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Research Article

A Two-Scale Discretization Scheme for Mixed Variational Formulation of Eigenvalue Problems

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This paper discusses highly efficient discretization schemes for mixed variational formulation of eigenvalue problems. A new finite element two-scale discretization scheme is proposed by combining the mixed finite element method with the shifted-inverse power method for solving matrix eigenvalue problems. With this scheme, the solution of an eigenvalue problem on a fine grid K^h is reduced to the solution of an eigenvalue problem on a much coarser grid K^H and the solution of a linear algebraic system on the fine grid K^h. Theoretical analysis shows that the scheme has high efficiency. For instance, when using the Mini element to solve Stokes eigenvalue problem, the resulting solution can maintain an asymptotically optimal accuracy by taking $H = O(\sqrt[4]{h})$, and when using the P_{k+1}-P_k element to solve eigenvalue problems of electric field, the calculation results can maintain an asymptotically optimal accuracy by taking $H = O(\sqrt[3]{h})$. Finally, numerical experiments are presented to support the theoretical analysis.

1. Introduction

To improve the efficiency of finite element method, Xu introduced a two-scale discretization scheme and applied it to nonsymmetric and nonlinear elliptic problems (see [1–3]). Later on, this scheme attracted the attention of academic circles and has been successfully applied to Stokes equations (see [4–7]), semilinear eigenvalue problems (see [8]), and linear differential operator eigenvalue problems, and so forth.

Up to now, two kinds of finite element two-scale discretization schemes have been developed for linear differential operator eigenvalue problems. The first kind is established by Xu and Zhou [9] in 2001, whose idea correlates with the iterative Galerkin method which was established by Lin and Xie [10] and Sloan [11], but it bases on the finite element spaces on two different scale grids. The scheme of Xu and Zhou has been applied to the conforming finite element method for electric structure eigenvalue problem (see [12–14]), the conforming/nonconforming finite element method for non-self-adjoint eigenvalue problem (see [15, 16]), the conforming/nonconforming finite element method for Steklov eigenvalue problem (see [17, 18]), the mixed finite element method for Stokes eigenvalue problem, and biharmonic equations eigenvalue problem (see [19, 20]). Another group of two-scale discretization scheme is proposed by Yang and Bi [21]. They established the conforming/nonconforming finite element two-scale discretization scheme based on the shifted-inverse power method.

With two-scale discretization schemes, the solution of an eigenvalue problem on a fine grid K^h is reduced to the solution of an eigenvalue problem on a much coarser grid K^H and the solution of a linear algebraic system on the fine grid K^h, and the resulting solution still maintains an asymptotically optimal accuracy. Thus, the computational efficiency is improved.

Influenced by the work mentioned above, in this paper we establish a new finite element two-scale discretization scheme for mixed variational formulation of eigenvalue problem and apply it to Stokes eigenvalue problem and eigenvalue problem of electric field. The research of this paper has the following features.

(1) Our two-scale discretization scheme is a combinative production of the mixed finite element method and the shifted-inverse power method (see [22, Algorithm 27.2]). Comparing with the scheme in [19, 20], the scheme in this paper is more efficient: the resulting solution obtained by our scheme can maintain an asymptotically optimal accuracy by taking $H = O(\sqrt[4]{h})$ when solving Stokes eigenvalue problem and $H = O(\sqrt[3]{h})$ when solving eigenvalue problem of electric field; however, with the scheme in [19, 20] the resulted solution maintains an asymptotically optimal accuracy by taking $H = O(\sqrt[2]{h})$.

(2) The literatures of high-efficient numerical method for eigenvalue problem of electric field are not too many by now, thus they seem to be very valuable. Our two-scale discretization scheme is a new and highly efficient method for eigenvalue problem of electric field.

The rest of this paper is organized as follows. Some preliminaries of finite element approximations for eigenvalue problems which are needed in this paper are provided in the next section. In Section 3, for eigenvalue problem mixed variational formulation (2.3)-(2.4) in general form, the finite element two-scale discretization scheme based on the shifted-inverse power method is established, and the validity of this scheme is proved theoretically. In Sections 4 and 5, the scheme established in Section 3 is applied to Stokes eigenvalue problem and eigenvalue problem of electric field, respectively. Finally, numerical experiments are presented in Section 6.

2. Preliminaries

Let V, W, and D be three real Hilbert spaces with inner products and norms $(\cdot,\cdot)_V$, $\|\cdot\|_V$, $(\cdot,\cdot)_W$, $\|\cdot\|_W$, and $(\cdot,\cdot)_D$, $\|\cdot\|_D$, respectively. We suppose that $V \hookrightarrow D$ (continuously imbedded), $a(\cdot,\cdot)$ is a symmetric, continuous, and V-elliptic bilinear form on $V \times V$, that is,

$$\begin{aligned} |a(q,\psi)| &\le M_1 \|q\|_V \|\psi\|_V, \quad \forall q,\psi \in V, \\ a(q,q) &\ge \nu \|q\|_V^2, \quad \forall 0 \ne q \in V, \end{aligned} \tag{2.1}$$

$b(\cdot,\cdot)$ is a continuous bilinear form on $V \times W$, that is,

$$\left|b(\psi,v)\right| \leq M_2\|\psi\|_V\|v\|_W, \quad \forall\psi \in V,\ v \in W. \tag{2.2}$$

In scientific and engineering computations, many eigenvalue problems for differential equation have the following mixed variational formulation.

Find $(\lambda, u, \sigma) \in \mathbf{R} \times V \times W$, $(u,\sigma) \neq (0,0)$, such that

$$a(u,\psi) + b(\psi,\sigma) = \lambda(u,\psi)_D, \quad \forall\psi \in V, \tag{2.3}$$

$$b(u,v) = 0, \quad \forall v \in W. \tag{2.4}$$

In order to solve problem (2.3)-(2.4), one should construct finite element spaces $V_h \subset V$ and $W_h \subset W$. Restricting (2.3)-(2.4) on $V_h \times W_h$ we get the conforming mixed finite element approximation as follows. Find $(\lambda_h, u_h, \sigma_h) \in \mathbf{R} \times V_h \times W_h$, $(u_h,\sigma_h) \neq (0,0)$, such that

$$a(u_h,\psi) + b(\psi,\sigma_h) = \lambda_h(u_h,\psi)_D, \quad \forall\psi \in V_h, \tag{2.5}$$

$$b(u_h,v) = 0, \quad \forall v \in W_h. \tag{2.6}$$

Consider the associated source and approximate source problems.

Given $f \in D$, find $(w,p) \in V \times W$ satisfying

$$a(w,\psi) + b(\psi,p) = (f,\psi)_D, \quad \forall\psi \in V, \tag{2.7}$$

$$b(w,v) = 0, \quad \forall v \in W. \tag{2.8}$$

Given $f \in D$, find $(w_h,p_h) \in V_h \times W_h$ satisfying

$$a(w_h,\psi) + b(\psi,p_h) = (f,\psi)_D, \quad \forall\psi \in V_h, \tag{2.9}$$

$$b(w_h,v) = 0, \quad \forall v \in W_h. \tag{2.10}$$

As for the mixed finite element method for boundary value problems, Brezzi and Babuska, and others have established a systematic theory. Denote

$$\begin{aligned} V_0 &= \{u \in V : b(u,v) = 0, \forall v \in W\}, \\ V_{h0} &= \{u \in V_h : b(u,v) = 0, \forall v \in W_h\}. \end{aligned} \tag{2.11}$$

Theorem 2.1 (Brezzi-Babuska Theorem). *Suppose that*

(1) $a(\cdot,\cdot)$ *and* $b(\cdot,\cdot)$ *are continuous bilinear forms, that is,*

$$\begin{aligned} \left|a(q,\psi)\right| &\leq M_1\|q\|_V\|\psi\|_V, \quad \forall q,\psi \in V, \\ \left|b(\psi,v)\right| &\leq M_2\|\psi\|_V\|v\|_W, \quad \forall\psi \in V,\ v \in W; \end{aligned} \tag{2.12}$$

(2) *there exists* $\nu_1 > 0$, *such that*

$$a(q,q) \geq \nu_1 \|q\|_V^2, \quad \forall 0 \neq q \in V_0; \tag{2.13}$$

(3) *inf-sup condition: there exists* $\nu_2 > 0$, *such that*

$$\sup_{\psi\in V, \psi\neq 0} \frac{b(\psi,v)}{\|\psi\|_V} \geq \nu_2 \|v\|_W, \quad \forall v \in W; \tag{2.14}$$

then there exists a unique solution to the problem (2.7)-(2.8), *and*

$$\|w\|_V + \|p\|_W \leq C\|f\|_D, \tag{2.15}$$

where constant C *just depends on* ν_1, ν_2 *and* M_1, M_2. *Furthermore, suppose that*

(4) *there exists a constant* $\widetilde{\mu}_1 > 0$, *such that*

$$a(q,q) \geq \widetilde{\mu}_1 \|q\|_V^2, \quad \forall 0 \neq q \in V_{h0}; \tag{2.16}$$

(5) *discrete inf-sup condition: there exists a constant* $\widetilde{\mu}_2 > 0$, *such that*

$$\sup_{\psi\in V_h, \psi\neq 0} \frac{b(\psi,v)}{\|\psi\|_V} \geq \widetilde{\mu}_2 \|v\|_W, \quad \forall v \in W_h. \tag{2.17}$$

Then there exists a unique solution (w_h, p_h) *to the problem* (2.9)-(2.10); *moreover, the following error estimate is valid:*

$$\|w - w_h\|_V + \|p - p_h\|_W \leq C\left\{ \inf_{q\in V_h} \|w - q\|_V + \inf_{v\in W_h} \|p - v\|_W \right\}, \tag{2.18}$$

where C *just depends on* $\widetilde{\mu}_1, \widetilde{\mu}_2$ *and* M_1, M_2.

