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Homogeneous Fourier multipliers of Marcinkiewicz type

Anthony Carbery and Andreas Seeger(1)

1. Introduction

Let $m\in L^\infty(\mathbf{R}^2)$ be homogeneous of degree zero. Then m is almost everywhere determined by $h_\pm(\xi_1)=m(\xi_1,\pm1)$. For $k\in\mathbf{Z}$ let $I_k=[2^{-k-1},2^{-k}]\cup[-2^{-k},-2^{-k-1}]$ and let h_+ and h_- satisfy the condition

$$\sup_{k\in\mathbf{Z}}\left(\int_{I_k}|sh'_\pm(s)|^r\frac{ds}{s}\right)^{1/r}<\infty. \tag{1.1}$$

Rubio de Francia posed the question whether a condition like (1.1) is sufficient to prove that m is a Fourier multiplier of $L^p(\mathbf{R}^2)$, $1<p<\infty$. An application of the Marcinkiewicz multiplier theorem with L^2-Sobolev hypotheses (cf. (1.3) and (1.5) below) and interpolation arguments already show that the answer is yes, provided $r>2$. Recently, Duoandikoetxea and Moyua [15] have shown that the same conclusion can be reached if $r=2$. On the other hand, since characteristic functions of halfspaces are Fourier multipliers of L^p, $1<p<\infty$, a simple averaging argument shows that the condition $h'\in L^1$ implies L^p-boundedness for $1<p<\infty$. Our first theorem shows that the weaker assumption (1.1) with $r=1$ implies boundedness in $L^p(\mathbf{R}^2)$, for $1<p<\infty$.

Theorem 1.1. *Suppose that h_+ and h_- satisfy the hypotheses of the Marcinkiewicz multiplier theorem on the real line, that is*

$$\sup_{k\in\mathbf{Z}}\int_{I_k}|dh_\pm(s)|\le A \tag{1.2}$$

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for $I_k=[2^{-k-1},2^{-k}]\cup[-2^{-k},-2^{-k-1}]$. *Let* $m\in L^\infty(\mathbf{R}^2)$ *be homogeneous of degree zero, such that for* $\xi_1\in\mathbf{R}$, $m(\xi_1,1)=h_+(\xi_1)$ *and* $m(\xi_1,-1)=h_-(\xi_1)$. *Then* m *is a Fourier multiplier of* $L^p(\mathbf{R}^2)$, $1<p<\infty$, *with norm* $\le CA$.

One can obtain a stronger result for fixed $p>1$ using the space V^q of functions of bounded q-variation. Given an interval I on the real line a function h belongs to $V^q(I)$ if for each partition $\{x_0<x_1<\ldots<x_N\}$ of I the sum $\sum_{\nu=1}^N |h(x_\nu)-h(x_{\nu-1})|^q$ is bounded and the upper bound of such sums is finite. We denote by $\|h\|_{V^q}^q$ the least upper bound. Then the following result is an immediate consequence of Theorem 1.1 and the interpolation argument in [8].

Corollary 1.2. *Let* m, $h_\pm$ *and* I_k *be as above and suppose that*

$$\|h_\pm\|_\infty+\sup_k \|h_\pm\|_{V^q(I_k)}<\infty.$$

Then m *is a Fourier multiplier of* $L^p(\mathbf{R}^2)$, *if* $|1/p-\frac{1}{2}|<1/2q$.

A slightly weaker result can be formulated in terms of Sobolev spaces. Let β be an even C^∞ function on the real line, supported in $\left(\frac{5}{8},\frac{8}{5}\right)\cup\left(-\frac{8}{5},-\frac{5}{8}\right)$ and positive in $(1/\sqrt{2},\sqrt{2})\cup(-\sqrt{2},-1/\sqrt{2})$; we shall assume that $\sum_{k\in\mathbf{Z}}\beta^2(2^k s)=1$ for $s\neq 0$. Let $L^q_\alpha(\mathbf{R}^d)$ denote the standard Sobolev space with norm $\|h\|_{L^q_\alpha}=\|\mathcal{F}^{-1}[(1+|\xi|^2)^{\alpha/2}\hat{h}]\|_q$. Then $L^q_\alpha(\mathbf{R})\subset V_q$ if $\alpha>1/q$ and therefore we obtain

