

## Research Article

# Warped Product Submanifolds of Riemannian Product Manifolds

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We study warped product of the type  $N_\theta \times_f N_T$  and  $N_\theta \times_f N_\perp$ , where  $N_\theta$ ,  $N_T$ , and  $N_\perp$  are proper slant, invariant, and anti-invariant submanifolds, respectively, and we prove some basic results and finally obtain some inequalities for squared norm of second fundamental form.

## 1. Introduction

Bishop and O'Neil [1] introduced the notion of warped product manifolds that occur naturally; for example, surface of revolution is a warped product manifold. With regard to physical applications of these manifolds, one may realize that space time around a massive star or a black hole can be modeled on warped product manifolds [2]. CR-warped product was introduced by Chen [3]; he studied warped product CR-submanifolds in the setting of Kaehler manifolds and showed that there does not exist warped product of the form  $N_\perp \times_f N_T$ ; therefore he considered warped product CR-submanifolds of type  $N_T \times N_\perp$  and established a relationship between the warping function  $f$  and the squared norm of second fundamental form [3]. In [4] Ateken studied semi-slant warped product of Riemannian product manifolds. In fact they proved that there exists no warped product if spheric submanifold of warped product submanifold is proper slant submanifold. On the other hand they proved the existence of warped product of the type  $N_\theta \times_f N_T$  and  $N_\theta \times_f N_\perp$  via some examples. In this continuation we have studied the warped product submanifolds in which proper slant submanifolds are totally geodesic; that is, we study the warped product of the types  $N_\theta \times N_T$  and  $N_\theta \times N_\perp$  and called them semi-slant warped product and hemi-slant warped product submanifolds, respectively.

## 2. Preliminaries

Let  $(M_1, g_1)$  and  $(M_2, g_2)$  be the Riemannian manifolds with dimensions  $m_1$  and  $m_2$ , respectively, and let  $M_1 \times M_2$  be Riemannian product manifold of  $M_1$  and  $M_2$ . We denote projection mapping of  $T(M_1 \times M_2)$  onto  $TM_1$  and  $TM_2$  by  $\sigma_*$  and  $\pi_*$ , respectively. Then we have  $\sigma_* + \pi_* = I$ ,  $\sigma_*^2 = \sigma_*$ ,  $\pi_*^2 = \pi_*$ , and  $\sigma_* \circ \pi_* = \pi_* \circ \sigma_* = 0$ , where  $\star$  denotes the differential.

Riemannian metric of the Riemannian product manifold  $M = M_1 \times M_2$  is defined by

$$g(X, Y) = g_1(\sigma_* X, \sigma_* Y) + g_2(\pi_* X, \pi_* Y), \quad (2.1)$$

for any  $X, Y \in T\overline{M}$ . If we set  $F = \sigma_* - \pi_*$ , then  $F^2 = I$ ,  $F \neq I$ , and  $g$  satisfies the condition

$$g(FX, Y) = g(X, FY), \quad (2.2)$$

for any  $X, Y \in T\overline{M}$ ; thus  $F$  defines an almost Riemannian product structure on  $\overline{M}$ . We denote Levi-Civita connection on  $\overline{M}$  by  $\overline{\nabla}$ ; then the covariant derivative of  $F$  is defined as

$$(\overline{\nabla}_X F)Y = \overline{\nabla}_X FY - F\overline{\nabla}_X Y, \quad (2.3)$$

for any  $X, Y \in T\overline{M}$ . We say that  $F$  is parallel with respect to the connection  $\overline{\nabla}$  if we have  $(\overline{\nabla}_X F)Y = 0$ . Here from [5], we know that  $F$  is parallel; that is,  $F$  is Riemannian product structure.