Since $a(\cdot,\cdot)$ is a symmetric, continuous, and V-elliptic bilinear form on $V \times V$, $b(\cdot,\cdot)$ is a continuous bilinear form, then conditions (1), (2), and (4) of Brezzi-Babuska Theorem hold. Suppose inf-sup condition and discrete inf-sup condition hold. Then by Brezzi-Babuska Theorem, we know that (2.7)-(2.8) and (2.9)-(2.10) are uniquely solvable for each $f \in D$. Thus we can define the corresponding linear bounded operators:
$T : D \to V,\ S : D \to W$: for all $f \in D$

$$a(Tf,\psi) + b(\psi, Sf) = (f,\psi)_D, \quad \forall \psi \in V, \tag{2.19}$$

$$b(Tf,v) = 0, \quad \forall v \in W, \tag{2.20}$$

$T_h : D \to V_h \subset V,\ S_h : D \to W_h \subset W$: for all $f \in D$,

$$a(T_h f, \varphi) + b(\varphi, S_h f) = (f, \varphi)_D, \quad \forall \varphi \in V_h, \tag{2.21}$$

$$b(T_h f, v) = 0, \quad \forall v \in W_h. \tag{2.22}$$

It is obvious that (2.3)-(2.4) have an equivalent operator form:

$$\lambda T u = u, \tag{2.23}$$

$$\sigma = S(\lambda u). \tag{2.24}$$

Equations (2.5)-(2.6) have an equivalent operator form:

$$\lambda_h T_h u_h = u_h, \tag{2.25}$$

$$\sigma_h = S_h(\lambda_h u_h). \tag{2.26}$$

It is easy to prove that $T : D \to D$, $T_h : D \to D$ are self-adjoint operators. In fact, for all $f, g \in D$, taking $\varphi = Tg,\ v = Sg$ in (2.7)-(2.8) we obtain

$$\begin{gathered} a(Tf, Tg) + b(Tg, Sf) = (f, Tg)_D, \\ b(Tf, Sg) = 0. \end{gathered} \tag{2.27}$$

Exchanging f and g, we get

$$a(Tg, Tf) + b(Tf, Sg) = (g, Tf)_D, \tag{2.28}$$

$$b(Tg, Sf) = 0, \tag{2.29}$$

then

$$\begin{aligned} (f, Tg)_D &= a(Tf, Tg) + b(Tg, Sf) + b(Tf, Sg) \\ &= a(Tg, Tf) + b(Tf, Sg) + b(Tg, Sf) = (g, Tf)_D + 0 = (Tf, g)_D. \end{aligned} \tag{2.30}$$

It shows that $T : D \to D$ is self-adjoint in the sense of inner product $(\cdot,\cdot)_D$. Analogously, it can be proved that $T_h : D \to D$ is self-adjoint in the sense of inner product $(\cdot,\cdot)_D$.

Assume that $V \hookrightarrow D$ (compact imbedded), then it is easy to prove that the operator $T : D \to D$ is completely continuous, $T : V \to V$ is completely continuous, and T_h is a finite rank operator. Combining (2.3)-(2.4), (2.5)-(2.6), and the V-ellipticity of $a(\cdot,\cdot)$, we deduce

$$\lambda = \frac{a(u,u)}{(u,u)_D} > 0, \qquad \lambda_h = \frac{a(u_h,u_h)}{(u_h,u_h)_D} > 0. \tag{2.31}$$

Then, from the spectral theory of self-adjoint and completely continuous operator we know that the eigenvalues of (2.3)-(2.4) can be sorted as

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \nearrow +\infty, \tag{2.32}$$

and the corresponding eigenfunctions are

$$u_1, \sigma_1, u_2, \sigma_2, \ldots, u_k, \sigma_k, \ldots, \tag{2.33}$$

where $(u_i, u_j)_D = \delta_{ij}$.

The eigenvalues of (2.5)-(2.6) can be sorted as

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{k,h} \leq \cdots \leq \lambda_{N_h,h}, \tag{2.34}$$

and the corresponding eigenfunctions are

$$u_{1,h}, \sigma_{1,h}, u_{2,h}, \sigma_{2,h}, \ldots, u_{k,h}, \sigma_{k,h}, \ldots, u_{N_h,h}, \sigma_{N_h,h}, \tag{2.35}$$

where $(u_{i,h}, u_{j,h})_D = \delta_{ij}$.

It is obvious that $a(\cdot,\cdot)$ is an inner product on V, $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$ and $\|\cdot\|_V$ are equivalent norms. Let $u_h = u_{i,h}$ in (2.5); then

$$a\left(u_{i,h}, u_{j,h}\right) + b\left(u_{j,h}, \sigma_h\right) = \lambda_{i,h}\left(u_{i,h}, u_{j,h}\right)_D. \tag{2.36}$$

From (2.6), we get $b(u_{j,h}, \sigma_h) = 0$. Then

$$a\left(u_{i,h}, u_{j,h}\right) = \lambda_{i,h}\left(u_{i,h}, u_{j,h}\right)_D = \lambda_{i,h}\delta_{ij}; \tag{2.37}$$

therefore, $\{u_{i,h}/\|u_{i,h}\|_a\}$ is a completely normal eigenvector system on V_h in the sense of inner product $a(\cdot,\cdot)$.

Denote $\lambda_k = 1/\mu_k$, $\lambda_{k,h} = 1/\mu_{k,h}$. In this paper, μ_k and $\mu_{k,h}$, λ_k, and $\lambda_{k,h}$ are all called eigenvalues. Let μ be the kth eigenvalue with algebraic multiplicity q, $\mu = \mu_k = \mu_{k+1} = \cdots = \mu_{k+q-1}$. $M(\mu)$ is the space spanned by all eigenfunctions $\{u_j\}_k^{k+q-1}$ corresponding to μ of T. $M_h(\mu)$ is the space spanned by all eigenfunctions $\{u_{j,h}\}_k^{k+q-1}$ corresponding to all eigenvalues of T_h that converge to μ. Let $\widehat{M}(\mu) = \{v : v \in M(\mu), \|v\|_a = 1\}$, $\widehat{M}_h(\mu) = \{v : v \in M_h(\mu), \|v\|_a = 1\}$. We call $\lambda = 1/\mu$ the kth eigenvalue, too. Denote $M(\lambda) = M(\mu)$, $M_h(\lambda) = M_h(\mu)$, and $\widehat{M}(\lambda) = \widehat{M}(\mu)$. Define

$$\begin{aligned} \left\|(T - T_h)|_{M(\lambda)}\right\|_D &= \max_{u \in M(\lambda), u \neq 0} \frac{\|(T - T_h)u\|_D}{\|u\|_D}, \\ \left\|(T - T_h)|_{M(\lambda)}\right\|_a &= \max_{u \in M(\lambda), u \neq 0} \frac{\|(T - T_h)u\|_a}{\|u\|_a}. \end{aligned} \tag{2.38}$$

The convergence and error estimate about mixed element method of eigenvalue problem have been studied by [23–25]. From these literatures we easily know that the following results are valid.

Lemma 2.2. *Suppose that the conditions of Brezzi-Babuska Theorem hold, and* $\|T-T_h\|_D \to 0\ (h \to 0)$*. Let* $(\lambda_h, u_h, \sigma_h)$ *be the kth eigenpair of* (2.5)-(2.6)*,* $\|u_h\|_a = 1$*, and* λ *the kth eigenvalue of* (2.3)-(2.4)*. Then* $\lambda_h \to \lambda\ (h \to 0)$*, and there exists an eigenfunction* (u, σ) *corresponding to* λ *such that*

$$|\lambda_h - \lambda| + \|u_h - u\|_D \leq C\left\|(T - T_h)|_{M(\lambda)}\right\|_D, \tag{2.39}$$

$$\|\sigma - \sigma_h\|_W \leq \|S_h(\lambda u) - S(\lambda u)\|_W + C\left\|(T - T_h)|_{M(\lambda)}\right\|_D, \tag{2.40}$$

$$\|u - u_h\|_a \leq C\left\|(T_h - T)|_{M(\lambda)}\right\|_a. \tag{2.41}$$

Let $u \in \widehat{M}(\lambda)$*; then there exists* $u_h \in M_h(\lambda)$ *such that*

$$\|u - u_h\|_a \leq C\left\|(T_h - T)|_{M(\lambda)}\right\|_a. \tag{2.42}$$

Proof. From the spectral approximation theory (see [23]) we have (2.39).

Let u satisfy (2.39), and $\sigma = S(\lambda u)$. Next we will prove that this eigenpair satisfies (2.40)-(2.41). From Brezzi-Babuska Theorem and (2.39), we get

$$\|S(\lambda u - \lambda_h u_h)\|_W \leq C\|\lambda u - \lambda_h u_h\|_D \leq C\left\|(T - T_h)|_{M(\lambda)}\right\|_D, \tag{2.43}$$

$$\|T(\lambda u - \lambda_h u_h)\|_V \leq C\|\lambda u - \lambda_h u_h\|_D \leq C\left\|(T - T_h)|_{M(\lambda)}\right\|_D. \tag{2.44}$$

Using the triangle inequality and (2.43), we deduce

$$\begin{aligned}
&\|\sigma - \sigma_h\|_W - \|S_h(\lambda u) - S(\lambda u)\|_W \\
&\quad = \left|\|S(\lambda u) - S_h(\lambda_h u_h)\|_W - \|S_h(\lambda u) - S(\lambda u)\|_W\right| \\
&\quad \leq \|S_h(\lambda u - \lambda_h u_h)\|_W \leq C\left\|(T - T_h)|_{M(\lambda)}\right\|_D,
\end{aligned} \tag{2.45}$$

that is, (2.40) is valid.