Corollary 1.3. *Let* $m\in L^\infty(\mathbf{R}^2)$ *be homogeneous of degree zero and* $h_\pm(\xi_1)=m(\xi_1,\pm 1)$. *Suppose that* $q>1$ *and that*

$$\sup_{t\in\mathbf{R}_+}\|\beta\, h_\pm(t\cdot)\|_{L^q_\alpha(\mathbf{R})}<\infty,\quad \alpha>\frac{1}{q}. \tag{1.3}$$

Then m *is a Fourier multiplier of* $L^p(\mathbf{R}^2)$ *if* $|1/p-\frac{1}{2}|<1/2q$.

We now compare these results with more standard multiparameter versions of the Hörmander–Marcinkiewicz multiplier theorem. In order to formulate them let

$$\mathcal{D}_j^\alpha g=\mathcal{F}^{-1}[(1+|\xi_j|^2)^{\alpha/2}\mathcal{F}g]$$

and, for $1<q<\infty$, let $\mathcal{H}^q_\alpha(\mathbf{R}^n)$ be the *multiparameter* Sobolev space of all functions g, such that

$$\|g\|_{\mathcal{H}^q_\alpha(\mathbf{R}^n)}:=\|\mathcal{D}_1^\alpha\ldots\mathcal{D}_n^\alpha g\|_{L^q(\mathbf{R}^n)}<\infty.$$

Let β be as above and denote by $\beta_{(i)}$ a copy of β as a function of the ξ_i-variable. Then if $q\ge 2$ the condition

$$\sup_{t\in(\mathbf{R}_+)^d}\|\beta_{(1)}\otimes\ldots\otimes\beta_{(d)}m(t_1\cdot,\ldots,t_d\cdot)\|_{\mathcal{H}^q_\alpha(\mathbf{R}^d)}<\infty,\quad \alpha>\frac{1}{q} \tag{1.4}$$

implies that m is a Fourier multiplier of L^p for $|1/p-\frac{1}{2}|<1/q$. For $q=2$ the proof of this result is a variant of Stein's proof of the Hörmander multiplier theorem (see [25, Ch. IV]) and the general case follows by an interpolation argument as in [9]. If we apply this result to homogeneous multipliers and set

$$m(\xi',\pm 1)=g_\pm(\xi'),\quad \xi'\in\mathbf{R}^{d-1} \tag{1.5}$$

we obtain by a straightforward computation

Corollary 1.4. *Suppose that $r\geq 2$,*

$$\sup_{t\in(\mathbf{R}_+)^{d-1}}\left\|\mathcal{D}_1^{2\gamma}\mathcal{D}_2^{\gamma}\dots\mathcal{D}_{d-1}^{\gamma}\left[\beta_{(1)}\otimes\dots\otimes\beta_{(d-1)}g_\pm(t_1\,\cdot,\dots,t_{d-1}\,\cdot)\right]\right\|_{L^r(\mathbf{R}^{d-1})}<\infty,\ \gamma>\frac{1}{r}, \tag{1.6}$$

and that the condition analogous to (1.6) *holds for all permutations of the variables* $(s_1,\dots,s_{d-1})$. *Let m be homogeneous of degree zero and related to $g_\pm$ by* (1.5). *Then m is a Fourier multiplier of $L^p(\mathbf{R}^d)$ if $|1/p-\frac{1}{2}|<1/r$.*

In two dimensions Corollary 1.4 says that if $\alpha>1/q$, $q\geq 1$, and $\beta g_\pm(t\,\cdot)\in\mathcal{H}_\alpha^{2q}(\mathbf{R})$, uniformly in $t>0$, then m is a Fourier multiplier of L^p if $|1/p-\frac{1}{2}|<1/2q$. Corollary 1.3 is stronger since a compactly supported function in $H_\alpha^{2q}(\mathbf{R})$ belongs to $H_\alpha^q(\mathbf{R})$.