Let  $\overline{M}$  be a Riemannian product manifold with Riemannian product structure  $F$  and  $M$  an immersed submanifold of  $\overline{M}$ ; we also denote by  $g$  the induced metric tensor on  $M$  as well as on  $\overline{M}$ . If  $\overline{\nabla}$  is the Levi-Civita connection on  $\overline{M}$ , then the Gauss and Weingarten formulas are given, respectively, as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.4)$$

$$\overline{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (2.5)$$

for any  $X, Y \in TM$  and  $V \in T^\perp M$ , where  $\nabla$  is the connection on  $M$  and  $\nabla^\perp$  is the connection in the normal bundle,  $h$  is the second fundamental form of  $M$ , and  $A_V$  is the shape operator of  $M$ . The second fundamental form  $h$  and the shape operator  $A_V$  are related by

$$g(A_V X, Y) = g(h(X, Y), V). \quad (2.6)$$

For any  $X \in TM$ , we can write

$$FX = TX + NX, \quad (2.7)$$

where  $TX$  and  $NX$  are the tangential and normal components of  $FX$ , respectively, and for  $V \in T^\perp M$ ,

$$FV = tV + nV, \quad (2.8)$$

where  $tV$  and  $nV$  are the tangential and normal components of  $FV$ , respectively, and the submanifold  $M$  is said to be invariant if  $N$  is identically zero. On the other hand  $M$  is said to be an anti-invariant submanifold if  $T$  is identically zero.

The covariant derivatives of  $T$ ,  $N$ ,  $t$ , and  $n$  are defined as

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (2.9)$$

$$(\bar{\nabla}_X N)Y = \nabla_X^\perp NY - N\nabla_X Y, \quad (2.10)$$

$$(\bar{\nabla}_X t)V = \nabla_X tV - t\nabla_X V, \quad (2.11)$$

$$(\bar{\nabla}_X n)V = \nabla_X^\perp nV - n\nabla_X^\perp V. \quad (2.12)$$

Using (2.4)–(2.9) we get

$$(\bar{\nabla}_X T)Y = A_{NY}X + th(X, Y), \quad (2.13)$$

$$(\bar{\nabla}_X N)Y = h(X, TY) - nh(X, Y). \quad (2.14)$$

Let  $M$  be an immersed submanifold of a Riemannian product manifold  $\bar{M}$ , for each nonzero vector  $X$  tangent to  $M$  at a point  $x$ , and we denote by  $\theta(x)$  the angle between  $FX$  and  $T_x M$ . The angle  $\theta(x)$  is called the slant angle of immersion.

Let  $M$  be an immersed submanifold of a Riemannian product manifold  $\bar{M}$ .  $M$  is said to be slant submanifold of Riemannian product manifold  $\bar{M}$  if the slant angle  $\theta(x)$  is constant which is independent of choice of  $x \in M$  and  $X \in TM$ .

Invariant and anti-invariant submanifolds are particular cases of slant submanifolds with angles  $\theta = 0$  and  $\theta = \pi/2$ , respectively. A slant submanifold which is neither invariant nor anti-invariant is called proper slant submanifold. The following characterization of slant submanifolds of Riemannian product manifolds is proved by Ateken [6].

**Theorem 2.1.** *Let  $M$  be an immersed submanifold of a Riemannian product manifold  $\bar{M}$ . Then  $M$  is a slant submanifold if and only if there exists a constant  $\lambda \in [0, 1]$  such that  $T^2 = \lambda I$ .*

Moreover, if  $\theta$  is the slant angle of  $M$ , then it satisfies  $\lambda = \cos^2 \theta$ .

Hence, for a slant submanifold we have the following relations which are consequences of the above theorem:

$$g(TX, TY) = \cos^2 \theta g(X, Y), \quad (2.15)$$

$$g(NX, NY) = \sin^2 \theta g(X, Y) \quad (2.16)$$

for any  $X, Y \in TM$ .

Papaghuic [7] introduced a class of submanifolds in almost Hermitian manifolds called semi-slant submanifolds; this class includes the class of proper CR-submanifolds and slant submanifolds. Cabrerizo et al. [8] initiated the study of contact version of semi-slant submanifolds and also gave the notion of Bi-slant submanifolds. A step forward Carriazo [9] defined and studied Bi-slant submanifolds and simultaneously gave the notion of anti-slant submanifolds; after that V. A. Khan and M. A. Khan [10] have studied anti-slant submanifolds with the name pseudo-slant submanifolds. Recently, Sahin [11] renamed these submanifolds and studied these submanifolds with the name hemi-slant submanifolds for their warped product.

*Definition 2.2.* A submanifold  $M$  of a Riemannian product manifold is said to be semi-slant submanifold if there exist two orthogonal complementary distributions  $D_T$  and  $D_\theta$  such that  $D_T$  is invariant and  $D_\theta$  is slant distribution with slant angle  $\theta \neq 0$ .

