

*Research Article*

# On Stable Perturbations of the Generalized Drazin Inverses of Closed Linear Operators in Banach Spaces

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We investigate the stable perturbation of the generalized Drazin inverses of closed linear operators in Banach spaces and obtain some new characterizations for the generalized Drazin inverses to have prescribed range and null space. As special cases of our results, we recover the perturbation theorems of Wei and Wang, Castro and Koliha, Rakocevic and Wei, Castro and Koliha and Wei.

## 1. Introduction and Preliminaries

Let  $X$  be a Banach spaces. Let  $L(X)$ ,  $C(X)$ , and  $B(X)$  denote the linear space of all linear operators, the homogeneous set of all densely defined closed linear operators and the Banach space of all bounded linear operators from  $X$  into itself, respectively. For any  $T \in L(X)$ , we denote by  $D(T)$ ,  $N(T)$ , and  $R(T)$  the domain, the null space, and, respectively, the range of  $T$ . The identity operator will be denoted by  $I$ .

As an important extension of the conventional Drazin inverse, the generalized Drazin inverse in Banach algebra was introduced firstly by Koliha [1]. Later, this notion was extended to closed linear operators by Koliha and Tran [2]. Recall that an operator  $S \in B(X)$  is said to be the generalized Drazin inverse of  $T \in C(X)$  if  $R(S) \subset D(T)$ ,  $R(I - TS) \subset D(T)$  and

$$ST = TS \quad \text{on} \quad D(T), \quad STS = S, \quad T(I - TS) \text{ is quasinilpotent.} \quad (1.1)$$

An operator  $T \in C(X)$  that possesses a generalized Drazin inverse is said to be generalized Drazin invertible, and its generalized Drazin inverse is denoted by  $T^d$ . From [3–5], we know that  $T$  is generalized Drazin invertible if and only if 0 is not an accumulation of  $\sigma(T)$ . In this case, we write  $T^\pi = I - TT^d$  for the spectral idempotent of  $T$ .

Perturbations of the conventional Drazin inverse and the generalized Drazin inverse were studied in many papers [3–13]. For example, Castro González and Koliha [3] introduced the concept of  $T$ -compatible and gave the stability theorem of generalized Drazin inverse under the inverse-compatible perturbation (Theorem 1.1). Let  $T \in C(X)$  be generalized Drazin invertible. An operator  $\delta T \in B(X)$  is called  $T$ -compatible if it commutes with  $T^\pi$  and if the operator  $T^\pi \delta T$  is quasinilpotent and commutes with  $T$ , that is,

$$T^\pi \delta T = \delta T T^\pi, \quad \sigma(T^\pi \delta T) = \{0\}, \quad T T^\pi \delta T = \delta T T T^\pi. \quad (1.2)$$

**Theorem 1.1** (see [3]). *Let  $T \in C(X)$  have the generalized Drazin inverse  $T^d \in B(X)$  and  $\delta T \in B(X)$  be  $T$ -compatible. If  $I + T^d \delta T : X \rightarrow X$  is invertible, then  $\bar{T} = T + \delta T$  is generalized Drazin invertible and  $\bar{T}^d = T^d (I + \delta T T^d)^{-1} = (I + T^d \delta T)^{-1} T^d$ .*

In this paper, we investigate the stable perturbation of the generalized Drazin inverses of closed linear operators in Banach spaces and obtain some new characterizations for the generalized Drazin inverses to have prescribed range and null space. As special cases of our results, we recover the perturbation theorems of Castro González and Koliha [3], Castro González et al. [4], Wei and Wang [6], Rakočević and Wei [7].

## 2. Main Results

First we will characterize the invertibility of  $I + \delta T T^d$ , which is inspired from [8].

**Lemma 2.1.** *Let  $T \in C(X)$  have a generalized Drazin inverse  $T^d \in B(X)$  and  $\delta T \in L(X)$  with  $D(T) \subset D(\delta T)$ . Then the following statements are equivalent:*

- (1)  $I + \delta T T^d : X \rightarrow X$  is bijective;
- (2)  $I + T^d \delta T : D(T) \rightarrow D(T)$  is bijective;
- (3)  $N(\bar{T}) \cap R(T^d) = \{0\}$  and  $X = \bar{T}R(T^d) + N(T^d)$ ,

where  $\bar{T} = T + \delta T$  and  $+$  denotes the algebraic direct sum.

