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Research Article

Common Fixed Point Theorems for Six Mappings in Generalized Metric Spaces

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By using the weakly commutative and weakly compatible conditions of self-mapping pairs, we prove some new common fixed point theorems for six self-mappings in the framework of generalized metric spaces. An example is provided to support our result. The results presented in this paper generalize the well-known comparable results in the literature due to Abbas, Nazir, Saadati, Mustafa, and Sims.

1. Introduction and Preliminaries

The study of fixed points of mappings satisfying certain conditions has been at the center of vigorous research activity. In 2006, Mustafa and Sims [1] introduced a new structure of generalized metric spaces, which are called G-metric spaces as follows.

Definition 1.1 (see [1]). Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if $x = y = z$;

(G_2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables;

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Since then the fixed point theory in G-metric spaces has been studied and developed by authors (see [2–43]). Fixed point problems have also been considered in partially ordered G-metric spaces (see [44–56]).

The purpose of this paper is to use the concept of weakly commuting mappings and weakly compatible mappings to discuss some new common fixed point problem for six self-mappings in G-metric spaces. The results presented in this paper extend and improve the corresponding results of Abbas et al. [2] and Mustafa and Sims [3].

We now recall some of the basic concepts and results in G-metric spaces.

Proposition 1.2 (see [1]). *Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in three of its variables.*

Definition 1.3 (see [1]). Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence (x_n), if $\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$, and we say that the sequence (x_n) is G-convergent to x or (x_n) G-convergent to x, that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geq N$ (throughout this paper we mean by $\mathbb{N}$ the set of all natural numbers).

Proposition 1.4 (see [1]). *Let (X, G) be a G-metric space, then the following are equivalent:*

(i) *(x_n) is G-convergent to x;*

(ii) *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$;*

(iii) *$G(x_n, x, x) \to 0$ as $n \to +\infty$;*

(iv) *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 1.5 (see [1]). Let (X, G) be a G-metric space. A sequence (x_n) is called G-cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $m, n, l \geq N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.6 (see [1]). *Let (X, G) be a G-metric space, then the following are equivalent:*

(i) *the sequence (x_n) is G-cauchy;*

(ii) *for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $m, n \geq N$.*

Definition 1.7 (see [1]). A G-metric space (X, G) is G-complete if every G-cauchy sequence in (X, G) is G-convergent in X.

Definition 1.8 (see [1]). Let (X, G) and (X', G') be G-metric spaces, and let $f : (X, G) \to (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that $x, y \in X$, and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Proposition 1.9 (see [1]). *Let (X, G) and (X', G') be G-metric spaces, then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.*

Definition 1.10 (see [4]). Two self-mappings f and g of a G-metric space (X, G) are said to be weakly commuting if $G(fgx, gfx, gfx) \leq G(fx, gx, gx)$ for all x in X.

Definition 1.11 (see [4]). Let f and g be two self-mappings from a G-metric space (X, G) into itself. Then the mappings f and g are said to be weakly compatible if $G(fgx, gfx, gfx) = 0$ whenever $G(fx, gx, gx) = 0$.

Proposition 1.12 (see [1]). *Let (X, G) be a G-metric space. Then, for all x, y, z, a in X, it follows that*

(i) *if $G(x, x, y) = 0$, then $x = y = z$;*

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;

(iii) $G(x, y, y) \leq 2G(y, x, x)$;

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;

(v) $G(x, y, z) \leq (2/3)(G(x, y, a) + G(x, a, z) + G(a, y, z))$;

(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

2. Common Fixed Point Theorems

Theorem 2.1. *Let (X, G) be a complete G-metric space, and let f, g, h, A, B, and C be six mappings of X into itself satisfying the following conditions:*

(i) $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$;

(ii) *for all $x, y, z \in X$,*

$$G(fx, gy, hz) \leq k \max \left\{ \begin{array}{c} G(Ax, gy, gy) + G(By, fx, fx), \\ G(By, hz, hz) + G(Cz, gy, gy), \\ G(Cz, fx, fx) + G(Ax, hz, hz) \end{array} \right\} \tag{2.1}$$

or

$$G(fx, gy, hz) \leq k \max \left\{ \begin{array}{c} G(Ax, Ax, gy) + G(By, By, fx), \\ G(By, By, hz) + G(Cz, Cz, gy), \\ G(Cz, Cz, fx) + G(Ax, Ax, hz) \end{array} \right\}, \tag{2.2}$$

where $k \in [0, 1/3)$. Then one of the pairs (f, A), (g, B), and (h, C) has a coincidence point in X. Further, if one of the following conditions is satisfied, then the mappings f, g, h, A, B, and C have a unique common fixed point in X.

(a) *Either f or A is G-continuous, the pair (f, A) is weakly commutative, the pairs (g, B) and (h, C) are weakly compatible;*

(b) *Either g or B is G-continuous, the pair (g, B) is weakly commutative, the pairs (f, A) and (h, C) are weakly compatible;*

(c) *Either h or C is G-continuous, the pair (h, C) is weakly commutative, the pairs (f, A) and (g, B) are weakly compatible.*

Proof. Suppose that mappings f, g, h, A, B, and C satisfy condition (2.1).

Let x_0 in X be arbitrary point, since $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$, there exist the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{3n} = fx_{3n} = Bx_{3n+1}, \qquad y_{3n+1} = gx_{3n+1} = Cx_{3n+2}, \qquad y_{3n+2} = hx_{3n+2} = Ax_{3n+3}, \tag{2.3}$$

for $n = 0, 1, 2, \ldots$.

If $y_n = y_{n+1}$ for some n, with $n = 3m$, then $p = x_{3m+1}$ is a coincidence point of the pair (g, B); if $y_{n+1} = y_{n+2}$ for some n, with $n = 3m$, then $p = x_{3m+2}$ is a coincidence point of the pair (h, C); if $y_{n+2} = y_{n+3}$ for some n, with $n = 3m$, then $p = x_{3m+3}$ is a coincidence point of the pair (f, A). Without loss of generality, we can assume that $y_n \neq y_{n+1}$ for all $n = 0, 1, 2 \ldots$.

Now we prove that $\{y_n\}$ is a G-cauchy sequence in X.