It is straight forward to see that semi-invariant submanifolds and slant submanifolds are semi-slant submanifolds with  $\theta = \pi/2$  and  $D_T = \{0\}$ , respectively.

If  $\mu$  is invariant subspace under  $F$  of the normal bundle  $T^\perp M$ , then in the case of semi-slant submanifold, the normal bundle  $T^\perp M$  can be decomposed as

$$T^\perp M = \mu \oplus ND_\theta. \quad (2.17)$$

A semi-slant submanifold  $M$  is called a semi-slant product if the distributions  $D_T$  and  $D_\theta$  are parallel on  $M$ . In this case  $M$  is foliated by the leaves of these distributions.

*Definition 2.3.* A submanifold  $M$  of a Riemannian product manifold is called hemi-slant submanifold if it is endowed with two orthogonal complementary distributions  $D_\perp$  and  $D_\theta$  such that  $D_\perp$  is totally real and  $D_\theta$  is slant distribution with slant angle  $\theta \neq \pi/2$ .

It is easy to see that semi-invariant submanifolds and slant submanifolds are semi-slant submanifolds with  $\theta = 0$  and  $D_\perp = \{0\}$ , respectively. The normal bundle  $T^\perp M$  can be decomposed as follows:

$$T^\perp M = \mu \oplus ND_\perp \oplus ND_\theta. \quad (2.18)$$

As  $D_\perp$  and  $D_\theta$  are orthogonal distributions on  $M$ , then it is easy to see that the distributions  $ND_\perp$  and  $ND_\theta$  are mutually perpendicular. In fact, the decomposition (2.18) is an orthogonal direct decomposition. A hemi-slant submanifold  $M$  is called a hemi-slant product if the distributions  $D_\perp$  and  $D_\theta$  are parallel on  $M$ .

As a generalization of product manifold and in particular of semi-slant product submanifolds (hemi-slant product submanifolds) one can consider warped product of manifolds which are defined as.

*Definition 2.4.* Let  $(B, g_B)$  and  $(F, g_F)$  be two Riemannian manifolds with Riemannian metric  $g_B$  and  $g_F$ , respectively, and  $f$  a positive differentiable function on  $B$ . The warped product of  $B$  and  $F$  is the Riemannian manifold  $(B \times F, g)$ , where

$$g = g_B + f^2 g_F. \quad (2.19)$$

For a warped product manifold  $N_1 \times_f N_2$ , we denote by  $D_1$  and  $D_2$  the distributions defined by the vectors tangent to the leaves and fibers, respectively. In other words,  $D_1$  is obtained by the tangent vectors of  $N_1$  via the horizontal lift and  $D_2$  is obtained by the tangent vectors of  $N_2$  via vertical lift. In case of semi-slant warped product submanifolds  $D_1$  and  $D_2$  are replaced by  $D_T$  and  $D_\theta$ , respectively.

The warped product manifold  $(B \times_f F, g)$  is denoted by  $B \times_f F$ . If  $X$  is the tangent vector field to  $M = B \times_f F$  at  $(p, q)$ , then

$$\|X\|^2 = \|d\pi_1 X\|^2 + f^2(p) \|d\pi_2 X\|^2. \quad (2.20)$$

Bishop and O'Neill [1] proved the following.

**Theorem 2.5.** *Let  $M = B \times_f F$  be warped product manifolds. If  $X, Y \in TB$  and  $V, W \in TF$ , then*

- (i)  $\nabla_X Y \in TB$ ,
- (ii)  $\nabla_X V = \nabla_V X = (Xf/f)V$ ,
- (iii)  $\nabla_V W = (-g(V, W)/f)\nabla f$ .

$\nabla f$  is the gradient of  $f$  and is defined as

$$g(\nabla f, X) = Xf, \quad (2.21)$$

for all  $X \in TM$ .

**Corollary 2.6.** *On a warped product manifold  $M = N_1 \times_f N_2$ , the following statements hold:*

- (i)  $N_1$  is totally geodesic in  $M$ ;
- (ii)  $N_2$  is totally umbilical in  $M$ .

Throughout, we denote by  $N_T$ ,  $N_\perp$ , and  $N_\theta$  invariant, anti-invariant, and slant submanifolds, respectively, of a Riemannian product manifold  $\overline{M}$ .