We are now going to discuss variants of Theorem 1.1 in higher dimensions. First if $g_\pm\in\mathcal{H}_\alpha^q(\mathbf{R}^{d-1})$, $\alpha>1/q$ and if $g_\pm$ are compactly supported in $\left[\frac{1}{2},2\right]^{d-1}$ then the homogeneous extension m is a Fourier multiplier of $L^p(\mathbf{R}^d)$ if $|1/p-\frac{1}{2}|<1/2q$. In fact by a simple averaging argument one sees that the condition $g_\pm\in\mathcal{H}_{1+\varepsilon}^1$ implies that m is an L^1 multiplier and the general case follows by interpolation. We remark that if $\alpha<|2/p-1|$ the condition $g_\pm\in\mathcal{H}_\alpha^q$ (any q) does not imply that m is a Fourier multiplier of L^p. Relevant counterexamples have been pointed out by López–Melero [22] and Christ [7].

Perhaps surprisingly, the situation in higher dimensions changes if one imposes dilation invariant conditions as in Theorem 1.1. One might want to just replace hypothesis (1.2) by the hypotheses of the Marcinkiewicz multiplier theorem in $\mathbf{R}^{d-1}$ ([25, p. 108]). However this assumption is not sufficient to deduce that m is a Fourier multiplier of L^p for any $p\neq 2$ (see Section 3 for the counterexample involving the Kakeya set). However, we do have

Theorem 1.5. *Let $m\in L^\infty(\mathbf{R}^d)$, $d\geq 2$, be homogeneous of degree zero and let $g_\pm$ be as in* (1.5). *Suppose that $q\geq 2$, and*

$$\sup_{t\in(\mathbf{R}_+)^{d-1}}\|\beta_{(1)}\otimes\dots\otimes\beta_{(d-1)}g_\pm(t_1\,\cdot,\dots,t_{d-1}\,\cdot)\|_{\mathcal{H}_\alpha^q(\mathbf{R}^{d-1})}<\infty,\quad \alpha>\frac{1}{q}. \tag{1.7}$$

Then m is a Fourier multiplier of $L^p(\mathbf{R}^d)$ if $|1/p-\frac{1}{2}|<1/2q$.

Interpolating Theorem 1.5 with Corollary 1.4 (with p close to 1) yields

Corollary 1.6. *Let $m\in L^\infty(\mathbf{R}^d)$, $d\geq 2$, be homogeneous of degree zero and let $g_\pm$ be as in (1.5). Suppose that $1<p<\frac{4}{3}$ and*

$$\sup_{t\in(\mathbf{R}_+)^{d-1}} \left\|\mathcal{D}_1^\alpha \mathcal{D}_2^\gamma \dots \mathcal{D}_{d-1}^\gamma \left[\beta_{(1)}\otimes\dots\otimes\beta_{(d-1)} g_\pm(t_1\,\cdot\,,\dots,t_{d-1}\,\cdot\,)\right]\right\|_{L^2(\mathbf{R}^{d-1})}<\infty,$$
$$\gamma>\frac{1}{2},\ \alpha>\frac{2}{p}-1$$

and that the analogous conditions obtained by permuting the $(s_1,\dots,s_{d-1})$-variables hold. Then m is a Fourier multiplier of $L^p(\mathbf{R}^d)$.

In particular if $\sup_{t\in(\mathbf{R}_+)^{d-1}}\|\beta_{(1)}\otimes\dots\otimes\beta_{(d-1)}g_\pm(t_1\,\cdot\,,\dots,t_{d-1}\,\cdot\,)\|_{\mathcal{H}^2_\alpha(\mathbf{R}^{d-1})}<\infty$ and $1<p<\frac{4}{3}$ then m is a Fourier multiplier of L^p provided that $\alpha>2/p-1$. This result is essentially sharp: in Section 3 we show that in order for

$$\sup_{t\in(\mathbf{R}_+)^{d-1}}\|\beta_{(1)}\otimes\dots\otimes\beta_{(d-1)}g_\pm(t_1\,\cdot\,,\dots,t_{d-1}\,\cdot\,)\|_{\mathcal{H}^q_\alpha(\mathbf{R}^{d-1})}<\infty$$

to imply that m is a Fourier multiplier of L^p we must necessarily have $\alpha\geq 2/p-\frac{3}{2}+1/q$ if $1<p<\frac{4}{3}$ and $\alpha>1/q$ if $\frac{4}{3}\leq p\leq 2$.