Using the triangle inequality and (2.44), we get

$$\begin{aligned}
&\|u - u_h\|_a - \|T_h(\lambda u) - T(\lambda u)\|_a \\
&\quad = \left|\|T(\lambda u) - T_h(\lambda_h u_h)\|_a - \|T_h(\lambda u) - T(\lambda u)\|_a\right| \\
&\quad \leq \|T_h(\lambda u - \lambda_h u_h)\|_a \\
&\quad \leq C\left\|(T - T_h)|_{M(\lambda)}\right\|_D \leq C\left\|(T - T_h)|_{M(\lambda)}\right\|_a,
\end{aligned} \tag{2.46}$$

which together with $\|T_h(\lambda u) - T(\lambda u)\|_a \leq C\|(T - T_h)|_{M(\lambda)}\|_a$ yields (2.41).

Let the eigenfunctions $\{u_{j,h}\}$ be an orthonormal system of $M_h(\lambda)$ in the sense of inner product $a(\cdot,\cdot)$. Then, from (2.41) and Lemma 3.1 we know that there exists a basis $\{u_j^0\}$ of $M(\lambda)$ satisfying $\|u_j^0\|_a = 1$ and the following result is valid:

$$\left\|u_j^0 - u_{j,h}\right\|_a \leq C\left\|(T_h - T)|_{M(\lambda)}\right\|_a. \tag{2.47}$$

For any $u \in \widehat{M}(\lambda)$, we write $u = \sum_{j=k}^{k+q-1} \alpha_j u_j^0$. By calculation, we get

$$1 = \|u\|_a^2 = \sum_{j=k}^{k+q-1} \alpha_j^2 + \sum_{i\neq j,i,j=k}^{k+q-1} \alpha_i\alpha_j a\left(u_i^0, u_j^0\right). \tag{2.48}$$

From (2.47), when $i \neq j$ we have $a(u_i^0, u_j^0) = a(u_i^0, u_j^0) - a(u_{i,h}, u_{j,h}) \rightarrow 0\,(h \rightarrow 0)$, and thus we get $\sum_{j=k}^{k+q-1} \alpha_j^2 \rightarrow 1\,(h \rightarrow 0)$.

Denote $u_h = \sum_{j=k}^{k+q-1} \alpha_j u_{j,h}$, then $u_h \in M_h(\lambda)$. From (2.47), we deduce

$$\|u - u_h\|_a \leq C\sum_{j=k}^{k+q-1}\left\|u_{j,h} - u_j^0\right\|_a \leq C\left\|(T_h - T)|_{M(\lambda)}\right\|_a, \tag{2.49}$$

that is, (2.42) is valid. The proof is completed. □

For $(u^*, \sigma^*) \in V \times W,\ u^* \neq 0$, define the Rayleigh quotient

$$\lambda^r = \frac{a(u^*, u^*) + 2b(u^*, \sigma^*)}{(u^*, u^*)_D}. \tag{2.50}$$

The following lemma is an extension of [23, Lemma 9.1].

Lemma 2.3. *Let (λ, u, σ) be an eigenpair of (2.3)-(2.4); then for all $(u^*, \sigma^*) \in V\times W$, $(u^*, \sigma^*) \neq (0,0)$ with its Rayleigh quotient satisfying*

$$\lambda^r - \lambda = \frac{a(u^* - u, u^* - u) + 2b(u^* - u, \sigma^* - \sigma)}{(u^*, u^*)_D} - \lambda\frac{(u^* - u, u^* - u)_D}{(u^*, u^*)_D}. \tag{2.51}$$

Proof. From (2.3)-(2.4), we deduce

$$\begin{aligned}
&a(u^*-u,u^*-u)+2b(u^*-u,\sigma^*-\sigma)-\lambda(u^*-u,u^*-u)_D \\
&\quad = a(u^*,u^*)-2a(u^*,u)+a(u,u)+2b(u^*,\sigma^*)-2b(u^*,\sigma)-2b(u,\sigma^*) \\
&\qquad +2b(u,\sigma)-\lambda(u^*,u^*)_D+2\lambda(u^*,u)_D-\lambda(u,u)_D. \\
&\quad = a(u^*,u^*)+2b(u^*,\sigma^*)-\lambda(u^*,u^*)_D-2(a(u^*,u)+b(u^*,\sigma)-\lambda(u^*,u)_D) \\
&\qquad -2b(u,\sigma^*)+(a(u,u)+b(u,\sigma)-\lambda(u,u)_D)+b(u,\sigma). \\
&\quad = a(u^*,u^*)+2b(u^*,\sigma^*)-\lambda(u^*,u^*)_D-0-0+0+0.
\end{aligned} \tag{2.52}$$

By dividing by $(u^*,u^*)_D$ on both sides of the above identity, we obtain (2.51). □

Taking $(u^*,\sigma^*)=(u_h,\sigma_h)$ in (2.51) and using (2.4) and (2.6), we derive

Lemma 2.4. *Let* (λ,u,σ) *and* (λ_h,u_h,σ_h) *be the* k*th eigenpair of* (2.3)-(2.4) *and* (2.5)-(2.6), *respectively; then*

$$\lambda_h-\lambda=\frac{a(u_h-u,u_h-u)+2b(u_h-u,v-\sigma)}{(u_h,u_h)_D}-\lambda\frac{(u_h-u,u_h-u)_D}{(u_h,u_h)_D},\quad \forall v\in W_h, \tag{2.53}$$

$$|\lambda_h-\lambda|\le C\Big(\|u_h-u\|_a^2+\|u_h-u\|_a\|\sigma-v\|_W\Big),\quad \forall v\in W_h. \tag{2.54}$$

3. The Two-Scale Discretization Scheme for Mixed Variational Formulation of Eigenvalue Problems

This paper establishes the following finite element two-scale discretization scheme based on the shifted-inverse power method.

Scheme 1. One has the following.

Step 1. Solve the eigenvalue problem (2.3)-(2.4) on a coarse grid K^H: find $(\lambda_H,u_H,\sigma_H)\in\mathbf{R}\times V_H\times W_H$, $\|u_H\|_a=1$ such that

$$\begin{aligned}
a(u_H,\psi)+b(\psi,\sigma_H)&=\lambda_H(u_H,\psi)_D,\quad \forall\psi\in V_H,\\
b(u_H,v)&=0,\quad \forall v\in W_H.
\end{aligned} \tag{3.1}$$

Step 2. Solve a equation on a fine grid K^h: find $(u',\sigma')\in V_h\times W_h$ such that

$$\begin{aligned}
a(u',\psi)+b(\psi,\sigma')-\lambda_H(u',\psi)_D&=(u_H,\psi)_D,\quad \forall\psi\in V_h,\\
b(u',v)&=0,\quad \forall v\in W_h.
\end{aligned} \tag{3.2}$$

Set $u^h=u'/\|u'\|_a$, $\sigma^h=\sigma'/\|u'\|_a$.

Step 3. Compute the Rayleigh quotient

$$\lambda^h = \frac{a(u^h, u^h)}{(u^h, u^h)_D}. \tag{3.3}$$

Next we will discuss the validity of Scheme 1.

Lemma 3.1. *For any nonzero elements $u, v \in V$,*

$$\left\| \frac{u}{\|u\|_a} - \frac{v}{\|v\|_a} \right\|_a \leq 2\frac{\|u-v\|_a}{\|u\|_a}, \qquad \left\| \frac{u}{\|u\|_a} - \frac{v}{\|v\|_a} \right\|_a \leq 2\frac{\|u-v\|_a}{\|v\|_a}. \tag{3.4}$$

Proof. See [21]. □

Denote $\operatorname{dist}(u, V) = \inf_{v\in V}\|u-v\|_a$.
Consider the eigenvalue problem (2.25) on the space V_h.

Lemma 3.2. *Suppose that μ and μ_h are the kth eigenvalue of T and T_h, respectively, and (μ_0, u_0) is an approximate eigenpair where μ_0 is not an eigenvalue of T_h, $u_0 \in V_h$, $\|u_0\|_a = 1$, $\operatorname{dist}(u_0, M_h(\mu)) \leq 1/2$, $\max_{k\leq j\leq k+q-1}|(\mu_{j,h}-\mu_h)/(\mu_0-\mu_{j,h})| \leq 1/2$, $|\mu_0 - \mu_{j,h}| \geq (\rho/2)(j \neq k, k+1, \ldots, k+q-1)$, and $u^s \in V_h$, $u^h \in V_h$ satisfy*

$$(\mu_0 - T_h)u^s = u_0, \qquad u^h = \frac{u^s}{\|u^s\|_a}. \tag{3.5}$$

Then

$$\operatorname{dist}\left(u^h, \widehat{M}_h(\mu)\right) \leq \frac{16}{\rho}\left|\mu_0 - \mu_h\right| \operatorname{dist}(u_0, M_h(\mu)), \tag{3.6}$$

where $\rho = \min_{\mu_j\neq\mu}|\mu_j - \mu|$ is the separation constant of the eigenvalue μ.

Proof. See [21]. □

Theorem 3.3. *Suppose that the conditions of Brezzi-Babuska Theorem hold and $\|T_h - T\|_D \to 0$ $(h \to 0)$. Let $(\lambda^h, u^h, \sigma^h)$ be the approximate eigenpair obtained by the two-scale discretization scheme and H small properly. Then there exists $u \in M(\lambda)$ such that*

$$\left\|u^h - u\right\|_a \leq C\left(|\lambda_H - \lambda|^2 + |\lambda_H - \lambda|\left\|(T-T_H)|_{M(\lambda)}\right\|_D + \left\|(T-T_h)|_{M(\lambda)}\right\|_a\right), \tag{3.7}$$

$$\left|\lambda^h - \lambda\right| \leq C\left(\left\|u^h - u\right\|_a^2 + \|u^h - u\|_a \inf_{v\in W_h}\|\sigma - v\|_W\right). \tag{3.8}$$

Proof. We use Lemma 3.2 in the proof.