Actually, using the condition (2.1) and (G_3), we have

$$\begin{aligned}
&G(y_{3n-1}, y_{3n}, y_{3n+1}) \\
&\quad = G(fx_{3n}, gx_{3n+1}, hx_{3n-1}) \\
&\quad \leq k \max\left\{\begin{matrix} G(Ax_{3n}, gx_{3n+1}, gx_{3n+1}) + G(Bx_{3n+1}, fx_{3n}, fx_{3n}), \\ G(Bx_{3n+1}, hx_{3n-1}, hx_{3n-1}) + G(Cx_{3n-1}, gx_{3n+1}, gx_{3n+1}), \\ G(Cx_{3n-1}, fx_{3n}, fx_{3n}) + G(A_{3n}, hx_{3n-1}, hx_{3n-1}) \end{matrix}\right\} \\
&\quad = k \max\left\{\begin{matrix} G(y_{3n-1}, y_{3n+1}, y_{3n+1}) + G(y_{3n}, y_{3n}, y_{3n}), \\ G(y_{3n}, y_{3n-1}, y_{3n-1}) + G(y_{3n-2}, y_{3n+1}, y_{3n+1}), \\ G(y_{3n-2}, y_{3n}, y_{3n}) + G(y_{3n-1}, y_{3n-1}, y_{3n-1}) \end{matrix}\right\} \\
&\quad \leq k \max\left\{\begin{matrix} G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n-2}, y_{3n-1}, y_{3n}), \\ G(y_{3n-1}, y_{3n}, y_{3n+1}) + [G(y_{3n-2}, y_{3n-1}, y_{3n-1}) + G(y_{3n-1}, y_{3n+1}, y_{3n+1})] \end{matrix}\right\} \\
&\quad \leq k \max\left\{\begin{matrix} G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n-2}, y_{3n-1}, y_{3n}), \\ 2G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n-2}, y_{3n-1}, y_{3n}) \end{matrix}\right\} \\
&\quad = k[2G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n-2}, y_{3n-1}, y_{3n})],
\end{aligned} \tag{2.4}$$

which further implies that

$$(1 - 2k)G(y_{3n-1}, y_{3n}, y_{3n+1}) \leq kG(y_{3n-2}, y_{3n-1}, y_{3n}). \tag{2.5}$$

Thus

$$G(y_{3n-1}, y_{3n}, y_{3n+1}) \leq \lambda G(y_{3n-2}, y_{3n-1}, y_{3n}), \tag{2.6}$$

where $\lambda = k/(1 - 2k)$. Obviously $0 \leq \lambda < 1$.

Similarly it can be shown that

$$\begin{gathered}
G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq \lambda G(y_{3n-1}, y_{3n}, y_{3n+1}), \\
G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \leq \lambda G(y_{3n}, y_{3n+1}, y_{3n+2}).
\end{gathered} \tag{2.7}$$

It follows from (2.6) and (2.7) that, for all $n \in \mathbb{N}$,

$$G(y_n, y_{n+1}, y_{n+2}) \le \lambda G(y_{n-1}, y_n, y_{n+1}) \le \lambda^2 G(y_{n-2}, y_{n-1}, y_n) \le \cdots \le \lambda^n G(y_0, y_1, y_2). \tag{2.8}$$

Therefore, for all $n, m \in \mathbb{N}$, $n < m$, by (G_3) and (G_5) we have

$$\begin{aligned} G(y_n, y_m, y_m) &\le G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + G(y_{n+2}, y_{n+3}, y_{n+3}) \\ &\quad + \cdots + G(y_{m-1}, y_m, y_m) \\ &\le G(y_n, y_{n+1}, y_{n+2}) + G(y_{n+1}, y_{n+2}, y_{n+3}) + \cdots + G(y_{m-1}, y_m, y_{m+1}) \\ &\le \left(\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \cdots + \lambda^{m-1}\right) G(y_0, y_1, y_2) \\ &\le \frac{\lambda^n}{1-\lambda} G(y_0, y_1, y_2) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.9}$$

Hence $\{y_n\}$ is a G-cauchy sequence in X, since X is complete G-metric space, there exists a point $u \in X$ such that $y_n \to u (n \to \infty)$.

Since the sequences $\{fx_{3n}\} = \{Bx_{3n+1}\}$, $\{gx_{3n+1}\} = \{Cx_{3n+2}\}$, and $\{hx_{3n-1}\} = \{Ax_{3n}\}$ are all subsequences of $\{y_n\}$, then they all converge to u, that is,

$$\begin{gathered} y_{3n} = fx_{3n} = Bx_{3n+1} \longrightarrow u, \qquad y_{3n+1} = gx_{3n+1} = Cx_{3n+2} \longrightarrow u, \\ y_{3n-1} = hx_{3n-1} = Ax_{3n} \longrightarrow u \quad (n \longrightarrow \infty). \end{gathered} \tag{2.10}$$

Now we prove that u is a common fixed point of f, g, h, A, B, and C under the condition (a).

First, we suppose that A is continuous, the pair (f, A) is weakly commutative, the pairs (g, B) and (h, C) are weakly compatible.

Step 1. We prove that $u = fu = Au$.

By (2.10) and weakly commutativity of mapping pair (f, A), we have

$$G(fAx_{3n}, Afx_{3n}, Afx_{3n}) \le G(fx_{3n}, Ax_{3n}, Ax_{3n}) \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.11}$$

Since A is continuous, then $A^2x_{3n} \to Au$ $(n \to \infty)$, $Afx_{3n} \to Au$ $(n \to \infty)$. By (2.11), we know that $fAx_{3n} \to Au$ $(n \to \infty)$.

From the condition (2.1), we get

$$\begin{aligned} &G(fAx_{3n}, gx_{3n+1}, hx_{3n+2}) \\ &\quad \le k \max \left\{ \begin{array}{l} G(A^2x_{3n}, gx_{3n+1}, gx_{3n+1}) + G(Bx_{3n+1}, fAx_{3n}, fAx_{3n}), \\ G(Bx_{3n+1}, hx_{3n+2}, hx_{3n+2}) + G(Cx_{3n+2}, gx_{3n+1}, gx_{3n+1}), \\ G(Cx_{3n+2}, fAx_{3n}, fAx_{3n}) + G(A^2x_{3n}, hx_{3n+2}, hx_{3n+2}) \end{array} \right\}. \end{aligned} \tag{2.12}$$

Letting $n \to \infty$ and using the Proposition 1.12(iii), we have

$$G(Au,u,u) \leq k\max\left\{\begin{array}{c} G(Au,u,u)+G(u,Au,Au), \\ G(u,u,u)+G(u,u,u), \\ G(u,Au,Au)+G(Au,u,u) \end{array}\right\}$$
$$= k[G(Au,u,u)+G(u,Au,Au)]$$
$$\leq 3kG(Au,u,u). \tag{2.13}$$

Hence, $G(Au,u,u)=0$ and $Au=u$, since $0 \leq k < 1/3$.

