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Research Article

Exact and Analytic-Numerical Solutions of Lagging Models of Heat Transfer in a Semi-Infinite Medium

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Different non-Fourier models of heat conduction have been considered in recent years, in a growing area of applications, to model microscale and ultrafast, transient, nonequilibrium responses in heat and mass transfer. In this work, using Fourier transforms, we obtain exact solutions for different lagging models of heat conduction in a semi-infinite domain, which allow the construction of analytic-numerical solutions with prescribed accuracy. Examples of numerical computations, comparing the properties of the models considered, are presented.

1. Introduction

Non-Fourier models of heat conduction have increasingly been considered in recent years to model microscale and ultrafast, transient, nonequilibrium responses in heat and mass transfer, where thermal lags and nonclassical phenomena are present (see, e.g., [1] and references therein). The growing area of applications of these models include, among other examples, the processing of thin-film engineering structures with ultrafast lasers [2, 3], the transfer of heat in nanofluids [4, 5], or the exchange of heat in biological tissues [6–8].

In the Dual-Phase-Lag (DPL) model [9–11], the equation relating the heat flux vector $\mathbf{q}$ and the temperature T, for time t and spatial point $\mathbf{r}$,

$$\mathbf{q}\left(\mathbf{r}, t+\tau_q\right) = -k \nabla T\left(\mathbf{r}, t+\tau_T\right), \tag{1}$$

where $k > 0$ is the thermal conductivity, incorporates two lags, τ_q for the heat flux and τ_T for the temperature gradient. When both lags are zero, the Fourier law is recovered, while for $\tau_q > 0$ and $\tau_T = 0$, it reduces to the Single-Phase-Lag (SPL) model [12].

Combining (1) with the principle of energy conservation,

$$-\nabla \cdot \mathbf{q}(\mathbf{r}, t) + Q(\mathbf{r}, t) = C_p T_t(\mathbf{r}, t), \tag{2}$$

where C_p is the volumetric heat capacity and Q, the volumetric heat source, in the absence of heat sources a partial differential equation with delay is obtained [13, 14] as

$$T_t\left(\mathbf{r}, t+\tau_q\right) = \alpha \Delta T\left(\mathbf{r}, t+\tau_T\right), \tag{3}$$

where $\alpha = k/C_p$ is the thermal diffusivity. When both lags are zero, the diffusion equation, a parabolic partial differential equation which represents the classical model for heat conduction and other transport phenomena, is obtained.

Using first-order approximations in (1),

$$\mathbf{q}(\mathbf{r}, t) + \tau_q \frac{\partial \mathbf{q}}{\partial t}(\mathbf{r}, t) \cong -k\left\{\nabla T(\mathbf{r}, t) + \tau_T \frac{\partial}{\partial t} \nabla T(\mathbf{r}, t)\right\}, \tag{4}$$

a hyperbolic equation is derived, commonly referred to as the DPL model [9], here denoted as DPL(1, 1),

$$T_t(\mathbf{r}, t) + \tau_q T_{tt}(\mathbf{r}, t) = \alpha\left\{\Delta T(\mathbf{r}, t) + \tau_T \Delta T_t(\mathbf{r}, t)\right\}, \tag{5}$$

which for $\tau_T = 0$ reduces to the Cattaneo-Vernotte (CV) model [15–17].

Approximations in (1) up to order two in τ_q and/or τ_T have also been considered [18, 19], leading to models that will be denoted as DPL(2, 1),

$$\begin{aligned} T_t(\mathbf{r}, t) + \tau_q T_{tt}(\mathbf{r}, t) + \frac{\tau_q^2}{2} T_{ttt}(\mathbf{r}, t) \\ = \alpha\left\{\Delta T(\mathbf{r}, t) + \tau_T \Delta T_t(\mathbf{r}, t)\right\}, \end{aligned} \tag{6}$$