In order to prove more refined results on $L^p(\mathbf{R}^d)$, $d\geq 3$, p close to 1, we shall use multiparameter Calderón–Zygmund theory. It turns out that it is useful (and easier) to first prove a result for the multiparameter Hardy space $H^p(\mathbf{R}^d)$, $0<p\leq 1$. The Hardy space H^p is defined in terms of square-functions invariant under the multiparameter family of dilations $\delta_t x=(t_1x_1,\dots,t_dx_d)$, $t\in(\mathbf{R}_+)^d$. Again we formulate the multiplier result using localized multiparameter Sobolev spaces invariant under multiparameter dilations. In order to include a sharp result also for $p<1$ we want to admit values of $q\leq 1$ in (1.2). To make this possible the definition of $\mathcal{H}^q_\alpha$ has to be modified. We may always assume that β above is such that $\sum_{r\in\mathbf{Z}}\beta^2(2^{-r}s)=1$ for $s\neq 0$. Let $\psi_r=\beta^2(2^{-r}\,\cdot\,)$ if $r\geq 1$ and $\psi_0=1-\sum_{r>0}\psi_r$. For $n=(n_1,\dots,n_{d-1})$, $n_i\geq 0$, $i=1,\dots,d-1$ set $\psi_n(\xi_1,\dots,\xi_{d-1})=\prod_{i=1}^{d-1}\psi_{n_i}(\xi_i)$. The decomposition

$$g=\sum_{n\in(\mathbf{N}_0)^{d-1}}\widehat{\psi}_n*g$$

is referred to as the inhomogeneous Littlewood–Paley decomposition of $\mathbf{R}^{d-1}$. Then

$$\|g\|_{\mathcal{H}^q_\alpha(\mathbf{R}^{d-1})}\approx\left\|\left(\sum_{n\in(\mathbf{N}_0)^{d-1}}2^{2(n_1+\dots+n_{d-1})\alpha}|\widehat{\psi}_n*g|^2\right)^{1/2}\right\|_{L^q(\mathbf{R}^{d-1})} \tag{1.9}$$

for $1<q<\infty$, and for $q\le 1$ we define $\mathcal{H}^q_\alpha(\mathbf{R}^{d-1})$ as the space of tempered distributions for which the quasinorm on the right hand side of (1.9) is finite. In this paper we shall always have $\alpha>1/q$; in this case $\mathcal{H}^q_\alpha$ is embedded in L^∞. This and other properties of the spaces $\mathcal{H}^q_\alpha$ may be proved by obvious modifications of the one-parameter case; for the latter we refer to [27].

Theorem 1.7. *Let $m\in L^\infty(\mathbf{R}^d)$ be homogeneous of degree zero and related to $g_\pm$ as in* (1.5). *Suppose that $0<r\le 1$ and*

$$\sup_{t\in(\mathbf{R}_+)^{d-1}} \|\beta_{(1)}\otimes\ldots\otimes\beta_{(d-1)}g_\pm(t_1\,\cdot,\ldots,t_{d-1}\,\cdot)\|_{\mathcal{H}^r_\alpha(\mathbf{R}^{d-1})}<\infty,\quad \alpha>\frac{2}{r}-1. \tag{1.10}$$

Moreover, if $d\ge 3$, suppose that

(1.11)
$$\sup_{t\in(\mathbf{R}_+)^{d-1}} \big\|\mathcal{D}_1^{2\gamma}\mathcal{D}_2^{\gamma}\ldots\mathcal{D}_{d-2}^{\gamma}\big[\beta_{(1)}\otimes\ldots\otimes\beta_{(d-1)}g_\pm(t_1\,\cdot,\ldots,t_{d-1}\,\cdot)\big]\big\|_{L^2(\mathbf{R}^{d-1})}<\infty,$$
$$\gamma>\frac{1}{r}-\frac{1}{2}$$

and that the analogous conditions obtained by permuting the $(s_1,\ldots,s_{d-1})$-variables hold. Then m is a Fourier multiplier of the multiparameter Hardy space $H^p(\mathbf{R}^d)$, $r\le p<\infty$.