Again by using condition (2.1), we have

$$G(fu,gx_{3n+1},hx_{3n+2}) \leq k\max\left\{\begin{array}{c} G(Au,gx_{3n+1},gx_{3n+1})+G(Bx_{3n+1},fu,fu), \\ G(Bx_{3n+1},hx_{3n+2},hx_{3n+2})+G(Cx_{3n+2},gx_{3n+1},gx_{3n+1}), \\ G(Cx_{3n+2},fu,fu)+G(Au,hx_{3n+2},hx_{3n+2}) \end{array}\right\}. \tag{2.14}$$

Letting $n \to \infty$, we have

$$G(fu,u,u) \leq kG(u,fu,fu). \tag{2.15}$$

From the Proposition 1.12(iii), we get

$$G(fu,u,u) \leq kG(u,fu,fu) \leq 2kG(fu,u,u). \tag{2.16}$$

Hence, $G(fu,u,u)=0$ and $fu=u$, since $0 \leq k < 1/3$.

So we have $u=Au=fu$.

Step 2. We prove that $u=gu=Bu$.

Since $f(X) \subset B(X)$ and $u=fu \in f(X)$, there is a point $v \in X$ such that $u=fu=Bv$. Again by using condition (2.1), we have

$$G(fu,gv,hx_{3n+2}) \leq k\max\left\{\begin{array}{c} G(Au,gv,gv)+G(Bv,fu,fu), \\ G(Bv,hx_{3n+2},hx_{3n+2})+G(Cx_{3n+2},gv,gv), \\ G(Cx_{3n+2},fu,fu)+G(Au,hx_{3n+2},hx_{3n+2}) \end{array}\right\}. \tag{2.17}$$

Letting $n \to \infty$, using $u=Au=fu$ and the Proposition 1.12(iii), we obtain

$$G(u,gv,u) \leq kG(u,gv,gv) \leq 2kG(u,gv,u). \tag{2.18}$$

Hence, $G(u,gv,u)=0$ and so $gv=u=Bv$, since $0 \leq k < 1/3$.

Since the pair (g,B) is weakly compatible, we have

$$gu=gBv=Bgv=Bu. \tag{2.19}$$

Again by using condition (2.1), we have

$$G(fu, gu, hx_{3n+2}) \leq k \max \left\{ \begin{array}{c} G(Au, gu, gu) + G(Bu, fu, fu), \\ G(Bu, hx_{3n+2}, hx_{3n+2}) + G(Cx_{3n+2}, gu, gu), \\ G(Cx_{3n+2}, fu, fu) + G(Au, hx_{3n+2}, hx_{3n+2}) \end{array} \right\}. \tag{2.20}$$

Letting $n \to \infty$, using $u = Au = fu$, $gu = Bu$ and the Proposition 1.12(iii), we have

$$G(u, gu, u) \leq k[G(u, gu, gu) + G(u, gu, u)] \leq 3kG(u, gu, u). \tag{2.21}$$

Hence, $G(u, gu, u) = 0$ and so $u = gu = Bu$, since $0 \leq k < 1/3$.

So we have $u = gu = Bu$.

Step 3. We prove that $u = hu = Cu$.

Since $g(X) \subset C(X)$ and $u = gu \in g(X)$, there is a point $w \in X$ such that $u = gu = Cw$. Again by using condition (2.1), we have

$$G(fu, gu, hw) \leq k \max \left\{ \begin{array}{c} G(Au, gu, gu) + G(Bu, fu, fu), \\ G(Bu, hw, hw) + G(Cw, gu, gu), \\ G(Cw, fu, fu) + G(Au, hw, hw) \end{array} \right\}. \tag{2.22}$$

Using $u = Au = fu$, $u = gu = Bu = Cw$ and the Proposition 1.12(iii), we obtain

$$G(u, u, hw) \leq kG(u, hw, hw) \leq 2kG(u, u, hw). \tag{2.23}$$

Hence, $G(u, u, hw) = 0$ and so $hw = u = Cw$, since $0 \leq k < 1/3$.

Since the pair (h, C) is weakly compatible, we have

$$hu = hCw = Chw = Cu. \tag{2.24}$$

Again by using condition (2.1), we have

$$G(fu, gu, hu) \leq k \max \left\{ \begin{array}{c} G(Au, gu, gu) + G(Bu, fu, fu), \\ G(Bu, hu, hu) + G(Cu, gu, gu), \\ G(Cu, fu, fu) + G(Au, hu, hu) \end{array} \right\}. \tag{2.25}$$

Using $u = Au = fu$, $u = gu = Bu$, $Cu = hu$ and the Proposition 1.12(iii), we have

$$G(u, u, hu) \leq k[G(u, hu, hu) + G(hu, u, u)] \leq 3kG(u, u, hu). \tag{2.26}$$

Hence, $G(u, u, hu) = 0$ and so $u = hu = Cu$, since $0 \leq k < 1/3$.

Therefore, u is the common fixed point of f, g, h, A, B, and C when A is continuous and the pair (f, A) is weakly commutative, the pairs (g, B) and (h, C) are weakly compatible.

Next, we suppose that f is continuous, the pair (f, A) is weakly commutative, the pairs (g, B) and (h, C) are weakly compatible.

Step 1. We prove that $u = fu$.

By (2.10) and weak commutativity of mapping pair (f, A), we have

$$G(fAx_{3n}, Afx_{3n}, Afx_{3n}) \leq G(fx_{3n}, Ax_{3n}, Ax_{3n}) \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.27}$$

Since f is continuous, then $f^2x_{3n} \to fu$ $(n \to \infty)$, $fAx_{3n} \to fu$ $(n \to \infty)$. By (2.10), we know that $Afx_{3n} \to fu$ $(n \to \infty)$.