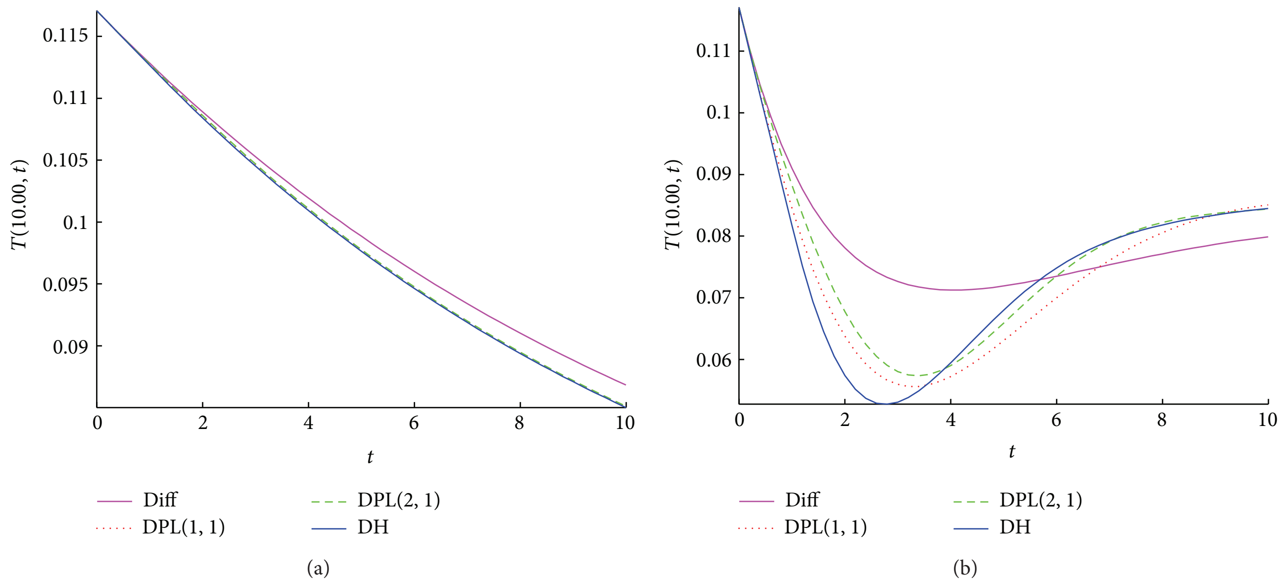

Figure 1: Temperature evolution, at $x = 10$, for DPL, DH, and classical diffusion (Diff) models with Dirichlet boundary conditions and parameters $\tau_T = 0$, $\tau_q = 1$, and initial function $\varphi(x, t) = 2(1 - \cos(x))/(\pi x)$, for $\alpha = 0.1$ (a) and $\alpha = 0.8$ (b).

and DPL(2, 2),

$$\begin{aligned} T_t(\mathbf{r}, t) + \tau_q T_{tt}(\mathbf{r}, t) + \frac{\tau_q^2}{2} T_{ttt}(\mathbf{r}, t) \\ = \alpha \left\{ \Delta T(\mathbf{r}, t) + \tau_T \Delta T_t(\mathbf{r}, t) + \frac{\tau_T^2}{2} \Delta T_{tt}(\mathbf{r}, t) \right\}. \end{aligned} \quad (7)$$

From the original formulation of the DPL model, as given in (1), for $\tau = \tau_q - \tau_T > 0$, a retarded partial differential equation is obtained [13, 14], referred to as the delayed heat conduction model (DH),

$$T_{t'}\left(\mathbf{r}, t'\right) = \alpha \Delta T\left(\mathbf{r}, t' - \tau\right), \quad (8)$$

where $t' = t + \tau_q$.

Exact solutions for some particular DPL models in different settings have been discussed (e.g., [11, 13, 14, 20–22]), and many specific methods to construct numerical solutions, usually in finite domains using finite difference schemes, have been developed (see, e.g., [23–27]).

In semi-infinite domains, some particular problems have also been considered. Solutions for heat propagation according to DPL(1, 1) model in a semi-infinite solid, produced by suddenly raising the temperature at the boundary, were obtained in [11, 20], using Laplace and Fourier transforms. Relations between the local values of heat flux and temperature, in the form of integral equations, in a semi-infinite solid were considered in [13, 28].

In this work, using Fourier transforms, explicit solutions for lagging models of heat conduction in a semi-infinite domain, with different types of boundary conditions, are obtained, allowing the construction of numerical solutions with bounded errors.

It should be noted that Fourier transforms can also be used in time-dependent problems (e.g., [29, 30]), and the approach of this work could also be useful for time-dependent DPL models, which have already been proposed [31].