### 3. Semi-Slant Warped Product Submanifolds

In this section we will consider the warped product of the type  $N_\theta \times_f N_T$ .

For the warped product of the type  $N_\theta \times_f N_T$  by Theorem 2.5 we have

$$\nabla_X Z = \nabla_Z X = Z \ln f X, \quad (3.1)$$

for any  $Z \in TN_\theta$  and  $X \in TN_T$ .

**Lemma 3.1.** *Let  $M = N_\theta \times_f N_T$  be a semi-slant warped product submanifold of a Riemannian product manifold; then*

- (i)  $g(h(X, Z), NW) = 0$ ,
- (ii)  $g(h(X, X), NZ) = TZ \ln f \|X\|^2$ ,

for any  $X \in TN_T$  and  $Z, W \in TN_\theta$ .

*Proof.* For any  $Z, W \in TN_\theta$ ,  $(\nabla_Z T)W \in TN_\theta$ ; then from (2.13)

$$(\nabla_Z T)W = A_{NW}Z + th(Z, W). \quad (3.2)$$

Taking inner product with  $X \in TN_T$  we have

$$g(h(X, Z), NW) = 0. \quad (3.3)$$

This is part (i) of the lemma.

Now for any  $X, Y \in TN_T$ , from (2.13) and (2.9),

$$\nabla_X TY - T\nabla_X Y = th(X, Y). \quad (3.4)$$

Taking inner product with  $Z \in TN_\theta$ , the above equation yields

$$-g(\nabla_X Z, TY) - g(\nabla_X TZ, Y) = g(h(X, Y), NZ). \quad (3.5)$$

Using (3.1), the above equation gives

$$-Z \ln f g(X, TY) + TZ \ln f g(X, Y) = g(h(X, Y), NZ). \quad (3.6)$$

In particular

$$TZ \ln f g(X, X) = g(h(X, X), NZ). \quad (3.7)$$

This proves part (ii) of the lemma. Now we have the following corollary.  $\square$

**Corollary 3.2.** *For the warped product of the type  $N_\theta \times_f N_T$  following statements are equivalent:*

- (i)  $H \in \mu$ ,
- (ii)  $\theta = \pi/2$  or the warping function  $f$  is constant; that is, there does not exist warped product.

*Proof.* Since  $N_T$  is totally umbilical, then from (3.7)

$$g(H, NZ) = TZ \ln f. \quad (3.8)$$

Replacing  $Z$  by  $TZ$  and using Theorem 2.1, we get

$$g(H, NTZ) = \cos^2 \theta Z \ln f. \quad (3.9)$$

The proof follows from (3.9).  $\square$

Now we have the following characterization for semi-slant warped product submanifolds.

**Theorem 3.3.** *A semi-slant submanifold  $M$  of Riemannian product manifolds  $\overline{M}$  with integrable invariant distribution  $D_T$  and the slant distribution  $D_\theta$  is locally a semi-slant warped product if and only if  $\nabla_X TZ \in D_\theta$  and there exist a  $C^\infty$ -function  $\alpha$  on  $M$  with  $X\alpha = 0$  for all  $X \in D_T$  such that*

$$A_{NZ}X = TZ \ln fX, \quad (3.10)$$

for all  $X \in D_T$  and  $Z \in D_\theta$ .

*Proof.* If  $M$  is a semi-slant warped product of the type  $N_\theta \times_f N_T$ , then for any  $X \in TN_T$  and  $Z \in TN_\theta$  from (2.9), (2.13), and (3.1), we have

$$TZ \ln fX - Z \ln fTX = A_{NZ}X + th(X, Z). \quad (3.11)$$

Taking inner product with  $X$ , the above equation gives

$$g(A_{NZ}X, X) = TZ \ln fg(X, X). \quad (3.12)$$

By part (i) of Lemma 3.1, we also have

$$g(A_{NZ}X, W) = 0. \quad (3.13)$$

From (3.12) and (3.13) we have the following equation:

$$A_{NZ}X = TZ \ln fX. \quad (3.14)$$

Conversely, let  $M$  be a semi-slant submanifold of  $\overline{M}$  satisfying the hypothesis of the theorem; then for any  $Z \in TN_\theta$  and  $Y \in TN_T$  we have

$$g(h(Z, Y), NZ) = g(A_{NZ}Y, Z) = 0. \quad (3.15)$$

This mean  $h(Z, Y) \in \mu$ .

