\mathfrak{B}$ of Y with $c(\mathfrak{B}) = 5$ which is α-off the origin and $r(\mathfrak{B}) \le \beta$. Then, X is a uniformly non-$l_3^{(1)}$ space and not smooth space.*

Proof. By Theorem 7, we obtain that X is a uniformly non-$l_3^{(1)}$ space. Suppose that X is a smooth space. Then, every 3-dimensional subspace Y of X is smooth. By Proposition 2, we obtain that $c(\mathfrak{B}_{\min}(Y)) = 4$. However, there exist two constants $\alpha, \beta > 0$ such that, for every 3-dimensional subspace Y of X, there exists a minimal ball-covering $\mathfrak{B}$ of Y with $c(\mathfrak{B}) = 5$ which is α-off the origin and $r(\mathfrak{B}) \le \beta$, a contradiction. Hence, we obtain that X is not a smooth space, which completes the proof. □

The following theorem (Theorem 15) shows that if a separable space X has the Radon-Nikodym property, then X^* has the ball-covering property. We first need a lemma.

Lemma 14 (see [5]). *Suppose that p is a Minkowski functional defined on the space X. Then, p is Gateaux differentiable at x and with the Gateaux derivative x^* if and only if x^* is a w^*-exposed point of C^* and exposed by x, where C^* is the polar of the level set $C = \{y \in X : p(y) \le 1\}$.*

Theorem 15. *Suppose that separable space X has the Radon-Nikodym property. Then, X^* has the ball-covering property.*

Proof. (a) First we will prove that there exists a sequence $\{x_n\}_{n=1}^\infty$ of w^*-exposed points of $B(X^{**})$ such that

$$\sup_n x^*\left(x_n\right) = \left\|x^*\right\|, \quad \forall x^* \in X^*. \quad (35)$$

Since X has the Radon-Nikodym property, then the closed convex hull $\mathrm{co}(E)$ of E is the whole $B(X)$, where E denotes strongly exposed points of $B(X)$.

Pick $y \in E$. Since $y \in E$ is a strongly exposed point of $B(X)$, there exists $y^* \in S(X^*)$ such that $x_n \to y$ whenever

$y^*(x_n) \to y^*(y) = \sup\{y^*(x) : x \in B(X)\} = 1$. Next we will prove that $y \in S(X^{**})$ is w^*-exposed point of $B(X^{**})$ and exposing by y^*. In fact, suppose that there exists $y^{**} \in S(X^{**})$ such that $y^*(y^{**}) = 1$. Since weak* topology is a Hausdorff topology, there exist a weak* neighbourhood U_y of y and a weak* neighbourhood $U_{y^{**}}$ of y^{**} such that $U_y \cap U_{y^{**}} = \emptyset$. Define the weak* neighbourhood as follows:

$$U_n = \left\{x^{**} \in X^{**} : \left|y^*\left(x^{**}\right) - y^*\left(y^{**}\right)\right| < \frac{1}{n}\right\} \bigcap U_{y^{**}}. \quad (36)$$

By the Goldstine theorem, there exists $y_n \in B(X^{**})$ such that $y_n \in U_n$. Hence, we have that $y^*(y_n) \to 1$ as $n \to \infty$. Since $y \in E$ is a strongly exposed point of $B(X)$, we obtain that $y_n \to y$ as $n \to \infty$. This implies that $y_n \in U_y$, when n is large enough. This contradicts the fact that $y_n \in U_n$. Hence, we obtain that $y \in S(X^{**})$ is a w^*-exposed point of $B(X^{**})$.

Since X is a separable space, then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\{x_n\}_{n=1}^{\infty}$ is a dense sequence in E. Noticing that $\overline{\mathrm{co}}(E) = B(X)$, we have

$$\begin{aligned} \left\|x^*\right\| &\geq \sup_n x^*\left(x_n\right) = \sup_{x\in E} x^*(x) \\ &= \sup_{x\in \mathrm{co}(E)} x^*(x) = \sup_{x\in B(X)} x^*(x) = \left\|x^*\right\|. \end{aligned} \quad (37)$$

(b) Next we will prove that X^* has the ball-covering property. By Lemma 14, for any $x_i \in \{x_n\}_{n=1}^{\infty} \subset S(X^{**})$, there exist $x_i^* \in S(X^*)$ such that $\|\cdot\|$ is Gateaux differentiable at x_i^* and with the Gateaux derivative x_i. For each fixed $1 < i < \infty$, let $B_{i,m}$ be the balls defined by

$$B_{i,m} = B\left(mx_i^*, m - \frac{1}{m}\right), \quad i = 1, 2, \ldots. \quad (38)$$