Select $\mu_0 = 1/\lambda_H$ and $u_0 = \lambda_H T_h u_H / \|\lambda_H T_h u_H\|_a$. Let $u^0 \in M(\lambda)$ such that $u_H - u^0$ satisfies (2.39) and (2.41). By calculation we deduce

$$\begin{aligned}\left\|\lambda_H T_h u_H - u^0\right\|_a &= \left\|\lambda_H T_h u_H - \lambda T u^0\right\|_a \\ &\leq C\left(|\lambda_H - \lambda| + \left\|u_H - u^0\right\|_D + \left\|(T - T_h)u^0\right\|_a\right) \\ &\leq C\left(|\lambda_H - \lambda| + \left\|(T - T_H)|_{M(\lambda)}\right\|_D + \left\|(T - T_h)|_{M(\lambda)}\right\|_a\right);\end{aligned} \tag{3.9}$$

thus, using Lemma 3.1, we get

$$\begin{aligned}\operatorname{dist}\left(u_0, \widehat{M}(\lambda)\right) &\leq \left\|u_0 - \frac{u^0}{\|u^0\|_a}\right\|_a \leq C\left\|\lambda_H T_h u_H - u^0\right\|_a \\ &\leq C\left(|\lambda_H - \lambda| + \left\|(T - T_H)|_{M(\lambda)}\right\|_D + \left\|(T - T_h)|_{M(\lambda)}\right\|_a\right).\end{aligned} \tag{3.10}$$

Using the triangle inequality and (2.42), we derive

$$\operatorname{dist}(u_0, M_h(\lambda)) \leq \operatorname{dist}\left(u_0, \widehat{M}(\lambda)\right) + C\left\|(T - T_h)|_{M(\lambda)}\right\|_a. \tag{3.11}$$

From Lemma 2.2 we know $\lambda_H \to \lambda$, $\lambda_{j,h} \to \lambda$; then

$$\left|\mu_0 - \mu_{j,h}\right| = \left|\frac{\lambda_H - \lambda + \lambda - \lambda_{j,h}}{\lambda_{j,h}\lambda_H}\right|. \tag{3.12}$$

When H is small enough, noting that $h \ll H$, from (3.11) and (3.10) we get

$$\operatorname{dist}(u_0, M_h(\lambda)) \leq \frac{1}{2}. \tag{3.13}$$

Having in mind that $\lambda = \lambda_{k+1} = \cdots = \lambda_{k+q-1}$ we have

$$\left|\mu_{j,h} - \mu_h\right| = \left|\frac{\lambda_h - \lambda_{j,h}}{\lambda_h \lambda_{j,h}}\right| = \left|\frac{\lambda_h - \lambda + \lambda_j - \lambda_{j,h}}{\lambda_h \lambda_{j,h}}\right|, \tag{3.14}$$

which together with (3.12), noting that $\lambda_{j,h} - \lambda$ is an infinitesimal of higher order comparing with $\lambda_H - \lambda$, yields

$$\max_{k \leq j \leq k+q-1}\left|\frac{\mu_{j,h} - \mu_h}{\mu_0 - \mu_{j,h}}\right| \leq \frac{1}{2}. \tag{3.15}$$

Since ρ is the separation constant, H is small enough, and $h \ll H$, there holds

$$\left|\mu_0 - \mu_{j,h}\right| \geq \frac{\rho}{2}, \quad j \neq k, k+1, \ldots, k+q-1. \tag{3.16}$$

For u' in Step 2 of Scheme 1, from the definition of T_h and S_h we have

$$a(T_h\lambda_H u', \psi) + b(\psi, S_h\lambda_H u') = \lambda_H (u', \psi)_D, \quad \forall \psi \in V_h, \tag{3.17}$$

$$b(T_h u', v) = 0, \quad \forall v \in W_h. \tag{3.18}$$

$$a(T_h u_H, \psi) + b(\psi, S_h u_H) = (u_H, \psi)_D, \quad \forall \psi \in V_h, \tag{3.19}$$

$$b(T_h u_H, v) = 0, \quad \forall v \in W_h. \tag{3.20}$$

Hence, Step 2 of Scheme 1 is equivalent to $(u', \sigma') \in V_h \times W_h$,

$$a(u', \psi) + b(\psi, \sigma') - \lambda_H a(T_h u', \psi) - \lambda_H b(\psi, S_h u') = a(T_h u_H, \psi) + b(\psi, S_h u_H), \quad \forall \psi \in V_h, \tag{3.21}$$

$$b(u', v) = 0, \quad \forall v \in W_h, \tag{3.22}$$

$u^h = u'/\|u'\|_a$, $\sigma^h = \sigma'/\|u'\|_a$.

From (3.21) we obtain

$$a(u' - \lambda_H T_h u' - T_h u_H, \psi) + b(\psi, \sigma' - \lambda_H S_h u' - S_h u_H) = 0, \quad \forall \psi \in V_h. \tag{3.23}$$

Combining (3.22), (3.18), and (3.20), we get

$$b(u' - \lambda_H T_h u' - T_h u_H, v) = 0, \quad \forall v \in W_h. \tag{3.24}$$

By (3.24), taking $\psi = u' - \lambda_H T_h u' - T_h u_H$ in (3.23), we obtain

$$a(u' - \lambda_H T_h u' - T_h u_H, u' - \lambda_H T_h u' - T_h u_H) = 0. \tag{3.25}$$

Thus

$$\left(\frac{1}{\lambda_H} - T_h\right)u' = \frac{1}{\lambda_H} T_h u_H, \qquad u^h = \frac{u'}{\|u'\|_a}. \tag{3.26}$$

From (3.26) we know that the first term on the left-hand side of (3.23) is equal to 0; thus

$$b(\psi, \sigma' - \lambda_H S_h u' - S_h u_H) = 0, \quad \forall \psi \in V_h; \tag{3.27}$$

then, using discrete inf-sup condition, we obtain

$$\sigma' = \lambda_H S_h u' + S_h u_H. \tag{3.28}$$

Thus Step 2 of Scheme 1 is equivalent to (3.26), (3.28), and $u^h = u'/\|u'\|_a$, $\sigma^h = \sigma'/\|u'\|_a$. Noting that $\lambda_H^{-1}T_h u_H = \|\lambda_H^{-1}T_h u_H\|_a u_0$ differs from u_0 by only a constant and denoting $u^s = u'/\|\lambda_H^{-1}T_h u_H\|_a$, we have

$$\left(\frac{1}{\lambda_H} - T_h\right)u^s = u_0, \qquad u^h = \frac{u^s}{\|u^s\|_a}. \tag{3.29}$$

By (3.13), (3.15), (3.16), and (3.29), we see that the conditions of Lemma 3.2 hold. Thus, substituting (3.11) and (3.12) into (3.6), we obtain

$$\operatorname{dist}\left(u^h, \widehat{M}_h(\lambda)\right) \le C|\lambda_H - \lambda|\left(\operatorname{dist}\left(u_0, \widehat{M}(\lambda)\right) + \|Tu - T_h u\|_a\right). \tag{3.30}$$

Let the eigenfunctions $\{u_{j,h}\}_k^{k+q-1}$ be an orthonormal system of $M_h(\lambda)$ (in the sense of $a(\cdot,\cdot)$). Then

$$\operatorname{dist}\left(u^h, M_h(\lambda)\right) = \left\| u^h - \sum_{j=k}^{k+q-1} a\left(u^h, u_{j,h}\right)u_{j,h}\right\|_a. \tag{3.31}$$

Let

$$u^* = \sum_{j=k}^{k+q-1} a\left(u^h, u_{j,h}\right)u_{j,h}, \tag{3.32}$$

and noting that $\|u^h - u^*\|_a \le \operatorname{dist}(u^h, \widehat{M}_h(\lambda))$, from (3.30) we deduce

$$\left\|u^h - u^*\right\|_a \le C|\lambda_H - \lambda|\left(\operatorname{dist}\left(u_0, \widehat{M}(\lambda)\right) + \|Tu - T_h u\|_a\right). \tag{3.33}$$

By Lemma 2.2, there exists $\{u_j^0\}_k^{k+q-1} \subset M(\lambda)$ such that $u_{j,h} - u_j^0$ satisfies (2.41). Let

$$u = \sum_{j=k}^{k+q-1} a\left(u^h, u_{j,h}\right)u_j^0; \tag{3.34}$$

then $u \in M(\lambda)$. Using (2.41) we deduce

$$\begin{aligned}\|u^* - u\|_a &= \left\|\sum_{j=k}^{k+q-1} a\left(u^h, u_{j,h}\right)\left(u_{j,h} - u_j^0\right)\right\|_a \\ &\le C\left(\sum_{j=k}^{k+q-1}\left\|u_{j,h} - u_j^0\right\|_a^2\right)^{1/2} \le C\left\|(T_h - T)|_{M(\lambda)}\right\|_a.\end{aligned} \tag{3.35}$$

Combining (3.33) with the previous inequality, we have

$$\left\|u^h - u\right\|_a \le C\left(|\lambda_H - \lambda| \operatorname{dist}\left(u_0, \widehat{M}(\lambda)\right) + \|(T_h - T)|_{M(\lambda)}\|_a\right). \tag{3.36}$$

Substituting (3.10) into (3.36), we get (3.7).