From (2.14), we have

$$-N\nabla_Z Y = fh(Z, Y) - h(Z, TY). \quad (3.16)$$

Comparing components of  $\mu$  and  $ND_\theta$ , we get

$$N\nabla_Z Y = 0. \quad (3.17)$$

It is evident from the above equation that  $\nabla_Z Y \in D_T$ ; this means  $\nabla_Z W \in D_\theta$  for any  $Z, W \in D_\theta$  and hence  $D_\theta$  is totally geodesic. Further, let  $N_T$  be a leaf of  $D_T$  and  $h^T$  a second fundamental form of the immersion  $N_T$  in  $M$ ; then for any  $X, Y \in D_T$  and  $Z, W \in D_\theta$

$$g(h^T(X, Y), FW) = g(\nabla_X Y, FW). \quad (3.18)$$

or

$$g(h^T(X, Y), FW) = -g(\nabla_X TW - A_{NW}X, Y). \quad (3.19)$$

Using the hypothesis, we get

$$g(h^T(X, Y), FW) = TW \ln f g(X, Y), \quad (3.20)$$

Finally, the above equation yields

$$h^T(X, Y) = g(X, Y) \nabla \alpha. \quad (3.21)$$

That is,  $N_T$  is totally umbilical and as  $X\alpha = 0$ , for all  $X \in D_T$ , this means that mean curvature vector of  $N_T$  is parallel; that is, the leaves of  $D_T$  are extrinsic spheres in  $M$ . Hence by virtue of result of [12] which says that if the tangent bundle of a Riemannian manifold  $M$  splits into an orthogonal sum  $TM = E_0 \oplus E_1$  of nontrivial vector subbundles such that  $E_1$  is spherical and its orthogonal complement  $E_0$  is auto parallel, then the manifold  $M$  is locally isometric to a warped product  $M_0 \times_f M_1$ , we can say  $M$  is locally semi-slant warped product submanifold  $N_\theta \times_f N_T$ , where warping function  $f = e^\alpha$ .

Let us denote by  $D_T$  and  $D_\theta$  the tangent bundles on  $N_T$  and  $N_\theta$ , respectively, and let  $\{X_1, \dots, X_p, X_{p+1} = FX_1, \dots, X_{2p} = FX_p\}$  and  $\{Z_1, \dots, Z_q, Z_{q+1} = TZ_1, \dots, Z_{2q} = TZ_q\}$  be local orthonormal frames of vector fields on  $N_T$  and  $N_\theta$ , respectively, with  $2p$  and  $2q$  being real dimensions:

$$\begin{aligned} \|h\|^2 &= \sum_{i,j=1}^{2p} g(h(X_i, X_j), h(X_i, X_j)) + \sum_{i=1}^{2p} \sum_{r=1}^{2q} g(h(X_i, Z_r), h(X_i, Z_r)) \\ &\quad + \sum_{r,s=1}^{2q} g(h(Z_r, Z_s), h(Z_r, Z_s)). \end{aligned} \quad (3.22)$$

Now, on a semi-slant warped product submanifold of a Riemannian product manifold, we prove the following.  $\square$

**Theorem 3.4.** *Let  $M = N_\theta \times_f N_T$  be a semi-slant warped product submanifold of a Riemannian product manifold  $\overline{M}$  with  $N_T$  and  $N_\theta$  invariant and slant submanifolds, respectively, of  $\overline{M}$ . Then the squared norm of the second fundamental form  $h$  satisfies*

$$\|h\|^2 \geq 4p(1 + \cos^2 \theta) \csc^2 \theta \|\nabla \ln f\|^2. \quad (3.23)$$

*Proof.* For  $N_\theta \times_f N_T$ , in view of decomposition (2.17), we may write

$$h(X, Y) = h_{ND_\theta}(X, Y) + h_\mu(X, Y), \quad (3.24)$$



for each  $X, Y \in TM$ , where  $h_{ND_\theta}(X, Y) \in ND_\theta$  and  $h_\mu(X, Y) \in \mu$  with

$$h_{ND_\theta}(X, Y) = \sum_{r=1}^{2q} h^r(X, Y)NZ_r, \quad (3.25)$$

where

$$h^r(X, Y) = \csc^2\theta g(h(X, Y), NZ_r), \quad (3.26)$$

for each  $Z \in TN_\theta$ . For any  $X \in TN_T$  and  $Z \in TN_\theta$ , by (3.25) we have

$$\begin{aligned} g(h_{ND_\theta}(X_i, X_i), h_{ND_\theta}(X_i, X_i)) &= g(h^r(X_i, X_i)NZ_r, h^r(X_i, X_i)NZ_r) \\ &\quad + \sum_{s \neq r} g(h^s(X_i, X_i)NZ_r, h^s(X_i, X_i)NZ_r). \end{aligned} \quad (3.27)$$