From the condition (2.1), we have

$$G\left(f^2x_{3n}, gx_{3n+1}, hx_{3n+2}\right) \leq k \max\left\{\begin{array}{c} G(Afx_{3n}, gx_{3n+1}, gx_{3n+1}) + G(Bx_{3n+1}, f^2x_{3n}, f^2x_{3n}), \\ G(Bx_{3n+1}, hx_{3n+2}, hx_{3n+2}) + G(Cx_{3n+2}, gx_{3n+1}, gx_{3n+1}), \\ G(Cx_{3n+2}, f^2x_{3n}, f^2x_{3n}) + G(Afx_{3n}, hx_{3n+2}, hx_{3n+2}) \end{array}\right\}. \tag{2.28}$$

Letting $n \to \infty$ and noting the Proposition 1.12(iii), we have

$$\begin{aligned} G(fu, u, u) &\leq k \max\left\{\begin{array}{c} G(fu, u, u) + G(u, fu, fu), \\ G(u, u, u) + G(u, u, u), \\ G(u, fu, fu) + G(fu, u, u) \end{array}\right\} \\ &= k\left[G(fu, u, u) + G(u, fu, fu)\right] \\ &\leq 3kG(fu, u, u). \end{aligned} \tag{2.29}$$

Hence, $G(fu, u, u) = 0$ and so $fu = u$, since $0 \leq k < 1/3$.

Step 2. We prove that $u = gu = Bu$.

Since $f(X) \subset B(X)$ and $u = fu \in f(X)$, there is a point $z \in X$ such that $u = fu = Bz$. Again by using condition (2.1), we have

$$G\left(f^2x_{3n}, gz, hx_{3n+2}\right) \leq k \max\left\{\begin{array}{c} G(Afx_{3n}, gz, gz) + G(Bz, f^2x_{3n}, f^2x_{3n}), \\ G(Bz, hx_{3n+2}, hx_{3n+2}) + G(Cx_{3n+2}, gz, gz), \\ G(Cx_{3n+2}, f^2x_{3n}, f^2x_{3n}) + G(Afx_{3n}, hx_{3n+2}, hx_{3n+2}) \end{array}\right\}. \tag{2.30}$$

Letting $n \to \infty$ and using $u = fu$ and the Proposition 1.12(iii), we have

$$G(u, gz, u) \leq kG(u, gz, gz) \leq 2kG(u, gz, u). \tag{2.31}$$

Hence $G(u, gz, u) = 0$ and so $gz = u = Bz$, since $0 \leq k < 1/3$.

Since the pair (g, B) is weakly compatible, we have

$$gu = gBz = Bgz = Bu. \tag{2.32}$$

Again by using condition (2.1), we have

$$G(fx_{3n}, gu, hx_{3n+2}) \leq k \max \left\{ \begin{array}{c} G(Ax_{3n}, gu, gu) + G(Bu, fx_{3n}, fx_{3n}), \\ G(Bu, hx_{3n+2}, hx_{3n+2}) + G(Cx_{3n+2}, gu, gu), \\ G(Cx_{3n+2}, fx_{3n}, fx_{3n}) + G(Ax_{3n}, hx_{3n+2}, hx_{3n+2}) \end{array} \right\}. \tag{2.33}$$

Letting $n \to \infty$ and using $u = fu$, $gu = Bu$ and the Proposition 1.12(iii), we have

$$G(u, gu, u) \leq k[G(u, gu, gu) + G(gu, u, u)] \leq 3kG(u, gu, u). \tag{2.34}$$

Hence, $G(u, gu, u) = 0$ and so $gu = u = Bu$, since $0 \leq k < 1/3$.

So we have $u = gu = Bu$.

Step 3. We prove that $u = hu = Cu$.

Since $g(X) \subset C(X)$ and $u = gu \in g(X)$, there is a point $t \in X$ such that $u = gu = Ct$. Again by using condition (2.1), we have

$$G(fx_{3n}, gu, ht) \leq k \max \left\{ \begin{array}{c} G(Ax_{3n}, gu, gu) + G(Bu, fx_{3n}, fx_{3n}), \\ G(Bu, ht, ht) + G(Ct, gu, gu), \\ G(Ct, fx_{3n}, fx_{3n}) + G(Ax_{3n}, ht, ht) \end{array} \right\}. \tag{2.35}$$

Letting $n \to \infty$ and using $u = gu = Bu$ and the Proposition 1.12(iii), we obtain

$$G(u, u, ht) \leq kG(u, ht, ht) \leq 2kG(u, u, ht). \tag{2.36}$$

Hence, $G(u, u, ht) = 0$ and so $ht = u = Ct$, since $0 \leq k < 1/3$.

Since the pair (h, C) is weakly compatible, we have

$$hu = hCt = Cht = Cu. \tag{2.37}$$

Again by using condition (2.1), we have

$$G(fx_{3n}, gu, hu) \leq k \max \left\{ \begin{array}{c} G(Ax_{3n}, gu, gu) + G(Bu, fx_{3n}, fx_{3n}), \\ G(Bu, hu, hu) + G(Cu, gu, gu), \\ G(Cu, fx_{3n}, fx_{3n}) + G(Ax_{3n}, hu, hu) \end{array} \right\}. \tag{2.38}$$

Letting $n \to \infty$ and using $u = gu = Bu$ and the Proposition 1.12(iii), we have

$$G(u, u, hu) \leq k[G(u, hu, hu) + G(u, u, hu)] \leq 3kG(u, u, hu). \tag{2.39}$$

Hence, $G(u, u, hu) = 0$ and so $hu = u = Cu$, since $0 \leq k < 1/3$.

Step 4. We prove that $u = Au$.

Since $h(X) \subset A(X)$ and $u = hu \in h(X)$, there is a point $p \in X$ such that $u = hu = Ap$. Again by using condition (2.1), we have

$$G(fp, gu, hu) \le k \max\left\{\begin{array}{c} G(Ap, gu, gu) + G(Bu, fp, fp), \\ G(Bu, hu, hu) + G(Cu, gu, gu), \\ G(Cu, fp, fp) + G(Ap, hu, hu) \end{array}\right\}. \tag{2.40}$$

Using $u = gu = Bu$, $u = hu = Cu$, and the Proposition 1.12(iii), we obtain

$$G(fp, u, u) \le kG(u, fp, fp) \le 2kG(fp, u, u). \tag{2.41}$$

Hence $G(fp, u, u) = 0$ and $fp = u = Ap$, since $0 \le k < 1/3$.