We know that $b(u^h, \sigma^h) = 0$ from Step 2 of two-scale scheme; then

$$\lambda^h = \frac{a(u^h, u^h)}{(u^h, u^h)_D} = \frac{a(u^h, u^h) + 2b(u^h, \sigma^h)}{(u^h, u^h)_D}. \tag{3.37}$$

Select $\lambda^r = \lambda^h,\ u^* = u^h,\ \sigma^* = \sigma^h$. From Lemma 2.3, we get

$$\lambda^h - \lambda = \frac{a(u^h - u, u^h - u) + 2b(u^h - u, \sigma^h - \sigma)}{(u^h, u^h)_D} - \lambda\frac{(u^h - u, u^h - u)_D}{(u^h, u^h)_D}. \tag{3.38}$$

Noting that, for all $v \in W_h,\ b(u^h - u, v) = 0$, we have

$$\lambda^h - \lambda = \frac{a(u^h - u, u^h - u) + 2b(u^h - u, v - \sigma)}{(u^h, u^h)_D} - \lambda\frac{(u^h - u, u^h - u)_D}{(u^h, u^h)_D}, \quad \forall v \in W_h. \tag{3.39}$$

Since $V \hookrightarrow D$ (continuously imbedded), $\|u^h - u\|_D \le C\|u^h - u\|_a$. Then from (3.39) we obtain (3.8). □

4. Two-Scale Discretization Scheme for Stokes Eigenvalue Problem

Consider the Stokes eigenvalue problem:

$$-\Delta\vec{u} + \nabla\sigma = \lambda\vec{u}, \quad \text{in } \Omega, \tag{4.1}$$

$$\operatorname{div} \vec{u} = 0, \quad \text{in } \Omega, \tag{4.2}$$

$$\vec{u} = 0, \quad \text{on } \partial\Omega, \tag{4.3}$$

where Ω is a polygonal domain in $\mathbf{R}^2$, $\vec{u} = (u_1, u_2)$ denotes the fluid velocity, and σ denotes the pressure.

In this paper, we use the symbol $\vec{\ }$ to stand for vector function. For the function σ in $H^m(\Omega)$, let

$$\|\sigma\|_m = \left(\sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha \sigma|^2\right)^{1/2}, \quad \alpha = \{\alpha_1, \alpha_2\},\ |\alpha| = \alpha_1 + \alpha_2. \tag{4.4}$$

For vector function $\vec{u} = (u_1, u_2)$, define

$$\|\vec{u}\|_m = \left(\|u_1\|_m^2 + \|u_2\|_m^2\right)^{1/2}, \qquad |\vec{u}|_m = \left(|u_1|_m^2 + |u_2|_m^2\right)^{1/2}. \tag{4.5}$$

Using Green's formula, we derive the mixed variational form associated with (4.1)–(4.3).

Find $(\lambda, \vec{u}, \sigma) \in \mathbf{R} \times H_0^1(\Omega)^2 \times L_0^2(\Omega)$ with $|\vec{u}|_1 = 1$ such that

$$\int_\Omega \sum_{i=1}^{2} \nabla u_i \cdot \nabla \varphi_i - \int_\Omega \operatorname{div} \vec{\varphi}\sigma = \lambda \int_\Omega \vec{u} \cdot \vec{\varphi}, \quad \forall \vec{\varphi} \in H_0^1(\Omega)^2, \tag{4.6}$$

$$-\int_\Omega \operatorname{div} \vec{u} v = 0, \quad \forall v \in L_0^2(\Omega). \tag{4.7}$$

Let $V_h \subset H_0^1(\Omega)^2$, $W_h \subset L_0^2(\Omega)$ be two mixed finite element spaces. The mixed finite element form is as follows.

Seek $(\lambda_h, \vec{u}_h, \sigma_h) \in \mathbf{R} \times V_h \times W_h$ with $|\vec{u}_h|_1 = 1$ such that

$$\int_\Omega \sum_{i=1}^{2} \nabla u_{hi} \cdot \nabla \varphi_i - \int_\Omega \operatorname{div} \vec{\varphi}_h \sigma_h = \lambda \int_\Omega \vec{u}_h \cdot \vec{\varphi}, \quad \forall \vec{\varphi} \in V_h, \tag{4.8}$$

$$-\int_\Omega \operatorname{div} \vec{u}_h v = 0, \quad \forall v \in W_h. \tag{4.9}$$

Denote

$$\begin{gathered} V = H_0^1(\Omega) \times H_0^1(\Omega), \\ W = L_0^2(\Omega) = \left\{ v \in L^2(\Omega) : \int_\Omega v = 0 \right\}, \qquad D = L^2(\Omega) \times L^2(\Omega), \\ a(\vec{u}, \vec{\varphi}) = \int_\Omega \sum_{i=1}^{2} \nabla u_i \cdot \nabla \varphi_i, \\ b(\vec{u}, v) = -\int_\Omega \operatorname{div} \vec{u} v, \qquad (\vec{u}, \vec{\varphi})_D = \int_\Omega \vec{u} \cdot \vec{\varphi}. \end{gathered} \tag{4.10}$$

Let $\|\vec{u}\|_a = \sqrt{a(\vec{u}, \vec{u})}$. It is clear that $\|\vec{u}\|_a = |\vec{u}|_1$ is a norm. Then (4.6)-(4.7) and (4.8)-(4.9) can be written in the forms of (2.3)-(2.4) and (2.5)-(2.6), respectively (we need to add $\vec{}$ for the vector function, e.g., u, φ should be written in the forms of $\vec{u}$, $\vec{\varphi}$).

We apply Scheme 1 to the Stokes eigenvalue problem (4.6)-(4.7). Adding the symbol $\vec{}$ for the vector function we get two-scale discretization scheme of mixed finite element for solving the Stokes eigenvalue problem (4.6)-(4.7), which is still called Scheme 1.

Consider the associated source and approximate source problems.

Find $(\vec{w}, p) \in V \times W$ such that

$$a(\vec{w}, \vec{\varphi}) + b(\vec{\varphi}, p) = \left(\vec{f}, \vec{\varphi}\right)_{D'}, \quad \forall \vec{\varphi} \in V, \tag{4.11}$$

$$b(\vec{w}, v) = 0, \quad \forall v \in W. \tag{4.12}$$

Seek $(\vec{w}_h, p_h) \in V_h \times W_h$ such that

$$a(\vec{w}_h, \vec{\varphi}) + b(\vec{\varphi}, p_h) = \left(\vec{f}, \vec{\varphi}\right)_{D'}, \quad \forall \vec{\varphi} \in V_h, \tag{4.13}$$

$$b(\vec{w}_h, v) = 0, \quad \forall v \in W_h. \tag{4.14}$$

From [26] we know that (4.11)-(4.12) satisfy conditions (1)–(3) in Brezzi-Babuska Theorem; therefore, there exists a unique solution $(\vec{w}, p) \in V \times W$ to the problem (4.11)-(4.12) and the following estimate is valid:

$$\|\vec{w}\|_1 + \|p\|_0 \le C_p \left\|\vec{f}\right\|_0. \tag{4.15}$$

Condition (4) in Brezzi-Babuska Theorem holds since $V_h \subset V$ and $W_h \subset W$. Suppose that condition (5) in Brezzi-Babuska Theorem (discrete inf-sup condition) is valid, then there exists a unique solution $(\vec{w}_h, p_h) \in V_h \times W_h$ to the problem (4.13)-(4.14), and the following error estimate is valid (see [27–29]):

$$\|\vec{w} - \vec{w}_h\|_1 + \|p - p_h\|_0 \le C\left(\inf_{\vec{\varphi} \in V_h} \|\vec{w} - \vec{\varphi}\|_1 + \inf_{v \in W_h} \|p - v\|_0\right). \tag{4.16}$$

We assume that the following a prior estimate holds: for any $\vec{f} \in D$, $\vec{w} \in H^{1+r}(\Omega) \times H^{1+r}(\Omega)$, $p \in H^r(\Omega)$, and

$$\|\vec{w}\|_{1+r} + \|p\|_r \le C\left\|\vec{f}\right\|_0, \tag{4.17}$$

where $0 < r \le 1$ is a number determined by the maximal inner angle ω of Ω. When $\omega < \pi$, $r = 1$ (see [30]).

Suppose that the following estimate holds: for any $\vec{w} \in H^{1+r}(\Omega) \times H^{1+r}(\Omega)$, and for any $p \in H^r(\Omega)$,

$$\inf_{\vec{\varphi} \in V_h} \|\vec{w} - \vec{\varphi}\|_V + \inf_{v \in W_h} \|p - v\|_W \le Ch^r \left(\|\vec{w}\|_{1+r} + \|p\|_r\right). \tag{4.18}$$

From Section 2, we know that (4.6)-(4.7) and (4.8)-(4.9) have the following equivalent operator forms, respectively,

$$\lambda T \vec{u} = \vec{u}, \qquad \sigma = S(\lambda \vec{u}), \tag{4.19}$$

$$\lambda_h T_h \vec{u}_h = \vec{u}_h, \qquad \sigma_h = S_h(\lambda_h \vec{u}_h). \tag{4.20}$$

Moreover, T and T_h are all self-adjoint compact operators.