*Proof.* (1)  $\Rightarrow$  (2). We first claim that  $I + T^d \delta T : D(T) \rightarrow D(T)$  is injective. In fact, if  $x \in D(T)$  satisfies  $(I + T^d \delta T)x = 0$ , then  $\delta T(I + T^d \delta T)x = 0$ , that is,

$$(I + \delta T T^d) \delta T x = 0. \quad (2.1)$$

Combining it with statement (1), we obtain  $\delta T x = 0$  and so  $x = -T^d \delta T x = 0$ . Next, we will show that  $I + T^d \delta T : D(T) \rightarrow D(T)$  is surjective, that is, for all  $q \in D(T)$ , we need to find a  $p \in D(T)$  such that  $(I + T^d \delta T)p = q$ . Let  $B = T^d(I + \delta T T^d)^{-1}$ , then

$$\begin{aligned} (I + T^d \delta T) B \delta T q &= (I + T^d \delta T) T^d (I + \delta T T^d)^{-1} \delta T q \\ &= T^d (I + \delta T T^d) (I + \delta T T^d)^{-1} \delta T q = T^d \delta T q \end{aligned} \quad (2.2)$$

and  $(I + T^d \delta T)(I - B \delta T)q = (I + T^d \delta T)q - T^d \delta T q = q$ . Set  $p = (I - B \delta T)q$ , then  $p \in D(T)$  and  $(I + T^d \delta T)p = q$ . Hence,  $I + T^d \delta T : D(T) \rightarrow D(T)$  is bijective.

(2)  $\Rightarrow$  (3). If  $I + T^d \delta T : D(T) \rightarrow D(T)$  is bijective, let  $x \in N(\bar{T}) \cap R(T^d)$ , then there exists  $y \in X$  satisfying  $x = T^d y$  and  $\bar{T} T^d y = \bar{T} x = 0$ . Hence

$$(I + T^d \delta T) T^d y = T^d y + T^d \delta T T^d y = T^d T T^d y + T^d \delta T T^d y = T^d \bar{T} T^d y = 0. \quad (2.3)$$

Thus  $T^d y = 0$ , that is,  $x = 0$ . Therefore,  $N(\bar{T}) \cap R(T^d) = \{0\}$ . Next we will show  $\bar{T} R(T^d) \cap N(T^d) = \{0\}$ . Let  $q \in \bar{T} R(T^d) \cap N(T^d)$ , then there exists  $p \in X$  with  $q = \bar{T} T^d p$ . Hence

$$\begin{aligned} 0 &= T^d q = T^d \bar{T} T^d p = T^d (T + \delta T) T^d p \\ &= T^d T T^d p + T^d \delta T T^d p = T^d p + T^d \delta T T^d p \\ &= (I + T^d \delta T) T^d p. \end{aligned} \quad (2.4)$$

Then  $T^d p = 0$  and  $q = \bar{T} T^d p = 0$ . Finally, we prove  $X = \bar{T} R(T^d) + N(T^d)$ . For all  $x \in X$ ,  $T^d x \in D(T)$ . Since  $I + T^d \delta T : D(T) \rightarrow D(T)$  is bijective, we can find  $y \in D(T)$  such that  $T^d x = (I + T^d \delta T)y$ , that is,  $T^d x = (I - T^d T)y + T^d \bar{T} y$ . Hence

$$T^d (x - \bar{T} y) = (I - T^d T)y \in R(T^d) \cap N(T^d T) = \{0\}. \quad (2.5)$$

Thus  $y = T^d T y \in R(T^d)$  and  $T^d x = T^d \bar{T} y$ , which implies  $x - \bar{T} y \in N(T^d)$ . Since  $x = \bar{T} y + (x - \bar{T} y) \in \bar{T} R(T^d) + N(T^d)$ , we obtain  $X = \bar{T} R(T^d) + N(T^d)$ .

(3)  $\Rightarrow$  (1). If  $N(\bar{T}) \cap R(T^d) = \{0\}$  and  $X = \bar{T} R(T^d) + N(T^d)$ , we first prove that  $I + \delta T T^d : X \rightarrow X$  is injective. In fact, if  $x \in X$  satisfies  $(I + \delta T T^d)x = 0$ , then

$$\bar{T} T^d x = T T^d x - x \in \bar{T} R(T^d) \cap N(T^d) = \{0\}, \quad (2.6)$$

which means  $x = T T^d x$  and  $\bar{T} T^d x = 0$ . Thus  $T^d x \in N(\bar{T}) \cap R(T^d)$ , that is,  $T^d x = 0$ . Hence  $x = T T^d x = 0$ . Next, we will show that  $I + \delta T T^d : X \rightarrow X$  is surjective. In fact, for all