Since the pair (f, A) is weakly compatible, we have

$$fu = fAp = Afp = Au = u. \tag{2.42}$$

Therefore, u is the common fixed point of f, g, h, A, B, and C when f is continuous and the pair (f, A) is weakly commutative, the pairs (g, B) and (h, C) are weakly compatible.

Similarly, we can prove the result that u is a common fixed point of f, g, h, A, B, and C when under the condition of (b) or (c).

Finally, we prove uniqueness of common fixed point u.

Let u and q be two common fixed points of f, g, h, A, B, and C, by using condition (2.1), we have

$$\begin{aligned} G(q, u, u) &= G(fq, gu, hu) \\ &\le k \max\left\{\begin{array}{c} G(Aq, gu, gu) + G(Bu, fq, fq), \\ G(Bu, hu, hu) + G(Cu, gu, gu), \\ G(Cu, fq, fq) + G(Aq, hu, hu) \end{array}\right\} \\ &= k\left[G(q, u, u) + G(u, q, q)\right] \\ &\le 3kG(q, u, u). \end{aligned} \tag{2.43}$$

Hence, $G(q, u, u) = 0$ and so $q = u$, since $0 \le k < 1/3$. Thus common fixed point is unique.

The proof using (2.2) is similar. This completes the proof. □

Now we introduce an example to support Theorem 2.1.

Example 2.2. Let $X = [0,1]$, and let (X,G) be a G-metric space defined by $G(x,y,z) = |x-y| + |y-z| + |z-x|$ for all $x, y,$ and z in X. Let f, g, h, A, B, and C be self-mappings defined by

$$fx = \begin{cases} 1, & x \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{7}{8}, & x \in \left(\dfrac{1}{2}, 1\right], \end{cases} \qquad gx = \begin{cases} \dfrac{10}{11}, & x \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{7}{8}, & x \in \left(\dfrac{1}{2}, 1\right], \end{cases} \qquad hx = \begin{cases} \dfrac{9}{10}, & x \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{7}{8}, & x \in \left(\dfrac{1}{2}, 1\right], \end{cases}$$

$$Ax = x, \qquad Bx = \begin{cases} 1, & x \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{7}{8}, & x \in \left(\dfrac{1}{2}, 1\right), \\ 0, & x = 1, \end{cases} \qquad Cx = \begin{cases} 1, & x \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{7}{8}, & x \in \left(\dfrac{1}{2}, 1\right), \\ \dfrac{10}{11}, & x = 1. \end{cases} \tag{2.44}$$

Note that A is G-continuous in X, and f, g, h, B, and C are not G-continuous in X.

Clearly we can get $f(X) \subset B(X)$, $g(X) \subset C(X)$, and $h(X) \subset A(X)$.

Actually, since $fX = \{7/8, 1\}$, $BX = \{0, 7/8, 1\}$, $gX = \{7/8, 10/11\}$, $CX = \{7/8, 10/11, 1\}$, $hX = \{7/8, 9/10\}$, and $AX = X = [0,1]$, so we know $f(X) \subset B(X)$, $g(X) \subset C(X)$, and $h(X) \subset A(X)$.

By the definition of the mappings of f and A, for all $x \in [0,1]$, $G(fAx, Afx, Afx) = G(fx, fx, fx) = 0 \le G(fx, Ax, Ax)$, so we can get the pair (f, A) is weakly commuting.

By the definition of the mappings of g and B, only for $x \in (1/2, 1)$, $gx = Bx = 7/8$, at this time $gBx = g(7/8) = 7/8 = B(7/8) = Bgx$, so $gBx = Bgx$, so we can obtain that the pair (g, B) is weakly compatible. Similarly, we can show that the pair (h, C) is also weakly compatible.

Now we proof that the mappings f, g, h, A, B, and C are satisfying the condition (2.1) of Theorem 2.1 with $k = 5/16 \in [0, 1/3)$. Let

$$M(x,y,z) = \max \left\{ \begin{array}{c} G(Ax, gy, gy) + G(By, fx, fx), \\ G(By, hz, hz) + G(Cz, gy, gy), \\ G(Cz, fx, fx) + G(Ax, hz, hz) \end{array} \right\}. \tag{2.45}$$

Case 1. If $x, y, z \in [0, 1/2]$, then

$$\begin{gathered} G(fx, gy, hz) = G\left(1, \frac{10}{11}, \frac{9}{10}\right) = \frac{1}{5}, \\ G(Ax, gy, gy) + G(By, fx, fx) = G\left(x, \frac{10}{11}, \frac{10}{11}\right) + G(1,1,1) = 2\left|x - \frac{10}{11}\right| \ge \frac{9}{11}. \end{gathered} \tag{2.46}$$

Thus, we have

$$G(fx, gy, hz) = \frac{1}{5} < \frac{5}{16} \cdot \frac{9}{11} \le k(G(Ax, gy, gy) + G(By, fx, fx)) \le kM(x,y,z). \tag{2.47}$$

Case 2. If $x, y \in [0, 1/2]$, $z \in (1/2, 1]$, then

$$\begin{gathered} G(fx, gy, hz) = G\left(1, \frac{10}{11}, \frac{7}{8}\right) = \frac{1}{4}, \\ G(Ax, gy, gy) + G(By, fx, fx) = G\left(x, \frac{10}{11}, \frac{10}{11}\right) + G(1,1,1) = 2\left|x - \frac{10}{11}\right| \geq \frac{9}{11}. \end{gathered} \tag{2.48}$$

Therefore, we get

$$G(fx, gy, hz) = \frac{1}{4} < \frac{5}{16} \cdot \frac{9}{11} \leq k(G(Ax, gy, gy) + G(By, fx, fx)) \leq kM(x, y, z). \tag{2.49}$$

Case 3. If $x, z \in [0, 1/2]$, $y \in (1/2, 1]$, then

$$\begin{gathered} G(fx, gy, hz) = G\left(1, \frac{7}{8}, \frac{9}{10}\right) = \frac{1}{4}, \\ G(Cz, fx, fx) + G(Ax, hz, hz) = G(1,1,1) + G\left(x, \frac{9}{10}, \frac{9}{10}\right) = 2\left|x - \frac{9}{10}\right| \geq \frac{4}{5}. \end{gathered} \tag{2.50}$$