Theorem 4.1. *Assume that discrete inf-sup condition,* (4.17) *and* (4.18) *hold; let H be properly small; and $(\lambda^h, \vec{u}^h, \sigma^h)$ an approximate eigenpair obtained by Scheme 1. Then there exists $\vec{u} \in M(\lambda)$ such that*

$$\left\|\vec{u}^h - \vec{u}\right\|_a \le C\left(H^{4r} + h^r\right), \tag{4.21}$$

$$\left|\lambda^h - \lambda\right| \le C\left(H^{8r} + h^{2r}\right). \tag{4.22}$$

Proof. From Brezzi-Babuska Theorem, (4.17), and (4.18), we deduce

$$\left\|T\vec{f} - T_h\vec{f}\right\|_V + \left\|S\vec{f} - S_h\vec{f}\right\|_W \le Ch^r\left\|\vec{f}\right\|_{D'}, \quad \forall \vec{f} \in D. \tag{4.23}$$

By virtue of Nitsche technique (see [29]) and (4.17), we derive

$$\begin{aligned}\left\|T\vec{f} - T_h\vec{f}\right\|_D &\le Ch^r\left(\inf_{q\in V_h}\left\|T\vec{f} - q\right\|_V + \inf_{v\in W_h}\left\|S\vec{f} - v\right\|_W\right)\\ &\le Ch^{2r}\left\|\vec{f}\right\|_{D'}, \quad \forall f \in D.\end{aligned} \tag{4.24}$$

Using (4.24), we have

$$\|T - T_h\|_D = \sup_{\vec{f}\in D, \|\vec{f}\|_D = 1}\left\|T\vec{f} - T_h\vec{f}\right\|_D \le Ch^{2r} \longrightarrow 0 \quad (h \longrightarrow 0). \tag{4.25}$$

Hence, the conditions in Lemma 2.2 and Theorem 3.3 hold.

From (2.23), we know that

$$|\lambda_h - \lambda| + \|\vec{u}_h - \vec{u}\|_D \le C\left\|(T - T_h)|_{M(\lambda)}\right\|_D \le Ch^{2r}. \tag{4.26}$$

From (4.23), (4.18), and (4.17), we have

$$\left\|(T - T_h)|_{M(\lambda)}\right\|_a = \sup_{\vec{u}\in M(\lambda), \vec{u}\ne 0}\frac{\|T\vec{u} - T_h\vec{u}\|_a}{\|\vec{u}\|_a} \le Ch^r, \tag{4.27}$$

$$\inf_{v\in W_h}\|\sigma - v\|_W \le Ch^r\|\sigma\|_r \le Ch^r\|\vec{u}\|_D. \tag{4.28}$$

Substituting (4.26) and (4.27) into (3.7), we obtain (4.21). Substituting (4.21) and (4.28) into (3.8), we obtain (4.22). □

4.1. Mini Mixed Finite Element

Consider two-scale discretization scheme of Mini mixed finite element for the Stokes eigenvalue problem (4.1)–(4.3) (Scheme 1).

Mini element was established by Arnold et al. in 1984 (see [31]). Let K^h be a regular triangulation of Ω under the meaning of paper [32], and

$$S^h = \left\{ v \in C\left(\overline{\Omega}\right) : v|_\kappa \in P_1,\ \kappa \in K^h \right\}, \qquad S_0^h = S^h \bigcap H_0^1(\Omega). \tag{4.29}$$

For any $\kappa \in K^h$, let N_1, N_2, and N_3 denote barycentric coordinates. Denote $B_h = \{v : v|_\kappa \in \operatorname{span}\{N_1 N_2 N_3\}, \kappa \in K^h\}$, and set

$$V_h = \left(S_0^h \bigoplus B_h\right)^2, \qquad W_h = S^h \bigcap L_0^2(\Omega). \tag{4.30}$$

From [31], we know that Mini element satisfies discrete inf-sup condition. From the interpolation theory in Sobolev space, we conclude that (4.18) is valid. Hence, for r satisfying (4.17), Scheme 1 for Mini mixed finite element is effective. Theorem 4.1 is valid.

4.2. P_1-P_1 *Mixed Finite Element*

Consider two-scale discretization scheme of P_1-P_1 mixed finite element for the Stokes eigenvalue problem (4.1)–(4.3) (Scheme 1).

Let K^{2h} be a regular triangulation of Ω, and K^h is the product of refining K^{2h} in the middle point. The P_1-P_1 mixed finite element space is defined by

$$V_h = \left[S^h \bigcap H_0^1(\Omega)\right]^2, \qquad W_h = S^{2h} \bigcap L_0^2(\Omega), \tag{4.31}$$

where S^{2h} and S^h are piecewise continuous linear polynomial spaces defined on K^{2h} and K^h, respectively.

From [33, Proposition 3.3], we know that P_1-P_1 element satisfies discrete inf-sup condition. By the interpolation theory in Sobolev space, we conclude that (4.18) holds. Therefore, for r satisfying (4.17), Scheme 1 for P_1-P_1 mixed finite element is effective. Theorem 4.1 is valid.

5. Two-Scale Discretization Scheme for Eigenvalue Problem of Electric Field

Consider the eigenvalue problem of electric field:

$$c^2 \operatorname{curl} \operatorname{curl} \vec{u} = \omega^2 \vec{u}, \quad \text{in } \Omega, \tag{5.1}$$

$$\operatorname{div} \vec{u} = 0, \quad \text{in } \Omega, \tag{5.2}$$

$$\vec{u} \times \vec{\gamma} = 0, \quad \text{on } \in \partial\Omega, \tag{5.3}$$

where Ω is a polyhedron in $\mathbf{R}^3$ and $\vec{\gamma}$ is the outward normal unit vector on $\partial\Omega$.

In physics, $\vec{u}$ in the above eigenvalue problem of electric field denotes electric field, ω denotes the time frequency, and c is the speed of the light. Usually we set $\lambda = \omega^2/c^2$ which is called eigenvalue.

The spaces $H(\mathrm{curl}, \Omega)$, $H_0(\mathrm{curl}, \Omega)$ are defined in the usual way:

$$\begin{aligned} H(\mathrm{curl}, \Omega) &= \left\{ \vec{q} \in L_2(\Omega)^3 : \mathrm{curl}\ \vec{q} \in L_2(\Omega)^3 \right\}, \\ H_0(\mathrm{curl}, \Omega) &= \left\{ \vec{q} \in H(\mathrm{curl}, \Omega) : \vec{q} \times \vec{\gamma}|_{\partial\Omega} = 0 \right\}. \end{aligned} \tag{5.4}$$

When Ω is convex polyhedron, we define the following function space:

$$\chi = \left\{ \vec{q} \in H_0(\mathrm{curl}, \Omega) : \mathrm{div}\, \vec{q} \in L^2(\Omega) \right\}. \tag{5.5}$$

Denote

$$\begin{aligned} (\vec{q}, \vec{\varphi})_0 &= \int_\Omega \vec{q} \cdot \vec{\varphi}\, dx, \qquad \|\vec{q}\|_0 = (\vec{q}, \vec{q})_0^{1/2}. \\ (\vec{q}, \vec{\varphi})_\chi &= (\mathrm{curl}\ \vec{q}, \mathrm{curl}\ \vec{\varphi})_0 + (\mathrm{div}\ \vec{q}, \mathrm{div}\ \vec{\varphi})_0, \qquad \|\vec{q}\|_\chi = (\vec{q}, \vec{q})_\chi^{1/2}. \end{aligned} \tag{5.6}$$

From [34, 35], we see that $\chi \subset H^1(\Omega)^3$; $(\vec{q}, \vec{\varphi})_\chi$ is a coercive bilinear form in χ, and $\|\vec{q}\|_\chi$ is a norm.

On the other hand, when Ω is nonconvex the maximal interior angle belongs to $(\pi, 2\pi)$. In this situation the problem is relatively complicated. Let E denote a set of reentrant edge with dihedral angles belonging to $(\pi, 2\pi)$, and let d denote the distance to the set E: $d(x) = \mathrm{dist}(x, \cup_{e \in E} \overline{e})$. We introduce a weight function ω_r which is a nonnegative smooth function with respect to x. It can be represented by d^r in reentrant edge and angular domain. We write $\omega_r \simeq d^r$. Define the weight function space:

$$\begin{aligned} L_r^2(\Omega) &= \left\{ v \in L_{\mathrm{loc}}^2(\Omega) : \omega_r v \in L_2(\Omega) \right\}, \\ \chi_r &= \left\{ \vec{q} \in H_0(\mathrm{curl}, \Omega) : \mathrm{div}\ \vec{q} \in L_r^2(\Omega) \right\}. \end{aligned} \tag{5.7}$$

Denote

$$\begin{aligned} (\vec{q}, \vec{\varphi})_{L_r^2} &= \int_\Omega \omega_r^2 \vec{q} \cdot \vec{\varphi}\, dx, \qquad \|\vec{q}\|_{L_r^2} = (\vec{q}, \vec{q})_{L_r^2}^{1/2}, \\ (\vec{q}, \vec{\varphi})_{\chi_r} &= (\mathrm{curl}\ \vec{q}, \mathrm{curl}\ \vec{\varphi})_0 + (\mathrm{div}\ \vec{q}, \mathrm{div}\ \vec{\varphi})_{L_r^2}, \qquad \|\vec{q}\|_{\chi_r} = (\vec{q}, \vec{q})_{\chi_r}^{1/2}. \end{aligned} \tag{5.8}$$

Let σ_{Δ}^{N} be the following smallest singular exponent in the Laplace problem with homogenous Dirichlet boundary condition:

$$\begin{gathered}\left\{\phi \in H_0^1(\Omega) : \Delta\phi \in L_2(\Omega)\right\} \subset \bigcap_{s<\sigma_{\Delta}^{D}} H^s(\Omega), \\ \left\{\phi \in H_0^1(\Omega) : \Delta\phi \in L_2(\Omega)\right\} \not\subset H^{\sigma_{\Delta}^{D}}(\Omega).\end{gathered} \tag{5.9}$$

From the regularity estimate we know $\sigma_{\Delta}^{D} \in (3/2, 2)$. Let $r_{\min} = 2 - \sigma_{\Delta}^{D}$.

From [36, 37], we know that, for all $r \in (r_{\min}, 1)$, the seminorm $|\vec{q}|_{\chi_r}$ is a norm in χ_r, and $\chi_r \cap H^1(\Omega)^3$ is dense in χ_r.

In the following discussion, we will use χ_r, $L_r^2(\Omega)$ both for non-convex and convex domain. We take $r \in (r_{\min}, 1)$ for non-convex domain; otherwise, we take $\chi_r = \chi$, $L_r^2(\Omega) = L^2(\Omega)$.