$y \in X$ , since  $X = \bar{T}R(T^d) + N(T^d)$ ,  $y$  can be expressed by  $y = \bar{T}T^d y_1 + y_2$ , where  $y_1 \in X$  and  $y_2 \in N(T^d)$ . Hence

$$\begin{aligned} (I + \delta TT^d)(TT^d y_1 + y_2) &= (TT^d y_1 + y_2) + (\bar{T} - T)T^d TT^d y_1 \\ &= TT^d y_1 + y_2 + \bar{T}T^d y_1 - TT^d y_1 \\ &= y_2 + \bar{T}T^d y_1 = y. \end{aligned} \quad (2.7)$$

Therefore,  $I + \delta TT^d : X \rightarrow X$  is bijective.  $\square$

**Lemma 2.2.** Let  $T \in C(X)$  have the generalized Drazin inverse  $T^d \in B(X)$ , and let  $\delta T \in L(X)$  satisfy  $D(T) \subset D(\delta T)$  and  $\bar{T} = T + \delta T \in C(X)$ , then  $I + \delta TT^d \in B(X)$ . Further, if  $I + \delta TT^d : X \rightarrow X$  is bijective, then its inverse  $(I + \delta TT^d)^{-1} \in B(X)$  and

$$B = T^d (I + \delta TT^d)^{-1} = (I + T^d \delta T)^{-1} T^d : X \rightarrow X \quad (2.8)$$

is a well-defined bounded linear operator with  $R(B) = R(T^d)$  and  $N(B) = N(T^d)$ .

*Proof.* Since  $T^d$  is bounded and  $\bar{T}$  is closed, by the Closed Operator Theorem, we know that  $\bar{T}T^d$  is bounded and so is  $I + \delta TT^d = I - TT^d + \bar{T}T^d$ . Further, if  $I + \delta TT^d : X \rightarrow X$  is bijective, it follows from the Inverse Operator Theorem that  $(I + \delta TT^d)^{-1} \in B(X)$ . By Lemma 2.1 and  $(I + T^d \delta T)T^d = T^d(I + \delta TT^d)$ , we can see that

$$B = T^d (I + \delta TT^d)^{-1} = (I + T^d \delta T)^{-1} T^d : X \rightarrow R(T^d) \quad (2.9)$$

is a well-defined bounded linear operator. Easily, we can obtain  $R(B) = R(T^d)$  and  $N(B) = N(T^d)$ .  $\square$

The following theorem points out that the generalized Drazin inverse with the prescribed range and null space must possess the expression  $T^d(I + \delta TT^d)^{-1}$ .

**Theorem 2.3.** Let  $T \in C(X)$  have the generalized Drazin inverse  $T^d \in B(X)$ , and let  $\delta T \in L(X)$  satisfy  $D(T) \subset D(\delta T)$  and  $\bar{T} = T + \delta T \in C(X)$ . Then the following statements are equivalent:

- (1)  $\bar{T}$  is generalized Drazin invertible with  $\bar{T}^\pi = T^\pi$ ;
- (2)  $\bar{T}$  is generalized Drazin invertible with  $N(\bar{T}^d) = N(T^d)$  and  $R(\bar{T}^d) = R(T^d)$ ;
- (3)  $I + \delta TT^d : X \rightarrow X$  is bijective and

$$B = T^d (I + \delta TT^d)^{-1} = (I + T^d \delta T)^{-1} T^d \quad (2.10)$$

is the generalized Drazin inverse of  $\bar{T}$ ;

- (4)  $\bar{T}$  is generalized Drazin invertible with  $T^d - \bar{T}^d = T^d \delta T \bar{T}^d = \bar{T}^d \delta TT^d$ ;
- (5)  $\bar{T}$  is generalized Drazin invertible with  $T^d - \bar{T}^d = \bar{T}^d \delta TT^d$ .

*Proof.* It is easy to see (1)  $\Rightarrow$  (2) and (4)  $\Rightarrow$  (5).