2. Solutions of DPL Models in a Semi-Infinite Domain

Consider a plate of infinite thickness, $x \in [0, \infty]$, that can be heated either at its surface, $x = 0$, or up to a certain depth, $x \in [0, l]$. We will consider, for $t \geq 0$, either Dirichlet, $T(0, t) = 0$, or Neumann, $T_x(0, t) = 0$, boundary conditions and also that

$$\lim_{x \to \infty} T(x, t) = 0, \quad t \geq 0. \quad (9)$$

Appropriate initial conditions must be provided for the different models. Thus, for DPL(1, 1) initial values for temperature and its time derivative have to be specified,

$$T(x, 0) = \varphi(x, 0), \quad T_t(x, 0) = \phi(x, 0), \quad 0 \leq x < \infty, \quad (10)$$

while for DPL(2, 1) and DPL(2, 2), its second derivative also has to be given,

$$T_{tt}(x, 0) = \psi(x, 0), \quad 0 \leq x < \infty, \quad (11)$$

and for the DH model, the initial condition for the temperature has to be specified for a time interval of τ amplitude,

$$T(x, t) = \varphi(x, t), \quad 0 \leq t \leq \tau, \ 0 \leq x < \infty. \quad (12)$$

For a wide class of initial functions [32, 33], the method of Fourier transform can be used to eliminate derivatives in the spatial domain and to obtain expressions for the exact solutions in the form of an infinite integral, either using Fourier sine transforms for Dirichlet conditions,

$$T(x, t) = \frac{2}{\pi} \int_0^\infty \mathcal{T}(w, t) \sin(wx)\, dw, \quad (13)$$

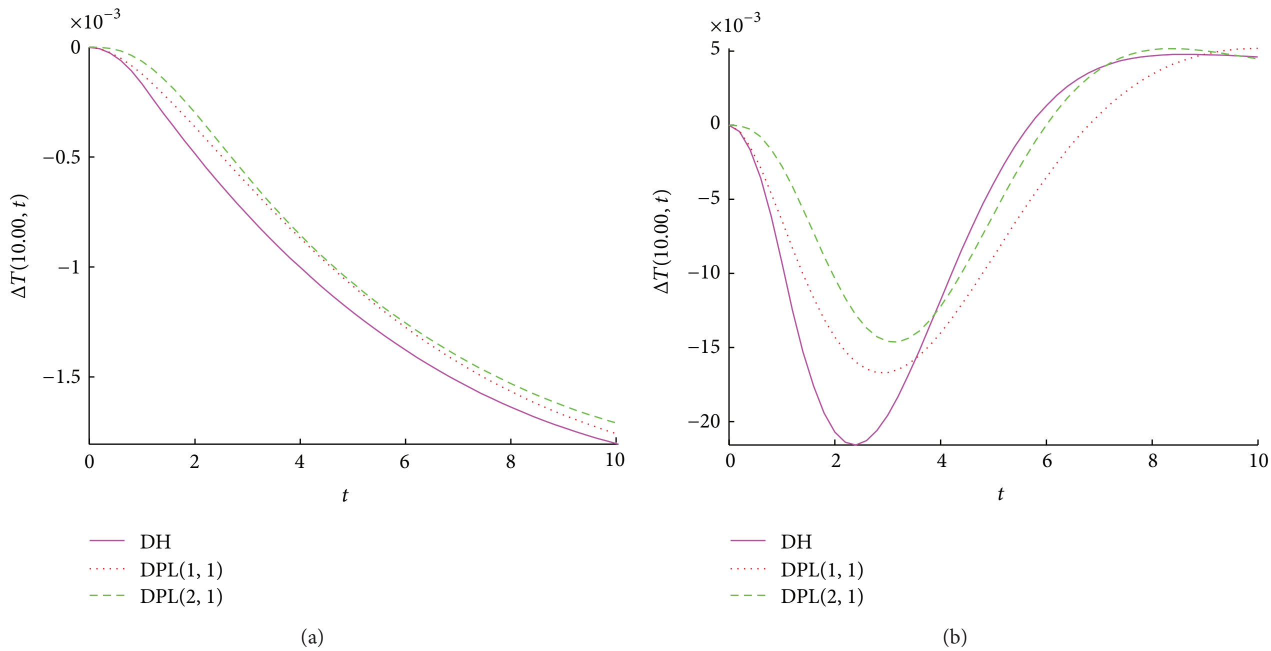

Figure 2: Differences from classical diffusion for models DPL and DH, for the data shown in Figure 1.