Clearly, every $B_{i,m}$ has the distance $1/m$ from the origin. We claim that

$$S\left(X^*\right) \subset \bigcup \left\{B_{i,m} : i = 1, 2, \ldots\ m = 1, 2, \ldots\right\}. \quad (39)$$

In fact, pick $\alpha \in (0, 1)$. Noticing that

$$\sup_n x^*\left(x_n\right) = \left\|x^*\right\|, \quad \forall x^* \in X^*, \quad (40)$$

we obtain that, for $y^* \in S(X^*)$, there exists $x \in \{x_n\}_{n=1}^{\infty}$ such that

$$y^*(x) \geq \alpha \left\|y^*\right\| = \alpha > 0. \quad (41)$$

We can assume that $x = x_j$ for some $1 \leq j < \infty$. Thus, there exist $\beta \geq \alpha$ and $h_j^* \in H_j^* = \{x^* \in X^* : x^*(x_j) = 0\}$ such that

$$y^* = \beta x^* + h_j^*. \quad (42)$$

We want to show that $y^* \in \cup_{m=1}^{\infty} B_{j,m}$. Otherwise, for every $m \in N$,

$$m - \frac{1}{m} \leq \left\|mx_j^* - y^*\right\| = \left\|(m-\beta)\, x_j^* - h_j^*\right\|. \quad (43)$$

Thus,

$$\begin{aligned} -\frac{1}{m} &\leq \left\|(m-\beta)\, x_j^* - h_j^*\right\| - m \\ &= \left\|(m-\beta)\, x_j^* - h_j^*\right\| - m\left\|x_j^*\right\| \\ &= (m-\beta)\left\{\left\|x_j^* - \frac{1}{m-\beta} h_j^*\right\| - \left\|x_j^*\right\|\right\} - \beta \\ &= \frac{\left\|x_j^* - t h_j^*\right\| - \left\|x_j^*\right\|}{t} - \beta, \end{aligned} \quad (44)$$

where $t = 1/(m-\beta)$. Letting $m \to \infty$, we observe that

$$0 \leq \left\|x_j^*\right\|'\left(h_j^*\right) - \beta \leq h_j^*\left(x_j\right) - \beta = -\beta < 0, \quad (45)$$

which is a contradiction. Therefore,

$$S\left(X^*\right) \subset \bigcup \left\{B_{i,m} : i = 1, 2, \ldots\ m = 1, 2, \ldots\right\}. \quad (46)$$

Hence, X^* has the ball-covering property, which completes the proof. □

Corollary 16. *If X^* is a separable space, then X^{**} has the ball-covering property.*

Proof. If X^* is separable, then X^* has the Radon-Nikodym property. By Theorem 13, we obtain that X^{**} has the ball-covering property, which completes the proof. □

3. Applications to Orlicz Function Spaces

It is easy to see that if X is separable, then X has the ball-covering property. Cheng [1] proved that the sequence space l^{∞} which is not separable has the ball-covering property. In this section, we obtain that there exists a nonseparable function space such that it has the ball-covering property.

Definition 17. $M : R \to R$ is called an N-function if it has the following properties:

(1) M is even, convex and $M(0) = 0$;

(2) $M(u) > 0$ for all $u \neq 0$;

(3) $\lim_{u\to 0} M(u)/u = 0$ and $\lim_{u\to\infty} M(u)/u = \infty$.

Let (G, Σ, μ) be a finite nonatomic and complete measure space. Denote by p and q the right derivative of M and N, respectively. We define

$$\begin{gathered} \rho_M(x) = \int_G M(x(t))\, dt, \\ L_M = \left\{x(t) : \rho_M(\lambda x) < \infty, \text{ for some } \lambda > 0\right\}, \\ E_M = \left\{x(t) : \rho_M(\lambda x) < \infty, \forall \lambda > 0\right\}. \end{gathered} \quad (47)$$

It is well known that the Orlicz function space L_M is a Banach space when it is equipped with the Luxemburg norm

$$\|x\| = \inf\left\{\lambda > 0 : \rho_M\left(\frac{x}{\lambda}\right) \leq 1\right\} \quad (48)$$

or equipped with the Amemiya-Orlicz norm

$$\|x\|^0 = \inf_{k>0} \frac{1}{k} \left(1 + \rho_M (kx)\right). \tag{49}$$

Let $p(u)$ denote the right derivative of $M(u)$ at $u \in R^+$ and let $q(v)$ be the generalized inverse function of $p(u)$ defined on R^+ by

$$q(v) = \sup_{u \geq 0} \{u \geq 0 : p(u) \leq v\}. \tag{50}$$

Then, we call $N(v) = \int_0^{|v|} q(s)ds$ the complementary function of M. It is well known that there holds the Young inequality $uv \leq M(u) + N(v)$ and $uv = M(u) + N(v) \Leftrightarrow u = |q(v)| \operatorname{sign} v$ or $v = |p(u)| \operatorname{sign} u$. Moreover, it is well known that M and N are complementary to each other.