(2)  $\Rightarrow$  (3). Suppose that  $\bar{T}$  is generalized Drazin invertible with  $N(\bar{T}^d) = N(T^d)$  and  $R(\bar{T}^d) = R(T^d)$ . We first show that  $I + \delta TT^d : X \rightarrow X$  is bijective. From Lemma 2.1, it suffices to show  $N(\bar{T}) \cap R(T^d) = \{0\}$  and  $X = \bar{T}R(T^d) + N(T^d)$ . Since  $\bar{T}^d$  is the generalized Drazin inverse of  $\bar{T}$ , we can obtain  $N(\bar{T}) \cap R(\bar{T}^d) = \{0\}$  and  $X = R(\bar{T}\bar{T}^d) \oplus N(\bar{T}\bar{T}^d) = \bar{T}R(\bar{T}^d) \oplus N(\bar{T}^d)$ . Hence

$$N(\bar{T}) \cap R(T^d) = \{0\}, \quad X = \bar{T}R(T^d) \oplus N(T^d). \quad (2.11)$$

Next, by  $N(\bar{T}^d) = N(T^d)$  and  $R(\bar{T}^d) = R(T^d)$ , we can obtain  $\bar{T}^d(I - TT^d) = 0$  and  $(\bar{T}^d\bar{T} - I)T^d = 0$ , that is,

$$\bar{T}^d = \bar{T}^d TT^d, \quad \bar{T}^d \bar{T} T^d = T^d. \quad (2.12)$$

Then  $\bar{T}^d + \bar{T}^d \bar{T} T^d - \bar{T}^d TT^d = T^d$ , that is,

$$\bar{T}^d (I + \delta TT^d) = \bar{T}^d [I + (\bar{T} - T)T^d] = T^d. \quad (2.13)$$

Therefore,  $\bar{T}^d = T^d (I + \delta TT^d)^{-1}$ . By Lemma 2.2, we have  $\bar{T}^d = (I + T^d \delta T)^{-1} T^d$ .

(3)  $\Rightarrow$  (4). Since the generalized Drazin inverse is unique, we get

$$\bar{T}^d = T^d (I + \delta TT^d)^{-1} = (I + T^d \delta T)^{-1} T^d. \quad (2.14)$$

Then  $\bar{T}^d (I + \delta TT^d) = T^d$  and  $(I + T^d \delta T) \bar{T}^d = T^d$ . Hence

$$T^d - \bar{T}^d = \bar{T}^d \delta TT^d = T^d \delta T \bar{T}^d. \quad (2.15)$$

(5)  $\Rightarrow$  (1). We only need to prove  $\overline{TT}^d = TT^d$ . In fact, by  $T^d - \bar{T}^d = \bar{T}^d \delta TT^d$ , we obtain  $T^d = \bar{T}^d (I + \delta TT^d)$  and  $\bar{T}^d = (I - \bar{T}^d \delta T) T^d$ . Hence

$$\begin{aligned} \bar{T}^d \bar{T} T^d &= \bar{T}^d \bar{T} \bar{T}^d (I + \delta TT^d) = \bar{T}^d (I + \delta TT^d) = T^d, \\ \bar{T}^d T T^d &= (I - \bar{T}^d \delta T) T^d T T^d = (I - \bar{T}^d \delta T) T^d = \bar{T}^d. \end{aligned} \quad (2.16)$$

Thus

$$\overline{TT}^d = \overline{\bar{T} T T^d} = \overline{\bar{T} T^d T} = \bar{T}^d \bar{T} T^d T = T^d T = TT^d \quad \text{on } D(T). \quad (2.17)$$

Since  $D(T)$  is dense in  $X$ ,  $\overline{TT}^d$  and  $TT^d$  are bounded, we can get  $\overline{TT}^d = TT^d$  and so  $\bar{T}^\pi = I - \overline{\bar{T} T^d} = I - TT^d = T^\pi$ .  $\square$

*Remark 2.4.* To guarantee  $\bar{T} \in C(X)$ , a general condition of  $T$  boundedness is often assumed, see [14].

**Theorem 2.5.** *Let  $T \in C(X)$  have the generalized Drazin inverse  $T^d \in B(X)$  and  $\delta T \in L(X)$  satisfy  $D(T) \subset D(\delta T)$  and  $\bar{T} = T + \delta T \in C(X)$ . Assume that  $I + \delta TT^d : X \rightarrow X$  is bijective, then the following statements are equivalent:*

(1)

$$B = T^d(I + \delta TT^d)^{-1} = (I + T^d\delta T)^{-1}T^d \quad (2.18)$$

is the generalized Drazin inverse of  $\bar{T}$ ;

- (2)  $\bar{T}T^d = TT^d\bar{T}T^d$ ,  $T^d\bar{T} = T^d\bar{T}T^dT$  and  $\lim_{n \rightarrow +\infty} \|\bar{T}^n(I - T^dT)\|^{1/n} = 0$ ;  
 (3)  $\bar{T}T^dT = TT^d\bar{T}$  and  $\lim_{n \rightarrow +\infty} \|\bar{T}^n(I - T^dT)\|^{1/n} = 0$ ;  
 (4)  $\bar{T}T^dT = TT^d\bar{T}$  and  $\bar{T}T^x$  is quasinilpotent;  
 (5)  $\bar{T}T^d = TT^d\bar{T}T^d$ ,  $T^d\bar{T} = T^d\bar{T}T^dT$  and  $\bar{T}T^x$  is quasinilpotent.