By introducing Lagrange multiplier σ, [36, 38, 39] changed (5.1)–(5.3) into the mixed variational formulation: find $(\lambda, \vec{u}, \sigma) \in \mathbf{R}^+ \times \chi_r \times L_r^2(\Omega)$ such that

$$\left(\vec{u}, \vec{\varphi}\right)_{\chi_r} + \left(\operatorname{div} \vec{\varphi}, \sigma\right)_{L_r^2} = \lambda\left(\vec{u}, \vec{\varphi}\right)_0, \quad \forall \vec{\varphi} \in \chi_r, \tag{5.10}$$

$$(\operatorname{div} \vec{u}, v)_{L_r^2} = 0, \quad \forall v \in L_r^2(\Omega). \tag{5.11}$$

Let K^h be a regular simplex partition (tetrahedral partition) of Ω with the mesh diameter h. Define the P_{k+1}-P_k finite element space as follows:

$$\begin{gathered}V_h = \left\{\vec{q} \in C^0\left(\overline{\Omega}\right)^3 : \vec{q} \times \vec{\gamma}|_{\partial\Omega} = 0,\ \vec{q}|_{\kappa} \in P_{k+1}(\kappa)^3, \forall\kappa \in K^h\right\}, \\ W_h = \left\{v \in C^0\left(\overline{\Omega}\right) : v|_{\kappa} \in P_k(\kappa),\ \forall\kappa \in K^h,\ v|_{E_h} = 0\right\}.\end{gathered} \tag{5.12}$$

Here we set $E_h = \cup_{\kappa\in K^h, \partial\kappa\cap E \neq \phi}\kappa$. $v|_{E_h} = 0$ means that v is zero on the tetrahedron where reentrant edge and angular point are adjacent.

Restricting (5.10)-(5.11) to the previous finite element space, we get discrete mixed variational form: find $(\lambda_h, \vec{u}_h, \sigma_h) \in \mathbf{R}^+ \times \chi_h \times M_h$ such that

$$\left(\vec{u}_h, \vec{\varphi}\right)_{\chi_r} + \left(\operatorname{div} \vec{\varphi}, \sigma_h\right)_{L_r^2} = \lambda_h\left(\vec{u}_h, \vec{\varphi}\right)_0, \quad \forall \vec{\varphi} \in \chi_h, \tag{5.13}$$

$$(\operatorname{div} \vec{u}_h, v)_{L_r^2} = 0, \quad \forall v \in M_h. \tag{5.14}$$

Set

$$\begin{gathered}V = \chi_r, \qquad \|\cdot\|_V = \|\cdot\|_{\chi_r}, \\ W = L_r^2(\Omega), \qquad \|\cdot\|_W = \|\cdot\|_{L_r^2}, \\ D = L_2(\Omega)^3, \qquad \|\cdot\|_D = \|\cdot\|_0, \\ a(\vec{q}, \vec{\varphi}) = \left(\vec{q}, \vec{\varphi}\right)_{\chi_r}, \qquad b(\vec{\varphi}, v) = \left(\operatorname{div} \vec{\varphi}, v\right)_{L_r^2}.\end{gathered} \tag{5.15}$$

Then (5.10)-(5.11) and (5.13)-(5.14) can be written in the forms of (2.3)-(2.4) and (2.5)-(2.6), respectively (we need to add $\vec{}$ for the vector function, e.g., u, φ should be written in the forms of $\vec{u}$, $\vec{\varphi}$).

We apply Scheme 1 to the eigenvalue problem of electric field (5.10)-(5.11). Adding the symbol $\vec{}$ for the vector function we get two-scale discretization scheme of mixed finite element for solving the eigenvalue problem of electric field (5.10)-(5.11) which is still called Scheme 1.

It is easy to know that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous bilinear forms on $V \times V$ and $V \times W$, respectively. V is compact embedded in D (when Ω is convex, it is valid obviously; when Ω is non-convex, see [36]).

Consider the source problem corresponding to (5.10)-(5.11).

Find $(\vec{w}, p) \in \chi_r \times L_r^2(\Omega)$ such that

$$(\vec{w},\vec{\varphi})_{\chi_r} + (\operatorname{div}\,\vec{\varphi}, p)_{L_r^2} = \left(\vec{f},\vec{\varphi}\right)_0, \quad \forall\vec{\varphi} \in \chi_r, \tag{5.16}$$

$$(\operatorname{div}\,\vec{w}, v)_{L_r^2} = 0, \quad \forall v \in L_r^2(\Omega). \tag{5.17}$$

For the problem (5.16)-(5.17) and its P_{k+1}-P_k element approximation, people have already proved the conditions in Brezzi-Babuska Theorem hold (see [38, 40, 41]).

Therefore, we can define operators T, S, T_h, S_h; moreover, (5.10)-(5.11) and (5.13)-(5.14) can be written in the forms of (2.23)-(2.24) and (2.25)-(2.26), respectively.

Lemma 5.1 is cited from the literature [36, 38].

Lemma 5.1. (5.1)–(5.3) *is equivalent to* (5.10)-(5.11), *and the solutions of* (5.10)-(5.11), $(\vec{u},\sigma)$, *satisfy* $\sigma = S(\lambda\vec{u}) = 0$ *and* $\vec{u} \in \chi_r$ *with* $\operatorname{div}\vec{u} = 0$.

Denote

$$\varepsilon_\lambda(h) = \sup_{\vec{u}\in\widehat{M}(\lambda)} \inf_{\vec{\varphi}\in V_h} \left\|T\vec{u} - \vec{\varphi}\right\|_a. \tag{5.18}$$

For the P_{k+1}-P_k element approximation of (5.16)-(5.17), [38] proved that the condition of [24, Theorem 1] (i.e., [38, Theorem 4.3]) is valid, hence, there holds the following.

Lemma 5.2. *For the* P_{k+1}-P_k *element approximation of* (5.16)-(5.17), *there exists* $r(h) > 0$, $r(h) \to 0\,(h \to 0)$ *such that*

$$\sup_{\vec{f}\in D, \vec{f}\neq 0} \frac{\left\|(T - T_h)\vec{f}\right\|_V}{\left\|\vec{f}\right\|_D} \leq r(h). \tag{5.19}$$

This lemma is very important. It tells us that $\|T - T_h\|_D \to 0\,(h \to 0)$. Based on this lemma, [38] also proved the following conclusion.

Lemma 5.3. *For the P_{k+1}-P_k element approximation of* (5.10)-(5.11), *the following estimate is valid:*

$$|\lambda - \lambda_h| \le C\varepsilon_\lambda(h)^2, \tag{5.20}$$

$$\|\vec{u}_h - \vec{u}\|_a \le C\varepsilon_\lambda(h). \tag{5.21}$$

Theorem 5.4. *Let $(\lambda^h, \vec{u}^h)$ be the approximate P_{k+1}-P_k element eigenpair obtained by Scheme 1; then there exists $\vec{u} \in M(\lambda)$ such that*

$$\left\|\vec{u}^h - \vec{u}\right\|_a \le C\left(\varepsilon_\lambda(H)^3 + \varepsilon_\lambda(h)\right), \tag{5.22}$$

$$\left|\lambda^h - \lambda\right| \le C\left(\varepsilon_\lambda(H)^6 + \varepsilon_\lambda(h)^2\right). \tag{5.23}$$

Proof. We use Theorem 3.3 to complete the proof. From (5.19) we see that the condition in Theorem 3.3 holds, and by Lemma 5.1 we know $\sigma = 0$.

From (2.18), we deduce

$$\begin{aligned} \left\|(T - T_h)|_{M(\lambda)}\right\|_a &= \sup_{\vec{u}\in\widehat{M}(\lambda)} \|T\vec{u} - T_h\vec{u}\|_a \\ &\le C \sup_{\vec{u}\in\widehat{M}(\lambda)} \inf_{\vec{\varphi}\in V_h} \left\|T\vec{u} - \vec{\varphi}\right\|_a \\ &\le C\varepsilon_\lambda(h). \end{aligned} \tag{5.24}$$

From (5.20), we derive

$$|\lambda_H - \lambda| \le C\varepsilon_\lambda(H)^2. \tag{5.25}$$

Substituting (5.24) and (5.25) into (3.7), we get (5.22).

Since $\sigma = 0$, (3.8) can be simplified to

$$\left|\lambda^h - \lambda\right| \le C\left\|\vec{u}^h - \vec{u}\right\|_a^2. \tag{5.26}$$

Substituting (5.22) into (5.26), we obtain (5.23). □

Let σ_Δ^N be the smallest singular exponent in the Laplace problem with homogenous Neumann boundary condition, then $\sigma_\Delta^N \in (3/2, 2)$. Denote $\tau = \min(r - r_{\min}, \sigma_\Delta^N - 1)$.