Note that in two dimensions Theorem 1.7 is a natural extension of Corollary 1.4 to H^p-spaces in product domains. The examples in Section 3 show that in higher dimensions additional assumptions such as (1.11) are necessary. When $d\ge 3$, Theorem 1.7 with $r=1$ serves as a substitute for Theorem 1.1. Notice that if $r=1$ condition (1.10) involves mixed derivatives in L^1 of order $d-1+\varepsilon$, and condition (1.11) involves derivatives in L^2 up to order $(d-1+\varepsilon)/2$. In comparison the hypotheses in Corollaries 1.3 and 1.6 involve L^2 derivatives up to order $(d+\varepsilon)/2$ if p is close to 1. As a consequence we obtain the following analogue of Corollary 1.4, formulated in terms of the standard *oneparameter* Sobolev space L^q_α.

Corollary 1.8. *Let $m\in L^\infty(\mathbf{R}^d)$ be homogeneous of degree zero and related to $g_\pm$ by* (1.5). *Suppose that $q>1$ and that*

$$\sup_{t\in(\mathbf{R}_+)^{d-1}} \|\beta_{(1)}\otimes\ldots\otimes\beta_{(d-1)}g_\pm(t_1\,\cdot,\ldots,t_{d-1}\,\cdot)\|_{L^q_\alpha(\mathbf{R}^{d-1})}<\infty,\quad \alpha>\frac{d-1}{q}.$$

Then m is a Fourier multiplier of $L^p(\mathbf{R}^2)$ if $\left|1/p-\frac{1}{2}\right|<1/2q$.

The counterexamples in [22], [7] show that the statement of the Corollary is false in the range $\left|1/p-\frac{1}{2}\right|>1/2q$. However in view of Theorems 1.5 and 1.7 one expects the following sharper result. Namely suppose that for some $q\in(1,2]$

$$\sup_{t\in(\mathbf{R}_+)^{d-1}} \|\beta_{(1)}\otimes\ldots\otimes\beta_{(d-1)}g_\pm(t_1\,\cdot\,,\ldots,t_{d-1}\,\cdot\,)\|_{\mathcal{H}^q_\alpha(\mathbf{R}^{d-1})}<\infty, \quad \alpha>\frac{1}{q}, \tag{1.12}$$

and in dimension $d\geq3$ suppose that

(1.13)

$$\sup_{t\in(\mathbf{R}_+)^{d-1}} \left\|\mathcal{D}_1^\alpha\mathcal{D}_2^\gamma\ldots\mathcal{D}_{d-2}^\gamma\left[\beta_{(1)}\otimes\ldots\otimes\beta_{(d-1)}g_\pm(t_1\,\cdot\,,\ldots,t_{d-1}\,\cdot\,)\right]\right\|_{L^2(\mathbf{R}^{d-1})}<\infty,$$
$$\gamma>\frac{1}{2},\ \alpha>\frac{1}{q}$$

as well as the analogous conditions obtained by permuting the $(s_1,\ldots,s_{d-1})$-variables. Then m should be a Fourier multiplier of $L^p(\mathbf{R}^d)$ if $|1/p-\frac{1}{2}|<1/2q$. In order to prove this one is tempted to use analytic interpolation and interpolate between the L^{p_0}-estimate of Theorem 1.7, for p_0 close to 1, and the $L^{4/3}$-estimate of Theorem 1.5. One would have to find the intermediate spaces for intersections of L^2 and L^q Sobolev spaces. However the intersection of the intermediate spaces does not need to be contained in the intermediate space of the intersections (for related counterexamples see [26]). It is actually possible to prove the result for $|1/p-\frac{1}{2}|<1/2q$ (assuming (1.12), (1.13)) by another approach. One has to use a general theorem for analytic families of operators acting on various kinds of atoms the proof of which relies heavily on multiparameter Calderón–Zygmund theory. We do not include the technical proof here but refer the reader to [5].