In this case,  $\bar{T}^x = T^x$  and

$$\|\bar{T}^d - T^d\| \leq \|T^d\| \cdot \|\delta TT^d\| \cdot \|(I + \delta TT^d)^{-1}\|. \quad (2.19)$$

*Proof.* (1)  $\Rightarrow$  (2). If  $B$  is the generalized Drazin inverse of  $\bar{T}$ , then by Theorem 2.3, we get  $\bar{T}^x = T^x$  and so  $\bar{T}B = TT^d$  and  $B\bar{T} = T^dT$ . Hence  $\bar{T}T^d = TT^d[I + (\bar{T} - T)T^d] = TT^d\bar{T}T^d$  and  $T^d\bar{T} = [I + T^d(\bar{T} - T)]T^dT = T^d\bar{T}T^dT$ . Also, for all  $x \in D(\bar{T})$ ,

$$\begin{aligned} [\bar{T}(I - \bar{T}B)]^2 x &= [\bar{T}(I - \bar{T}B)\bar{T}(I - \bar{T}B)]x \\ &= \bar{T}(I - \bar{T}B)\bar{T}(I - B\bar{T})x \\ &= \bar{T}\bar{T}(I - B\bar{T})(I - B\bar{T})x \\ &= \bar{T}^2(I - B\bar{T})x \\ &= \bar{T}^2(I - T^dT)x. \end{aligned} \quad (2.20)$$

Thus we obtain  $\bar{T}(I - B\bar{T})x = \bar{T}x - \bar{T}B\bar{T}x = (I - \bar{T}B)\bar{T}x \in D(\bar{T})$  and

$$\begin{aligned} [\bar{T}(I - \bar{T}B)]^3 x &= [\bar{T}(I - \bar{T}B)]^2 [\bar{T}(I - \bar{T}B)]x \\ &= [\bar{T}(I - \bar{T}B)]^2 [\bar{T}(I - B\bar{T})]x \end{aligned}$$

$$\begin{aligned}
&= \bar{T}^2 (I - B\bar{T}) \bar{T} (I - B\bar{T}) x && \text{(by (2.20))} \\
&= \bar{T}^2 (I - \bar{T}B) \bar{T} (I - B\bar{T}) x \\
&= \bar{T}^3 (I - B\bar{T}) (I - B\bar{T}) x \\
&= \bar{T}^3 (I - B\bar{T}) x \\
&= \bar{T}^3 (I - T^d T) x.
\end{aligned} \tag{2.21}$$

Hence for all  $n \in \mathbb{N}$ ,

$$[\bar{T}(I - \bar{T}B)]^n = \bar{T}^n (I - T^d T) \quad \text{on } D(T). \tag{2.22}$$

Therefore  $\lim_{n \rightarrow +\infty} \|\bar{T}^n (I - T^d T)\|^{1/n} = \lim_{n \rightarrow +\infty} \|[\bar{T}(I - \bar{T}B)]^n\|^{1/n} = 0$ .

(2)  $\Rightarrow$  (1). It follows from Lemma 2.2 that  $B = T^d (I + \delta T T^d)^{-1} = (I + T^d \delta T)^{-1} T^d$  is a well-defined bounded linear operator with  $R(B) = R(T^d) \subset D(T) = D(\bar{T})$ . By  $\bar{T} T^d = T T^d \bar{T} T^d$  and  $T^d \bar{T} = T^d \bar{T} T^d T$ , we can verify

$$\begin{aligned}
\bar{T}B &= \bar{T} T^d (I + \delta T T^d)^{-1} \\
&= T T^d \bar{T} T^d (I + \delta T T^d)^{-1} \\
&= T T^d (T + \delta T) T^d (I + \delta T T^d)^{-1} \\
&= (T T^d + T T^d \delta T T^d) (I + \delta T T^d)^{-1} \\
&= T T^d (I + \delta T T^d) (I + \delta T T^d)^{-1} = T T^d
\end{aligned} \tag{2.23}$$