Corollary 5.5. *Under the condition of Theorem 5.4, if Ω is a convex polyhedron there holds*

$$\left\|\vec{u}^h - \vec{u}\right\|_a \le C\left(H^3 + h\right), \tag{5.27}$$

$$\left|\lambda^h - \lambda\right| \le C\left(H^6 + h^2\right); \tag{5.28}$$

if Ω is a non-convex polyhedron, then

$$\left\|\vec{u}^h - \vec{u}\right\|_a \leq C\left(H^{3\mu} + h^{\mu}\right), \quad \forall \mu \in (0, \tau), \tag{5.29}$$

$$\left|\lambda^h - \lambda\right| \leq C\left(H^{6\mu} + h^{2\mu}\right), \quad \forall \mu \in (0, \tau). \tag{5.30}$$

Proof. When Ω is a convex polyhedron, for any $\vec{u} \in M(\lambda)$ we have $\vec{u} = T(\lambda\vec{u}) \in H^2(\Omega)$ (see [35], or [42, equation (44)]); therefore

$$\varepsilon_\lambda(H) \leq CH, \qquad \varepsilon_\lambda(h) \leq Ch. \tag{5.31}$$

Substituting these two formulae into (5.22) and (5.23), we obtain (5.27) and (5.28), respectively. When Ω is non-convex, for all $\vec{u} \in M(\lambda)$, from [38, equation (36)] we know that

$$\varepsilon_\lambda(H) \leq CH^{\mu}, \qquad \varepsilon_\lambda(h) \leq Ch^{\mu}. \tag{5.32}$$

Substituting the above two formulae into (5.22) and (5.23), we obtain (5.29) and (5.30), respectively. □

6. Numerical Experiments

In the following two examples, let $\lambda_{1,H}, \lambda_{2,H}, \ldots, \lambda_{4,H}$ be the first four eigenvalues computed by using mixed finite method directly on coarse mesh K^H and $\lambda_{1,h}, \lambda_{2,h}, \ldots, \lambda_{4,h}$ the first four eigenvalues computed by using mixed finite method directly on fine mesh K^h. Let $\lambda_1^h, \lambda_2^h, \ldots, \lambda_4^h$ denote the first four eigenvalues computed by Scheme 1 on the meshes K^H and K^h.

Example 6.1. Consider the Stokes eigenvalue problem (4.1)–(4.3), where $\Omega \subset \mathbf{R}^2$ is a unit square domain. The smallest eigenvalue λ_1 is approximately equal to 52.3446911 for this problem.

We adopt a uniform isosceles right triangulation for the domain Ω (the edge in each element is along three fixed directions), and we give an initial mesh in Figure 1 and refine the initial mesh in a uniform way (each triangle is divided into four congruent triangles) repeatedly to get meshes K^H and K^h.

We solve this problem by Scheme 1 with Mini element. The numerical results are shown in Table 1.

From Table 1, we conclude that Scheme 1 can efficiently solve Stokes eigenvalue problem.

Example 6.2. Consider the eigenvalue problem of electric field (5.1)–(5.3), where Ω is a square domain $[0, \pi] \times [0, \pi]$ or an L-shaped domain $[-1, 0] \times [-1, 0] \cup [-1, 1] \times [0, 1]$. For the square domain, the first four exact eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = 4$ for this problem. For the L-shaped domain, the first four approximate eigenvalues are $\lambda_1 \approx 1.475622$, $\lambda_2 \approx 3.534031$, $\lambda_3 \approx 9.869604$, $\lambda_4 \approx 9.869604$.

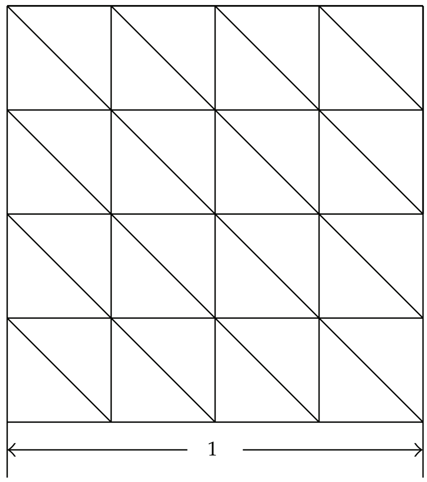

Figure 1

Table 1: The results on the square by Scheme 1 (Mini element) for Stokes eigenvalue problem.

k	H	h	$\lambda_{k,H}$	$\lambda_{k,h}$	λ_k^h
1	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{64}$	57.50602342019	52.42255785241	52.42147553267
1	$\frac{\sqrt{2}}{10}$	$\frac{\sqrt{2}}{100}$	55.61324143591	52.37622690147	52.37604993774
1	$\frac{\sqrt{2}}{12}$	$\frac{\sqrt{2}}{144}$	54.59828590012	52.35983048690	52.35978975974
2	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{64}$	103.5158846668	92.30714521284	92.29553836981
2	$\frac{\sqrt{2}}{10}$	$\frac{\sqrt{2}}{100}$	99.36963738012	92.19619098643	92.19429134425
2	$\frac{\sqrt{2}}{12}$	$\frac{\sqrt{2}}{144}$	97.12904494102	92.15848299868	92.15804753694
3	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{64}$	110.7048253409	92.43050152401	92.39531304531
3	$\frac{\sqrt{2}}{10}$	$\frac{\sqrt{2}}{100}$	103.8058661064	92.24032470162	92.23510195554
3	$\frac{\sqrt{2}}{12}$	$\frac{\sqrt{2}}{144}$	100.1402638604	92.17887213187	92.17771885852
4	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{64}$	159.7476076282	128.8864896622	128.7085892117
4	$\frac{\sqrt{2}}{10}$	$\frac{\sqrt{2}}{100}$	148.7986378758	128.4767873556	128.4133574197
4	$\frac{\sqrt{2}}{12}$	$\frac{\sqrt{2}}{144}$	142.5976379771	128.3089668708	128.3076834410

Table 2: The results on the square by Scheme 1 (P_2-P_1 element) for eigenvalue problem of electric field ($r = 0$).

k	H	h	$\lambda_{k,H}$	$\lambda_{k,h}$	λ_k^h
1	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	1.000489112	1.000000005	1.000000005
1	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	1.000099609	1.000000005	1.000000005
1	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	1.000031944	1.000000005	1.000000005
2	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	1.000490210	1.000000005	1.000000005
2	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	1.000099685	1.000000005	1.000000005
2	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	1.000031956	1.000000005	1.000000005
3	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	2.006497695	2.000000070	2.000000070
3	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	2.001365268	2.000000070	2.000000070
3	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	2.000442889	2.000000070	2.000000070
4	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	4.028293524	4.000000321	4.000000333
4	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	4.006078511	4.000000321	4.000000321
4	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	4.001988912	4.000000321	4.000000321

We adopt a uniform isosceles right triangulation for Ω (the edge in each element is along three fixed directions, see Figure 2 for the L-shaped domain, and see Figure 1 for the square domain $[0,\pi] \times [0,\pi]$) to produce the meshes K^H and K^h.

We use P_2-P_1 mixed finite element to solve this problem. The definition of P_2-P_1 mixed finite element space is given by

$$\begin{aligned} V_h &= \left\{ \vec{q} \in C^0\left(\overline{\Omega}\right)^2 : \vec{q} \times \vec{\gamma}|_{\partial\Omega} = 0,\ \vec{q}|_\kappa \in P_2(\kappa)^2,\ \forall \kappa \in K^h \right\}, \\ W_h &= \left\{ v \in C^0\left(\overline{\Omega}\right) : v|_\kappa \in P_1(\kappa),\ \forall \kappa \in K^h,\ v|_{E_h} = 0 \right\}. \end{aligned} \tag{6.1}$$

We compute the first four approximate eigenvalues by using Scheme 1 with P_2-P_1 element on the mesh K^H and K^h. The numerical results are listed in Tables 2, 3, and 4.

From Tables 2–4, we conclude that Scheme 1 can efficiently solve the eigenvalue problem of electric field.

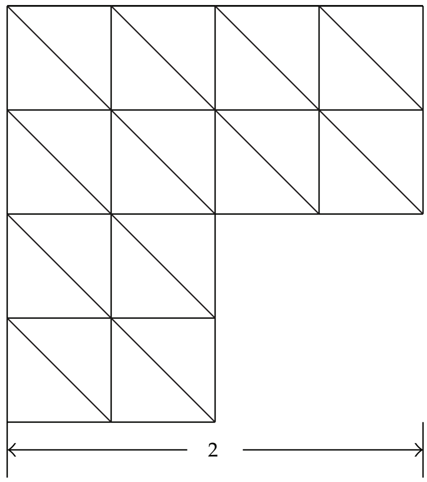

Figure 2

Table 3: The results on the L-shaped domain by Scheme 1 (P_2-P_1 element) for eigenvalue problem of electric field ($r = 0.5$).

k	H	h	$\lambda_{k,H}$	$\lambda_{k,h}$	λ_k^h
1	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	3.004954998	2.557217952	2.557241250
1	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	2.896205625	2.557217952	2.557225625
1	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	2.842087274	2.557217952	2.557221723
2	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	3.686002173	3.536967001	3.536967768
2	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	3.623341929	3.536967001	3.536967079
2	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	3.595097794	3.536967001	3.536967017
3	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	9.938579092	9.869605192	9.869605219
3	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	9.884458574	9.869605192	9.869605192
3	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	9.874470046	9.869605192	9.869605192
4	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	9.938761731	9.869605192	9.869605221
4	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	9.884463697	9.869605192	9.869605192
4	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	9.874472304	9.869605192	9.869605192

Table 4: The results on the L-shaped domain by Scheme 1 (P_2-P_1 element) for eigenvalue problem of electric field ($r = 0.95$).

k	H	h	$\lambda_{k,H}$	$\lambda_{k,h}$	λ_k^h
1	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	1.913411786	1.517146260	1.517159275
1	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	1.779172346	1.517146260	1.517148955
1	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	1.723917701	1.517146260	1.517147418
2	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	3.589241343	3.534187594	3.534187611
2	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	3.558323120	3.534187594	3.534187595
2	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	3.547766251	3.534187594	3.534187595
3	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	9.937421407	9.869605190	9.869605213
3	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	9.884270121	9.869605190	9.869605191
3	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	9.874420606	9.869605190	9.869605191
4	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{72}$	9.937916802	9.869605191	9.869605216
4	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{2}}{72}$	9.884282799	9.869605191	9.869605190
4	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{72}$	9.874424604	9.869605191	9.869605190

In this paper, C denotes a positive constant independent of h, which may stand for different values at its different occurrences.

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