Now, using (3.26), (3.7), and (2.16), the above equation takes the form

$$g(h_{ND_\theta}(X_i, X_i), h_{ND_\theta}(X_i, X_i)) = \csc^2\theta (TZ_r \ln f)^2 + \sin^2\theta \sum_{s \neq r} (h^s(X_i, X_i))^2. \quad (3.28)$$

Now summing over  $i = 1, \dots, 2p$ ,  $r, s = 1, \dots, 2q$  and again using (3.7) and (3.26), we have

$$g(h_{ND_\theta}(X_i, X_i), h_{ND_\theta}(X_i, X_i)) = 2p \left(1 + \cos^2\theta\right) \csc^2\theta \|\nabla \ln f\|^2 \left(1 + \sin^2\theta \csc^2\theta\right), \quad (3.29)$$

or

$$g(h_{ND_\theta}(X_i, X_i), h_{ND_\theta}(X_i, X_i)) = 4p \left(1 + \cos^2\theta\right) \csc^2\theta \|\nabla \ln f\|^2, \quad (3.30)$$

By similar calculation, from (3.25), (3.26), (3.3), and (2.16) it is easy to see that

$$g(h_{ND_\theta}(X_i, Z_r), h_{ND_\theta}(X_i, Z_r)) = 0. \quad (3.31)$$

The result follows from (3.22), (3.30), and (3.31).  $\square$

#### 4. Hemi-Slant Warped Product Submanifolds

In this section we will study the warped product of the type  $N_\theta \times_f N_\perp$ . For warped product of type  $N_\theta \times_f N_\perp$  from Theorem 2.5 we have

$$\nabla_X Z = \nabla_Z X = X \ln f Z, \quad (4.1)$$

for any  $X \in TN_\theta$  and  $Z \in TN_\perp$ .

Now we have the following lemma.

**Lemma 4.1.** *Let  $M = N_{\theta} \times_f N_{\perp}$  be a hemi-slant warped product submanifold of a Riemannian product manifold; then*

- (i)  $g(h(X, Z), NY) + g(h(X, Y), NZ) = 0$ ,
- (ii)  $g(h(X, Z), NZ) = 0$ ,
- (iii)  $g(h(Z, Z), NX) = TX \ln f \|Z\|^2$ ,

for any  $X, Y \in TN_{\theta}$  and  $Z \in TN_{\perp}$ .

*Proof.* For any  $X, Y \in TN_{\theta}$ ,  $(\bar{\nabla}_X T)Y \in TN_{\theta}$ ; then from (2.13) we have

$$g\left((\bar{\nabla}_X T)Y, Z\right) = g(A_{NY}X, Z) + g(th(X, Y), Z), \quad (4.2)$$

or equivalently the above equation gives

$$g(h(X, Z), NY) + g(h(X, Y), NZ) = 0, \quad (4.3)$$

which proves part (i).

From (2.9), (2.13), we have

$$A_{NZ}X + th(X, Z) = 0. \quad (4.4)$$

Taking inner product with  $Z \in TN_{\perp}$  the above equation is reduced to

$$g(h(X, Z), NZ) = 0. \quad (4.5)$$

Using (2.9), (2.13), and (4.1), we derive

$$TX \ln f Z = A_{NX}Z + th(X, Z). \quad (4.6)$$

Taking inner product with  $Z \in TN_{\perp}$  and using (4.5),

$$g(h(Z, Z), NX) = TX \ln f \|Z\|^2. \quad (4.7)$$

Now we have the following corollary. □

**Corollary 4.2.** *For the warped product of the type  $N_{\theta} \times_f N_{\perp}$  following statements are equivalent:*

- (i)  $H \in \mu \oplus ND^{\perp}$ ,
- (ii) the warping function  $f$  is constant; that is, there does not exist warped product.