and by Lemma 2.1,

$$\begin{aligned}
B\bar{T} &= (I + T^d \delta T)^{-1} T^d \bar{T} \\
&= (I + T^d \delta T)^{-1} T^d \bar{T} T^d T \\
&= (I + T^d \delta T)^{-1} T^d (T + \delta T) T^d T \\
&= (I + T^d \delta T)^{-1} (T^d T + T^d \delta T) T^d T \\
&= (I + T^d \delta T)^{-1} (I + T^d \delta T) T^d T = T^d T.
\end{aligned} \tag{2.24}$$

Thus  $\bar{T}B = B\bar{T}$  on  $D(\bar{T})$ . Obviously,  $R(I - \bar{T}B) = R(I - TT^d) \subset D(T) = D(\bar{T})$ . Also, we can see

$$\begin{aligned}
B\bar{T}B &= T^d(I + \delta TT^d)^{-1} \bar{T}T^d(I + \delta TT^d)^{-1} \\
&= T^d(I + \delta TT^d)^{-1} (T + \delta T)T^d(I + \delta TT^d)^{-1} \\
&= T^d(I + \delta TT^d)^{-1} (TT^d + \delta TT^d TT^d) (I + \delta TT^d)^{-1} \\
&= T^d(I + \delta TT^d)^{-1} (I + \delta TT^d) TT^d (I + \delta TT^d)^{-1} \\
&= T^d(I + \delta TT^d)^{-1} = B.
\end{aligned} \tag{2.25}$$

To the aim, we need show  $\bar{T}(I - \bar{T}B)$  is quasinilpotent. In fact, as in (1)  $\Rightarrow$  (2), we can obtain (2.22) and so

$$\lim_{n \rightarrow +\infty} \left\| \left[ \bar{T}(I - \bar{T}B) \right]^n \right\|^{1/n} = \lim_{n \rightarrow +\infty} \left\| \bar{T}^n (I - T^d T) \right\|^{1/n} = 0. \tag{2.26}$$

Therefore,  $B$  is the generalized Drazin inverse of  $\bar{T}$ .

(2)  $\Leftrightarrow$  (3). If  $\bar{T}T^d = TT^d \bar{T}T^d$  and  $T^d \bar{T} = T^d \bar{T}T^d T$ , then

$$\bar{T}T^d T = TT^d \bar{T}T^d T = TT^d \bar{T}. \tag{2.27}$$

Conversely, if  $\bar{T}T^d T = TT^d \bar{T}$ , then

$$\bar{T}T^d = \bar{T}T^d TT^d = TT^d \bar{T}T^d, \quad T^d \bar{T} = T^d TT^d \bar{T} = T^d \bar{T}T^d T. \tag{2.28}$$

(3)  $\Leftrightarrow$  (4). It follows from  $\bar{T}T^d T = TT^d \bar{T}$  that

$$\begin{aligned}
(\bar{T}T^x)^2 &= \bar{T}T^x \bar{T}T^x = \bar{T}(I - TT^d) \bar{T}T^x \\
&= \bar{T}(\bar{T} - TT^d \bar{T})T^x = \bar{T}(\bar{T} - \bar{T}T^d T)T^x \\
&= \bar{T}^2(I - T^d T)T^x = \bar{T}^2(I - T^d T) \quad \text{on } D(T)
\end{aligned} \tag{2.29}$$

and then for all  $n \in \mathbb{N}$ ,

$$(\bar{T}T^x)^n = \bar{T}^n (I - T^d T) \quad \text{on } D(T). \tag{2.30}$$

Therefore,  $\bar{T}T^x$  is quasinilpotent if and only if  $\lim_{n \rightarrow +\infty} \left\| \bar{T}^n (I - T^d T) \right\|^{1/n} = 0$ .

(4)  $\Leftrightarrow$  (5). It is similar to (2)  $\Leftrightarrow$  (3). In this case,

$$\begin{aligned} \|B - T^d\| &= \left\| T^d (I + \delta T T^d)^{-1} - T^d \right\| \\ &\leq \left\| T^d \delta T T^d (I + \delta T T^d)^{-1} \right\| \\ &\leq \|T^d\| \cdot \|\delta T T^d\| \cdot \left\| (I + \delta T T^d)^{-1} \right\|. \end{aligned} \tag{2.31}$$

□

*Remark 2.6.* Theorems 2.3 and 2.5 generalize the main results in [3–8, 10, 12, 13].