Hence, we have

$$G(fx, gy, hz) = \frac{1}{4} = \frac{5}{16} \cdot \frac{4}{5} \leq k(G(Cz, fx, fx) + G(Ax, hz, hz)) \leq kM(x, y, z). \tag{2.51}$$

Case 4. If $y, z \in [0, 1/2], x \in (1/2, 1]$, then

$$\begin{gathered} G(fx, gy, hz) = G\left(\frac{7}{8}, \frac{10}{11}, \frac{9}{10}\right) = \frac{3}{44}, \\ G(By, hz, hz) + G(Cz, gy, gy) = G\left(1, \frac{9}{10}, \frac{9}{10}\right) + G\left(1, \frac{10}{11}, \frac{10}{11}\right) = \frac{21}{55}. \end{gathered} \tag{2.52}$$

So we get

$$G(fx, gy, hz) = \frac{3}{44} < \frac{5}{16} \cdot \frac{21}{55} \leq k(G(By, hz, hz) + G(Cz, gy, gy)) \leq kM(x, y, z). \tag{2.53}$$

Case 5. If $x \in [0, 1/2], y, z \in (1/2, 1]$, then

$$G(fx, gy, hz) = G\left(1, \frac{7}{8}, \frac{7}{8}\right) = \frac{1}{4}. \tag{2.54}$$

If $y \in (1/2, 1)$, then

$$G(Ax, gy, gy) + G(By, fx, fx) = G\left(x, \frac{7}{8}, \frac{7}{8}\right) + G\left(\frac{7}{8}, 1, 1\right) = 2\left|x - \frac{7}{8}\right| + \frac{1}{4} \geq \frac{3}{4} + \frac{1}{4} = 1. \tag{2.55}$$

If $y = 1$, then

$$G(Ax, gy, gy) + G(By, fx, fx) = G\left(x, \frac{7}{8}, \frac{7}{8}\right) + G(0,1,1) = 2\left|x - \frac{7}{8}\right| + 2 \geq \frac{3}{4} + 2 = \frac{11}{4}. \tag{2.56}$$

And so we have

$$G(Ax, gy, gy) + G(By, fx, fx) \geq 1 \tag{2.57}$$

for all $y \in (1/2, 1]$. Hence we have

$$G(fx, gy, hz) = \frac{1}{4} < \frac{5}{16} \cdot 1 \leq k(G(Ax, gy, gy) + G(By, fx, fx)) \leq kM(x, y, z). \tag{2.58}$$

Case 6. If $y \in [0, 1/2], x, z \in (1/2, 1]$, then

$$\begin{aligned} G(fx, gy, hz) &= G\left(\frac{7}{8}, \frac{10}{11}, \frac{7}{8}\right) = \frac{3}{44}, \\ G(Ax, gy, gy) + G(By, fx, fx) &= G\left(x, \frac{10}{11}, \frac{10}{11}\right) + G\left(1, \frac{7}{8}, \frac{7}{8}\right) = 2\left|x - \frac{10}{11}\right| + \frac{1}{4} \geq \frac{1}{4}. \end{aligned} \tag{2.59}$$

Thus, we have

$$G(fx, gy, hz) = \frac{3}{44} < \frac{5}{16} \cdot \frac{1}{4} \leq k(G(Ax, gy, gy) + G(By, fx, fx)) \leq kM(x, y, z). \tag{2.60}$$

Case 7. If $z \in [0, 1/2], x, y \in (1/2, 1]$, then

$$G(fx, gy, hz) = G\left(\frac{7}{8}, \frac{7}{8}, \frac{9}{10}\right) = \frac{1}{20}. \tag{2.61}$$

If $y \in (1/2, 1)$, then

$$G(By, hz, hz) + G(Cz, gy, gy) = G\left(\frac{7}{8}, \frac{9}{10}, \frac{9}{10}\right) + G\left(1, \frac{7}{8}, \frac{7}{8}\right) = \frac{3}{10}. \tag{2.62}$$

If $y = 1$, then

$$G(By, hz, hz) + G(Cz, gy, gy) = G\left(0, \frac{9}{10}, \frac{9}{10}\right) + G\left(1, \frac{7}{8}, \frac{7}{8}\right) = \frac{41}{20}. \tag{2.63}$$

And so we have

$$G(By,hz,hz)+G(Cz,gy,gy)\geq\frac{3}{10} \tag{2.64}$$

for all $y\in(1/2,1]$. Hence we have

$$G(fx,gy,hz)=\frac{1}{20}<\frac{5}{16}\cdot\frac{3}{10}\leq k(G(By,hz,hz)+G(Cz,gy,gy))\leq kM(x,y,z). \tag{2.65}$$

Case 8. If $x,y,z\in(1/2,1]$, then

$$G(fx,gy,hz)=G\left(\frac{7}{8},\frac{7}{8},\frac{7}{8}\right)=0\leq\frac{5}{16}M(x,y,z)=kM(x,y,z). \tag{2.66}$$

Then in all the above cases, the mappings f, g, h, A, B, and C are satisfying the condition (2.1) of Theorem 2.1 with $k=5/16$ so that all the conditions of Theorem 2.1 are satisfied. Moreover, $7/8$ is the unique common fixed point for all of the mappings f, g, h, A, B, and C.

In Theorem 2.1, if we take $A=B=C=I$ (I is identity mapping, the same as below), then we have the following corollary.

Corollary 2.3 (see [2, Theorem 2.4]). *Let (X,G) be a complete G-metric space, and let f, g, and h be three mappings of X into itself satisfying the following conditions:*

$$G(fx,gy,hz)\leq k\max\left\{\begin{array}{c}G(x,gy,gy)+G(y,fx,fx),\\ G(y,hz,hz)+G(z,gy,gy),\\ G(z,fx,fx)+G(x,hz,hz)\end{array}\right\} \tag{2.67}$$

or

$$G(fx,gy,hz)\leq k\max\left\{\begin{array}{c}G(x,x,gy)+G(y,y,fx),\\ G(y,y,hz)+G(z,z,gy),\\ G(z,z,fx)+G(x,x,hz)\end{array}\right\} \tag{2.68}$$

for all $x,y,z\in X$, where $k\in[0,1/3)$. Then f, g, and h have a unique common fixed point in X.

Also, if we take $f=g=h$ and $A=B=C=I$ in Theorem 2.1, then we get the following.