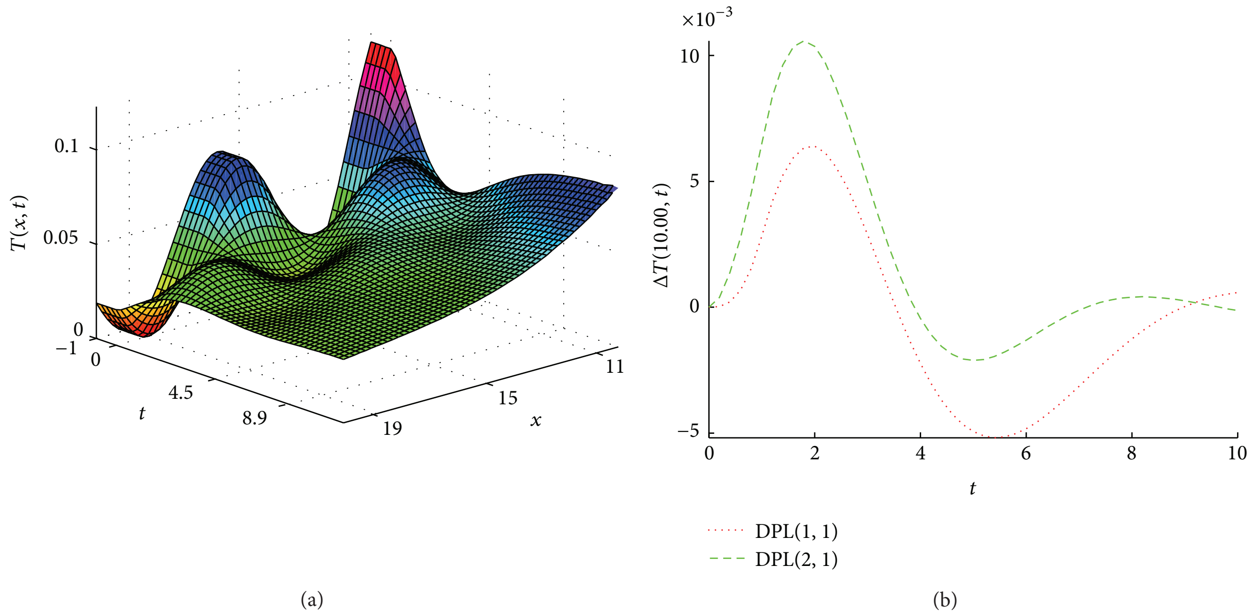

Figure 3: Temperature evolution, for $(x,t) \in [10,20]\times[0,10]$, for the DH model with Dirichlet boundary conditions and parameters $\tau_T = 0$, $\tau_q = 1$, $\alpha = 0.8$, and initial function $\varphi(x,t) = 2(1-\cos(x))/(\pi x)$ (a), and differences from DH of DPL(1, 1) and DPL(2, 1) at $x = 10$ (b).

or cosine transforms for Neumann conditions,

$$T(x,t) = \frac{2}{\pi}\int_0^{\infty} \mathcal{T}(w,t)\cos(wx)\,dw, \tag{14}$$

where the functions $\mathcal{T}(w,t)$, which are the corresponding Fourier sine or cosine transforms of $T(x,t)$, are obtained as solutions of the transformed temporal problems, depending on the continuous set of eigenvalues w^2.

For the family of DPL approximations, the transformed problems are initial-value problems for linear differential equations with constant coefficients. Thus, for DPL(1, 1), one gets

$$\begin{gathered}\tau_q \mathcal{T}''(w,t) + \left(1 + w^2\alpha\tau_T\right)\mathcal{T}'(w,t) + w^2\alpha\mathcal{T}(w,t) = 0,\\ \mathcal{T}(w,0) = F(w), \qquad \mathcal{T}'(w,0) = G(w),\end{gathered} \tag{15}$$