As applications, we recover some known results in [3, 6, 7].

**Corollary 2.7** (see [3]). *Let  $T \in C(X)$  have the generalized Drazin inverse  $T^d \in B(X)$  and  $\delta T \in B(X)$  be  $T$ -compatible. If  $I + T^d \delta T : X \rightarrow X$  is invertible, then  $\bar{T} = T + \delta T$  is generalized Drazin invertible and  $\bar{T}^d = T^d (I + \delta T T^d)^{-1} = (I + T^d \delta T)^{-1} T^d$ .*

*Proof.* It is easy to see  $\bar{T} \in C(X)$  and  $\bar{T} T^d T = T T^d \bar{T}$ . Since

$$\bar{T} T^\pi = T T^\pi + \delta T T^\pi \tag{2.32}$$

is the sum of two commuting quasinilpotent operators, we know that  $\bar{T} T^\pi$  is quasinilpotent. Hence by Theorem 2.5, we can get what we desired. □

**Corollary 2.8.** *Let  $T \in C(X)$  have the generalized Drazin inverse  $T^d \in B(X)$  and  $\delta T \in L(X)$  satisfy  $D(T) \subset D(\delta T)$  and  $\bar{T} = T + \delta T \in C(X)$ . Assume that  $I + \delta T T^d : X \rightarrow X$  is bijective and*

$$\delta T = T T^d \delta T = \delta T T^d T, \tag{2.33}$$

*then  $B = T^d (I + \delta T T^d)^{-1} = (I + T^d \delta T)^{-1} T^d$  is the generalized Drazin inverse of  $\bar{T} = T + \delta T$ .*

*Proof.* If  $\delta T = T T^d \delta T = \delta T T^d T$ , then  $\bar{T} T^d T = T T^d \bar{T}$  and  $\bar{T} T^\pi = T T^\pi$  is quasinilpotent. From Theorem 2.5,  $B$  is the generalized Drazin inverse of  $\bar{T}$ . □

*Remark 2.9.* The condition  $\delta T = T T^D \delta T = \delta T T^D T$  is called the condition (W) in [6, 7], which appear as a sufficient condition to ensure that  $T^D (I + \delta T T^D)^{-1}$  is the Drazin inverse of  $\bar{T}$ .

The following theorem shows that the above condition (W) is exactly the stable characterization of group inverse. Recall that an operator  $S \in B(X)$  is said to be the group inverse of  $T \in C(X)$  if  $S$  satisfies  $R(S) \subset D(T)$  and

$$T S T = T; \quad S T S = S, \quad S T = T S \quad \text{on} \quad D(T). \tag{2.34}$$

The group inverse of  $T$  is unique if it exists, which is always denoted by  $T^\#$ .

**Theorem 2.10.** Let  $T \in C(X)$  have the group inverse  $T^\# \in B(X)$  and let  $\delta T \in L(X)$  satisfy  $D(T) \subset D(\delta T)$  and  $\bar{T} = T + \delta T \in C(X)$ . Assume that  $I + \delta T T^\# : X \rightarrow X$  is bijective, then

$$B = T^\# (I + \delta T T^\#)^{-1} = (I + T^\# \delta T)^{-1} T^\# \quad (2.35)$$

is the group inverse of  $\bar{T}$  if and only if

$$\delta T = T T^\# \delta T = \delta T T^\# T. \quad (2.36)$$

In this case,  $\bar{T}^x = T^x$ ,  $R(\bar{T}) = R(T)$  and  $N(\bar{T}) = N(T)$ .

*Proof.* If  $\delta T = T T^\# \delta T = \delta T T^\# T$ , then  $\bar{T} = \bar{T} T^\# T = T T^\# \bar{T}$  and

$$\bar{T} T^x = \bar{T} (I - T T^\#) = \bar{T} (I - T^\# T) = 0 \quad \text{on } D(T). \quad (2.37)$$

Hence  $\bar{T} T^x = 0$ . And by (4) in Theorem 2.5, we know that  $B$  is the generalized Drazin inverse of  $\bar{T}$  with  $\bar{T} B = T T^\#$  and  $B \bar{T} = T^\# T$ . Hence  $\bar{T} B \bar{T} = T T^\# \bar{T} = \bar{T}$ . Therefore,  $B$  is also the group inverse of  $\bar{T}$ . Conversely, if  $B$  is the group inverse of  $\bar{T}$ , then

$$R(\bar{T}) = R(\bar{T} B) = \overline{R(B \bar{T})} = \overline{R(T^\# T)} = R(T T^\#) = R(T), \quad (2.38)$$

$$N(\bar{T}) = N(B \bar{T}) = N(\bar{T} B) \cap D(\bar{T}) = N(T T^\#) \cap D(T) = N(T^\# T) = R(I - T^\# T). \quad (2.39)$$