Corollary 2.4 (see [3, Theorem 2.4]). *Let (X,G) be a complete G-metric space, and let f be a mapping of X into itself satisfying the following conditions:*

$$G(fx,fy,fz)\leq k\max\left\{\begin{array}{c}G(x,fy,fy)+G(y,fx,fx),\\ G(y,fz,fz)+G(z,fy,fy),\\ G(z,fx,fx)+G(x,fz,fz)\end{array}\right\} \tag{2.69}$$

or

$$G(fx, fy, fz) \le k \max \left\{ \begin{array}{c} G(x, x, fy) + G(y, y, fx), \\ G(y, y, fz) + G(z, z, fy), \\ G(z, z, fx) + G(x, x, fz) \end{array} \right\}, \tag{2.70}$$

for all $x, y, z \in X$, where $k \in [0, 1/3)$. Then, f has a unique fixed point in X.

Remark 2.5. Theorem 2.1 and Corollaries 2.3 and 2.4 generalize and extend the corresponding results of Abbas and Rhoades [5] and Mustafa et al. [6].

Remark 2.6. In Theorem 2.1, if we take: (1) $f = g = h$; (2) $A = B = C$; (3) $g = h$ and $B = C$; (4) $g = h$, $B = C = I$, several new results can be obtained.

Theorem 2.7. *Let (X, G) be a complete G-metric space, and let f, g, h, A, B, and C be six mappings of X into itself satisfying the following conditions:*

(i) *$f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$;*

(ii) *the pairs (f, A), (g, B), and (h, C) are commutative mappings;*

(iii) *for all $x, y, z \in X$,*

$$G(f^m x, g^m y, h^m z) \le k \max \left\{ \begin{array}{c} G(Ax, g^m y, g^m y) + G(By, f^m x, f^m x), \\ G(By, h^m z, h^m z) + G(Cz, g^m y, g^m y), \\ G(Cz, f^m x, f^m x) + G(Ax, h^m z, h^m z) \end{array} \right\} \tag{2.71}$$

or

$$G(f^m x, g^m y, h^m z) \le k \max \left\{ \begin{array}{c} G(Ax, Ax, g^m y) + G(By, By, f^m x), \\ G(By, By, h^m z) + G(Cz, Cz, g^m y), \\ G(Cz, Cz, f^m x) + G(Ax, Ax, h^m z) \end{array} \right\}, \tag{2.72}$$

where $k \in [0,\ 1/2)$, $m \in \mathbb{N}$, then f, g, h, A, B, and C have a unique common fixed point in X.

Proof. Suppose that mappings f, g, h, A, B, and C satisfy the condition (2.71). Since $f^m X \subset f^{m-1} X \subset \cdots \subset fX$, $fX \subset BX$ so that $f^m X \subset BX$. Similarly, we can show that $g^m X \subset CX$ and $h^m X \subset AX$. From the the Theorem 2.1, we see that f^m, g^m, h^m, A, B, and C have a unique common fixed point u.

Since $fu = f(f^m u) = f^{m+1} u = f^m(fu)$, so that

$$G(f^m fu, g^m u, h^m u) \le k \max \left\{ \begin{array}{c} G(Afu, g^m u, g^m u) + G(Bu, f^m fu, f^m fu), \\ G(Bu, h^m u, h^m u) + G(Cu, g^m u, g^m u), \\ G(Cu, f^m fu, f^m fu) + G(Afu, h^m u, h^m u) \end{array} \right\}, \tag{2.73}$$

note that $Afu = fAu = fu$ and the Proposition 1.12(iii), we obtain

$$\begin{aligned} G(fu,u,u) &\leq k\max\left\{\begin{matrix} G(fu,u,u)+G(u,fu,fu), \\ G(u,u,u)+G(u,u,u), \\ G(u,fu,fu)+G(fu,u,u) \end{matrix}\right\} \\ &= k[G(u,fu,fu)+G(fu,u,u)] \\ &\leq 3kG(fu,u,u). \end{aligned} \tag{2.74}$$

Since $k \in [0,1/3)$, hence $G(fu,u,u) = 0$ and so $fu = u$.

By the same argument, we can prove that $gu = u$ and $hu = u$. Thus, we have $u = fu = gu = hu = Au = Bu = Cu$ so that f, g, h, A, B, and C have a common fixed point u in X. Let v be any other common fixed point of f, g, h, A, B, and C, then by using condition (2.71), we have

$$\begin{aligned} G(u,u,v) &= G(f^m u, g^m u, h^m v) \\ &\leq k\max\left\{\begin{matrix} G(Au,g^m u,g^m u)+G(Bu,f^m u,f^m u), \\ G(Bu,h^m v,h^m v)+G(Cv,g^m u,g^m u), \\ G(Cv,f^m u,f^m u)+G(Au,h^m v,h^m v) \end{matrix}\right\} \\ &= k\max\left\{\begin{matrix} G(u,u,u)+G(u,u,u), \\ G(u,v,v)+G(v,u,u), \\ G(v,u,u)+G(u,v,v) \end{matrix}\right\} \\ &= k[G(u,v,v)+G(v,u,u)] \\ &\leq 3kG(u,u,v). \end{aligned} \tag{2.75}$$

Hence, $G(u,u,v) = 0$ and so $u = v$, since $0 \leq k < 1/3$. Thus, common fixed point is unique.

The proof using (2.72) is similar. This completes the proof. □

In Theorem 2.7, if we take $A = B = C = I$, then we have the following corollary.

Corollary 2.8 (see [2, Corollary 2.5]). *Let (X,G) be a complete G-metric space, and let f, g, and h be three mappings of X into itself satisfying the following conditions:*

$$G(f^m x, g^m y, h^m z) \leq k\max\left\{\begin{matrix} G(x,g^m y,g^m y)+G(y,f^m x,f^m x), \\ G(y,h^m z,h^m z)+G(z,g^m y,g^m y), \\ G(z,f^m x,f^m x)+G(x,h^m z,h^m z) \end{matrix}\right\} \tag{2.76}$$

or

$$G(f^m x, g^m y, h^m z) \leq k\max\left\{\begin{matrix} G(x,x,g^m y)+G(y,y,f^m x), \\ G(y,y,h^m z)+G(z,z,g^m y), \\ G(z,z,f^m x)+G(x,x,h^m z) \end{matrix}\right\} \tag{2.77}$$

for all $x, y, z \in X$, *where* $k \in [0, 1/3)$, $m \in \mathbb{N}$; *then* f, g, *and* h *have a unique common fixed point in* X.