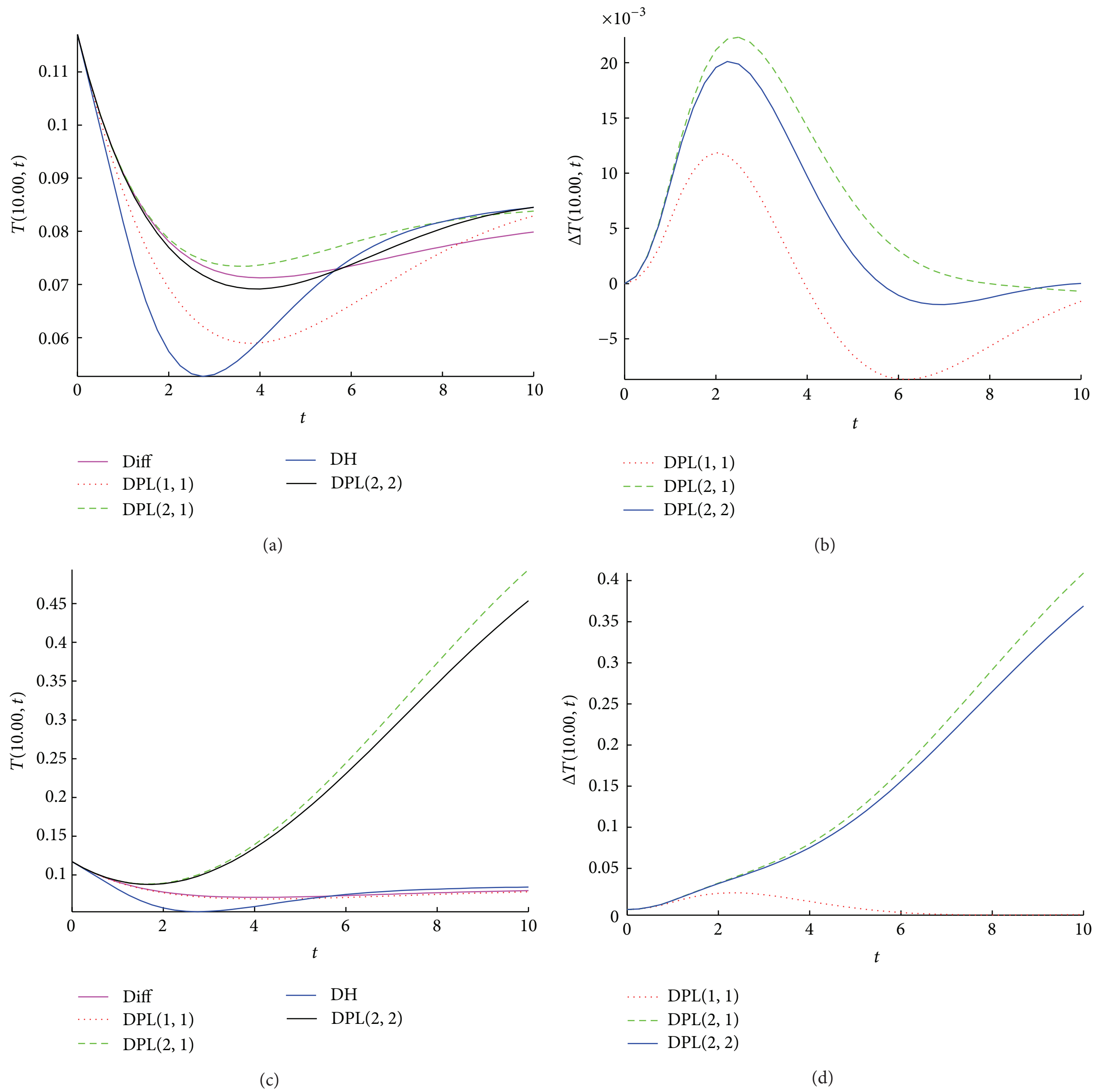

FIGURE 4: Temperature evolution for DPL, DH, and classical diffusion (Diff) models (left), and differences from DH of DPL approximations (right), at $x = 10$, with Dirichlet boundary conditions, initial function $\varphi(x,t) = 2(1-\cos(x))/(\pi x)$, and parameters $\alpha = 0.8$, $\tau_T = 1$, and $\tau_q = 1$ (top), or $\tau_T = 19$ and $\tau_q = 20$ (down).