*Proof.* As  $N_{\perp}$  is totally umbilical, then from (4.7) and from Theorem 2.5

$$g(H, NTX) = \cos^2 \theta X \ln f. \quad (4.8)$$

Since  $\theta \neq \pi/2$ , hence from the previous equation it is easy to see that statements (i) and (ii) are equivalent.

Let us denote by  $D_\theta$  and  $D_\perp$  the tangent bundles on  $N_\theta$  and  $N_\perp$ , respectively, and let  $\{X_1, X_2, \dots, X_q, X_{q+1} = TX_1, \dots, X_{2q} = TX_q\}$  and  $\{Z_1, Z_2, \dots, Z_p\}$  be local orthonormal frames of vector fields on  $N_\theta$  and  $N_\perp$ , respectively, with  $2p$  and  $q$  being their real dimensions; then

$$\begin{aligned} \|h\|^2 &= \sum_{r,s=1}^{2q} g(h(X_r, X_s), h(X_r, X_s)) + \sum_{r=1}^{2q} \sum_{i=1}^p g(h(X_r, Z_i), h(X_r, Z_i)) \\ &\quad + \sum_{i,j=1}^p g(h(Z_i, Z_j), h(Z_i, Z_j)). \end{aligned} \quad (4.9)$$

Now, on a hemi-slant warped product submanifold of a Riemannian product manifold, we prove the following inequality.  $\square$

**Theorem 4.3.** *Let  $M = N_\theta \times_f N_\perp$  be a hemi-slant warped product submanifold of a Riemannian product manifold  $\overline{M}$  with  $N_\perp$  and  $N_\theta$  anti-invariant and slant submanifolds, respectively, of  $\overline{M}$ . Then the squared norm of the second fundamental form  $h$  satisfies*

$$\|h\|^2 \geq 2p(1 + \cos^2 \theta) \csc^2 \theta \|\nabla \ln f\|^2. \quad (4.10)$$

*Proof.* In view of decomposition (2.18), the second fundamental form can be decomposed as follows:

$$h(X, Y) = h_{ND_\theta}(X, Y) + h_{ND_\perp}(X, Y) + h_\mu(X, Y), \quad (4.11)$$

for each  $X, Y \in TM$ , where  $h_{ND_\theta}(X, Y) \in ND_\theta$ ,  $h_{ND_\perp}(X, Y) \in ND_\perp$ , and  $h_\mu(X, Y) \in \mu$  with

$$h_{ND_\theta}(X, Y) = \sum_{r=1}^{2q} h^r(X, Y) NX'_r, \quad (4.12)$$

where

$$h^r(X, Y) = \csc^2 \theta g(h(X, Y), NX'_r), \quad (4.13)$$

for each  $X'_r \in TN_\theta$ .

Now making use of (2.18), (4.12), (4.13), and (4.7) we have

$$g(h_{ND_\theta}(Z_i, Z_i), h_{ND_\theta}(Z_i, Z_i)) = \csc^2 \theta (TX_r \ln f)^2 + \sin^2 \theta \sum_{s \neq r} (h^s(Z_i, Z_i))^2, \quad (4.14)$$

for any  $X \in TN_\theta$  and  $Z \in TN_\perp$ .

Again using (4.12) and (4.13), the previous equation gives

$$g(h_{ND_\theta}(Z_i, Z_i), h_{ND_\theta}(Z_i, Z_i)) = \csc^2\theta (TX_r \ln f)^2 + \sin^2\theta \csc^4\theta (TX_s \ln f)^2. \quad (4.15)$$

Summing over  $i = 1, \dots, p$  and  $r, s = 1, \dots, 2q$ , we have

$$g(h_{ND_\theta}(Z_i, Z_i), h_{ND_\theta}(Z_i, Z_i)) = 2p(1 + \cos^2\theta) \csc^2\theta \|\nabla \ln f\|^2. \quad (4.16)$$

Similarly, for any  $X \in TN_\theta$  and  $Z \in TN_\perp$  by (4.12), (4.13), and (4.5) it is easy to see that

$$g(h_{ND_\theta}(X_r, Z_i), h_{ND_\theta}(X_r, Z_i)) = 0. \quad (4.17)$$

The result follows from (4.9), (4.16), and (4.17).  $\square$

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