Also, if we take $f = g = h$ and $A = B = C = I$ in Theorem 2.7, then we get the following.

Corollary 2.9 (see [3, Corollary 2.5]). *Let* (X, G) *be a complete G-metric space, and let* f *be a mapping of* X *into itself satisfying the following conditions:*

$$G(f^m x, f^m y, f^m z) \leq k \max \left\{ \begin{array}{c} G(x, f^m y, f^m y) + G(y, f^m x, f^m x), \\ G(y, f^m z, f^m z) + G(z, f^m y, f^m y), \\ G(z, f^m x, f^m x) + G(x, f^m z, f^m z) \end{array} \right\} \tag{2.78}$$

or

$$G(f^m x, f^m y, f^m z) \leq k \max \left\{ \begin{array}{c} G(x, x, f^m y) + G(y, y, f^m x), \\ G(y, y, f^m z) + G(z, z, f^m y), \\ G(z, z, f^m x) + G(x, x, f^m z) \end{array} \right\} \tag{2.79}$$

for all $x, y, z \in X$, *where* $k \in [0, 1/3)$, $m \in \mathbb{N}$; *then* f *has a unique fixed point in* X.

Remark 2.10. In Theorem 2.7, if we take: (1) $f = g = h$; (2) $g = h$ and $B = C$; (3) $g = h$, $B = C = I$, several new results can be obtained.

Corollary 2.11. *Let* (X, G) *be a complete G-metric space, and let* f, g, h, A, B, *and* C *be six mappings of* X *into itself satisfying the following conditions:*

(i) $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$;

(ii) *for all* $x, y, z \in X$,

$$\begin{aligned} G(fx, gy, hz) \leq{} & a\{G(Ax, gy, gy) + G(By, fx, fx)\} \\ & + b\{G(By, hz, hz) + G(Cz, gy, gy)\} \\ & + c\{G(Cz, fx, fx) + G(Ax, hz, hz)\} \end{aligned} \tag{2.80}$$

or

$$\begin{aligned} G(fx, gy, hz) \leq{} & a\{G(Ax, Ax, gy) + G(By, By, fx)\} \\ & + b\{G(By, By, hz) + G(Cz, Cz, gy)\} \\ & + c\{G(Cz, Cz, fx) + G(Ax, Ax, hz)\}, \end{aligned} \tag{2.81}$$

where $0 \leq a + b + c < 1/3$. *Then one of the pairs* (f, A), (g, B), *and* (h, C) *has a coincidence point in* X. *Further, if one of the following conditions is satisfied, then the mappings* f, g, h, A, B, *and* C *have a unique common fixed point in* X.

(a) *Either* f *or* A *is G-continuous, the pair* (f, A) *is weakly commutative, the pairs* (g, B) *and* (h, C) *are weakly compatible;*

(b) *Either g or B is G-continuous, the pair (g, B) is weakly commutative, the pairs (f, A) and (h, C) are weakly compatible;*

(c) *Either h or C is G-continuous, the pair (h, C) is weakly commutative, the pairs (f, A) and (g, B) are weakly compatible.*

Proof. Suppose that mappings f, g, h, A, B, and C satisfy the condition (2.80). For $x, y, z \in X$, let

$$M(x,y,z) = \max\left\{\begin{array}{c} G(Ax,gy,gy)+G(By,fx,fx), \\ G(By,hz,hz)+G(Cz,gy,gy), \\ G(Cz,fx,fx)+G(Ax,hz,hz) \end{array}\right\}. \tag{2.82}$$

Then

$$\begin{aligned} a\{G(Ax,gy,gy)+G(By,fx,fx)\}&+b\{G(By,hz,hz)+G(Cz,gy,gy)\} \\ &+c\{G(Cz,fx,fx)+G(Ax,hz,hz)\} \\ &\leq (a+b+c)M(x,y,z). \end{aligned} \tag{2.83}$$

So, if

$$\begin{aligned} G(fx,gy,hz) \leq\ & a\{G(Ax,gy,gy)+G(By,fx,fx)\} \\ &+b\{G(By,hz,hz)+G(Cz,gy,gy)\} \\ &+c\{G(Cz,fx,fx)+G(Ax,hz,hz)\}, \end{aligned} \tag{2.84}$$

then $G(fx,gy,hz) \leq (a+b+c)M(x,y,z)$. Taking $k = a+b+c$ in Theorem 2.1, the conclusion of Corollary 2.11 can be obtained from Theorem 2.1 immediately.

The proof using (2.81) is similar. This completes the proof. □

Corollary 2.12. *Let (X, G) be a complete G-metric space, and let f, g, h, A, B, and C be six mappings of X into itself satisfying the following conditions:*

(i) $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$;

(ii) *the pairs (f, A), (g, B) and (h, C) are commutative mappings;*

(iii) *for all $x, y, z \in X$,*

$$\begin{aligned} G(f^m x, g^m y, h^m z) \leq\ & a\{G(Ax, g^m y, g^m y)+G(By, f^m x, f^m x)\} \\ &+b\{G(By, h^m z, h^m z)+G(Cz, g^m y, g^m y)\} \\ &+c\{G(Cz, f^m x, f^m x)+G(Ax, h^m z, h^m z)\} \end{aligned} \tag{2.85}$$

or

$$\begin{aligned} G(f^m x, g^m y, h^m z) \leq a\{&G(Ax, Ax, g^m y) + G(By, By, f^m x)\} \\ &+ b\{G(By, By, h^m z) + G(Cz, Cz, g^m y)\} \\ &+ c\{G(Cz, Cz, f^m x) + G(Ax, Ax, h^m z)\}, \end{aligned} \tag{2.86}$$

where $0 \leq a + b + c < 1/3$, $m \in \mathbb{N}$; *then* f, g, h, A, B, *and* C *have a unique common fixed point in* X.

Proof. The proof follows from Corollary 2.11, and from an argument similar to that used in Theorem 2.7. □

Remark 2.13. In Corollaries 2.11 and 2.12, if we take: (1) $A = B = C = I$; (2) $f = g = h$; (3) $A = B = C$; (4) $g = h$ and $B = C$; (5) $g = h$, $B = C = I$, several new results can be obtained.

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