Hence

$$\bar{T} - T T^\# \bar{T} = (I - T T^\#) \bar{T} = 0, \quad (2.40)$$

$$\bar{T} - \bar{T} T^\# T = \bar{T} (I - T^\# T) = 0. \quad (2.41)$$

Thus  $\bar{T} = T T^\# \bar{T} = \bar{T} T^\# T$ , that is,  $\delta T = T T^\# \delta T = \delta T T^\# T$ . □

The following corollary extends the Proposition 4.2 in [3] to the case of closed linear operators. Recall that  $T \in C(X)$  is an EP (Equal Projections) operator if  $T$  has the Moore-Penrose inverse  $T^\dagger \in B(X)$  satisfying  $T^\dagger T = T T^\dagger$  on  $D(T)$ .

**Corollary 2.11.** Let  $X$  be a Hilbert space and  $T \in C(X)$  be an EP operator. Let  $\delta T \in L(X)$  satisfy  $D(T) \subset D(\delta T)$  and  $\bar{T} = T + \delta T \in C(X)$ . Assume that  $I + \delta T T^\dagger : X \rightarrow X$  is bijective, then  $\bar{T}$  is also an EP operator and

$$B = T^\dagger (I + \delta T T^\dagger)^{-1} = (I + T^\dagger \delta T)^{-1} T^\dagger \quad (2.42)$$

is its Moore-Penrose inverse if and only if

$$\delta T = TT^+\delta T = \delta TT^+T. \quad (2.43)$$

In this case,  $B\bar{T} = \bar{T}B$  on  $D(\bar{T})$ ,  $\bar{T}^\pi = T^\pi$ ,  $R(\bar{T}) = R(T)$  and  $N(\bar{T}) = N(T)$ .

*Proof.* If  $\bar{T}$  is an EP operator and  $B$  is its Moore-Penrose inverse, then  $B$  is also its group inverse. By Theorem 2.10, we get  $\delta T = TT^+\delta T = \delta TT^+T$ . Conversely, if  $\delta T = TT^+\delta T = \delta TT^+T$ , then by Theorem 2.10 again,  $B$  is the group inverse of  $\bar{T}$ ,  $\bar{T}B = TT^+$  and  $B\bar{T} = T^+T$ . Therefore,  $B$  is the Moore-Penrose inverse of  $\bar{T}$ .  $\square$

To illustrate our results, we give an example of a closed EP operator.

*Example 2.12* (see [15]). Let

$$L^2[0,1] = \left\{ f, f \text{ measurable complex-valued on } [0,1], \int_{[0,1]} |f(x)|^2 dx < \infty \right\} \quad (2.44)$$

be the Hilbert space with the inner product:

$$\langle f, g \rangle = \int_{[0,1]} \overline{f(x)}g(x)dx, \quad f, g \in L^2[0,1]. \quad (2.45)$$

Set  $t : [0,1] \rightarrow \mathbb{C}$  by

$$t(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{\sqrt{x}} & 0 < x \leq 1 \end{cases} \quad (2.46)$$

and define the maximal operator of multiplication  $T$  by  $t$  on  $L^2[0,1]$ , that is,

$$Tf = tf, \quad \text{for } f \in D(T) = \left\{ f \in L^2[0,1], tf \in L^2[0,1] \right\}, \quad (2.47)$$

then  $T$  is a densely defined closed linear operator [16]. Since  $|t(x)| \geq 1$ , for all  $x \in [0,1]$ ,  $R(T) = L^2[0,1]$  and  $T$  has a bounded inverse  $T^{-1} : L^2[0,1] \rightarrow L^2[0,1]$  defined by

$$T^{-1}g = t_1g, \quad \forall g \in L^2[0,1], \quad (2.48)$$

where

$$t_1(x) = \begin{cases} 1 & x = 0, \\ \sqrt{x} & 0 < x \leq 1. \end{cases} \quad (2.49)$$

Therefore,  $T$  is a closed operator on  $L^2[0,1]$  with a bounded inverse  $T^{-1}$ , which is both the Moore-Penrose inverse and the group inverse of  $T$ .

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