We say that an N-function $M \in \Delta_2 (N \in \nabla_2)$ if there exist $K > 2$ and $u_0 \geq 0$ such that

$$M(2u) \leq KM(u) \quad (u \geq u_0). \tag{51}$$

By [13], we know that $L_M(L_M^0)$ is separable, $\Leftrightarrow$ $L_M(L_M^0)$ has the Radon-Nikodym property $\Leftrightarrow$ $M \in \Delta_2$, and $M \in \Delta_2 \Leftrightarrow L_M = E_M(L_M^0 = E_M^0) \Leftrightarrow L_M(L_M^0)$ is separable. Moreover, by [13], we know that $(E_M)^* = L_M^0$ and $(E_M^0)^* = L_M$.

Theorem 18. *If $M \in \nabla_2$ or $M \in \Delta_2$, then $L_M(L_M^0)$ has the ball-covering property.*

Proof. By [13], we know that $(E_M)^* = L_M^0, (E_M^0)^* = L_M$. Using Theorem 15, we obtain that if $M \in \nabla_2$, then $L_M(L_M^0)$ has the ball-covering property. Moreover, by $M \in \Delta_2$, we obtain that $L_M(L_M^0)$ is separable. Hence, $L_M(L_M^0)$ has the ball-covering property, which completes the proof. □

Remark 19. It is well known that there exists an N-function M such that $M \in \nabla_2$ and $M \notin \Delta_2$. This means that $L_M(L_M^0)$ is not a separable space. However, by Theorem 18, we obtain that $L_M(L_M^0)$ has ball-covering property.

Acknowledgment

This research is supported by The Fundamental Research Funds for the Central Universities, DL12BB36 and by Heilongjiang Provincial Department of Education Funds 12521070.

References

[1] L. Cheng, "Ball-covering property of Banach spaces," *Israel Journal of Mathematics*, vol. 156, pp. 111–123, 2006.

[2] L. Cheng, "Erratum to: ball-covering property of Banach spaces," *Israel Journal of Mathematics*, vol. 184, pp. 505–507, 2011.

[3] L. Cheng, Q. Cheng, and X. Liu, "Ball-covering property of Banach spaces that is not preserved under linear isomorphisms," *Science in China A*, vol. 51, no. 1, pp. 143–147, 2008.

[4] L. Cheng, H. Shi, and W. Zhang, "Every Banach space with a w-separable dual has a $1+\varepsilon$-equivalent norm with the ball covering property," *Science in China A*, vol. 52, no. 9, pp. 1869–1874, 2009.

[5] L. Cheng, Q. Cheng, and H. Shi, "Minimal ball-coverings in Banach spaces and their application," *Studia Mathematica*, vol. 192, no. 1, pp. 15–27, 2009.

[6] L. X. Cheng, Z. H. Luo, X. F. Liu, and W. Zhang, "Several remarks on ball-coverings of normed spaces," *Acta Mathematica Sinica*, vol. 26, no. 9, pp. 1667–1672, 2010.

[7] D. P. Giesy, "On a convexity condition in normed linear spaces," *Transactions of the American Mathematical Society*, vol. 125, pp. 114–146, 1966.

[8] R. C. James, "Uniformly non-square Banach spaces," *Annals of Mathematics*, vol. 80, pp. 542–550, 1964.

[9] R. C. James, "A nonreflexive Banach space that is uniformly nonoctahedral," *Israel Journal of Mathematics*, vol. 18, pp. 145–155, 1974.

[10] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, vol. 1364 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 1989.

[11] V. P. Fonf and C. Zanco, "Covering spheres of Banach spaces by balls," *Mathematische Annalen*, vol. 344, no. 4, pp. 939–945, 2009.

[12] M. Denker and H. Hudzik, "Uniformly non-$l_n^{(1)}$ Musielak-Orlicz sequence spaces," *Proceedings of the Indian Academy of Sciences*, vol. 101, no. 2, pp. 71–86, 1991.

[13] S. T. Chen, "Geometry of Orlicz spaces," *Dissertationes Mathematicae*, vol. 356, pp. 1–204, 1996.