for DPL(2, 1), the problem reads

$$\frac{\tau_q^2}{2}\mathcal{T}'''(w,t) + \tau_q \mathcal{T}''(w,t) + \left(1 + w^2\alpha\tau_T\right)\mathcal{T}'(w,t) + w^2\alpha\mathcal{T}(w,t) = 0, \tag{16}$$

with initial conditions

$$\mathcal{T}(w,0) = F(w), \qquad \mathcal{T}'(0) = G(w), \qquad \mathcal{T}''(0) = H(w), \tag{17}$$

and, with the same initial conditions as in DPL(2, 1), for DPL(2, 2), one gets

$$\left(\frac{\tau_q^2}{2}\right)\mathcal{T}'''(w,t) + \left(\frac{\tau_q + w^2\alpha\tau_T^2}{2}\right)\mathcal{T}''(w,t) + \left(1 + w^2\alpha\tau_T\right)\mathcal{T}'(w,t) + w^2\alpha\mathcal{T}(w,t) = 0, \tag{18}$$

where $F(w)$, $G(w)$, and $H(w)$ are the Fourier sine or cosine transforms, according to the type of boundary conditions, of the initial functions $\varphi(x,t)$, $\phi(x,t)$, and $\psi(x,t)$, respectively.

Hence, these problems can be solved, obtaining expressions for $\mathcal{T}(w,t)$ in terms of the roots of the corresponding characteristic equation, and thus explicit expressions for the exact solutions for these models, in the form of (13) or (14), can be obtained.

For the DH model, the transformed temporal problems are initial-value problems for delay differential equations with general initial functions,

$$\begin{aligned} &\mathcal{T}'(w,t) + w^2\alpha\mathcal{T}(w,t-\tau) = 0, \quad t > \tau, \\ &\mathcal{T}(w,t) = F(w,t), \quad 0 \le t \le \tau, \end{aligned} \tag{19}$$

where $F(w,t)$ is the appropriate Fourier transform, according to the boundary conditions, of the initial function $\varphi(x,t)$. To obtain constructive solutions for this problem, a combination of the steps method and a convolution integral can be applied [34, 35], producing the following expression, for $t \in [p\tau, (p+1)\tau]$,

$$\begin{aligned} \mathcal{T}(w,t) = {} & F(w,\tau) + F(w,0)\sum_{k=1}^{p}\frac{\left(-w^2\right)^k\alpha^k(t-k\tau)^k}{k!} \\ & + \sum_{k=1}^{p-1}\frac{\left(-w^2\right)^k\alpha^k}{k!}\int_0^{\tau}(t-k\tau-s)^k F_s(w,s)\,ds \\ & + \frac{\left(-w^2\right)^p\alpha^p}{p!}\int_0^{t-p\tau}(t-p\tau-s)^p F_s(w,s)\,ds. \end{aligned} \tag{20}$$

The solutions obtained with the Fourier transforms, as given in (13) or (14), can be shown to converge and provide exact solutions under adequate integrability and regularity conditions on the initial functions. Numerical integration is required in general to compute numerical approximations of these solutions, with errors that can be bounded in finite spatial and temporal domains by controlling errors in the numerical integrators or by appropriately truncating the infinite integrals. However, for some particular initial functions, the solutions given by (13) or (14) may reduce to finite integrals.

3. Numerical Examples

Numerical examples are presented in the following figures, where, in order to properly compare DPL and DH models, the initial interval for DH, where the initial function $\varphi(x,t)$ is given, is displaced to $[-\tau, 0]$, and the initial functions for DPL models are set so that the values of temperature and its first derivative at $t = 0$, and also its second derivative for DPL(2, 1) and DPL(2, 2), are matched to those of the DH model. The classical diffusion model, whose solution is available and readily obtained [36], is also included as reference.

First, we consider models with $\tau_T = 0$, so that DPL(2, 1) and DPL(2, 2) are equal, and an initial function with damped temperature oscillations, thought to be the result of a modulated heat source that is switched off at $t = 0$, showing the transient behavior for the different DPL models for different values of α (Figure 1), as well as their differences from classical diffusion (Figure 2).

In Figure 3, a more detailed view of the spatiotemporal behavior of the DH model (Figure 3(a)) and differences from DH of DPL(1, 1) and DPL(2, 1) (Figure 3(b)) are presented.

In Figure 4, different values of τ_T and τ_q, such that $\tau = \tau_q - \tau_T$ is kept constant, are used, so that variations in the temperature evolution are observed in the DPL approximate models, but not in the DH model, which only depends on the value of τ.

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