The paper is organized as follows: In Section 2 we prove Theorem 1.1 using weighted norm inequalities and variants of the maximal operator with respect to lacunary directions. Examples demonstrating the sharpness of our results in higher dimensions are discussed in Section 3. The proof of Theorem 1.5 is in Section 4; it relies on weighted norm inequalities which involve variants of the Kakeya maximal function. In Section 5 we prove the Hardy space estimates of Theorem 1.7.

As a convention we shall refer to the quasi-norms in H^p and $\mathcal{H}^p_\alpha$ as "norms" although for $p<1$ these spaces are not normed spaces. By M_p, $1\leq p\leq\infty$, we denote the standard space of Fourier multipliers of L^p. It will always be assumed that the even function $\beta\in C_0^\infty$ defined above satisfies $\sum_{\varkappa\in\mathbf{Z}}[\beta(2^\varkappa s)]^2=1$ for $s\neq0$. If $\mathfrak{a}\in\{1,\ldots,d\}$ and k, $\tilde{k}$ in $\mathbf{R}^\mathfrak{a}$ then we shall use the notation $k\leq\tilde{k}$ if $k_i\leq\tilde{k}_i$ for all $i\in\mathfrak{a}$. Similarly define $k\geq\tilde{k}$ etc. C will always be an abstract constant which may assume different values in different lines.

2. L^p-estimates in the plane

In the proof of Theorem 1.1 there is no loss of generality in assuming that m

is supported in the quadrant where $\xi_1>0$ and $\xi_2>0$. By a limiting argument as in Stein's book [25, p. 109], it suffices to prove the theorem under the formally stronger assumption

$$\|h\|_\infty+\sup_{k\in\mathbf{Z}}\int_{I_k}|h'(s)|\,ds\le A.$$

Let β be the smooth bump function defined in the introduction (supported in $\pm\left[\frac{5}{8},\frac{8}{5}\right]$). Let $\gamma\in C^\infty(\mathbf{R}^2\setminus\{0\})$ be homogeneous of degree 1 such that $\gamma(\xi)=1$ if $|\xi_1/\xi_2|\in\left[\frac{25}{64},\frac{64}{25}\right]$ (in particular on the support of $\beta\otimes\beta$) and such that $\gamma(\xi)=0$ if $|\xi_1/\xi_2|\notin\left(\frac{1}{4},4\right)$. Set

$$h_\varkappa(\xi)=\gamma(\xi)h(2^{-\varkappa}\xi_1/\xi_2).$$

Then we may split

$$m=\sum_{k\in\mathbf{Z}^2}[\beta\otimes\beta m_k](2^{k_1}\,\cdot\,,2^{k_2}\,\cdot\,),$$

where

$$m_k(\xi)=\beta(\xi_1)\beta(\xi_2)h_{k_1-k_2}(\xi_1/\xi_2)=\beta(\xi_1)\beta(\xi_2)\int_0^{\xi_1/\xi_2}h'_{k_1-k_2}(s)\,ds. \tag{2.1}$$

and $h_\varkappa$ is supported in $\left(\frac{1}{4},4\right)$, for all $\varkappa\in\mathbf{Z}$. Also set

$$\widehat{T_kf}(\xi)=[\beta\otimes\beta m_k](2^{k_1}\xi_1,2^{k_2}\xi_2)]\hat f(\xi).$$

Then by standard multiparameter Littlewood–Paley theory and duality, to establish Theorem 1.1 for $p\in[2,p_0)$, $p_0<\infty$, it suffices to obtain an inequality

$$\int|T_kf|^2\omega\le CA^2\int|f|^2\mathfrak{M}\omega \tag{2.2}$$

for a certain operator $\omega\mapsto\mathfrak{M}\omega$ which is bounded on $L^q(\mathbf{R}^2)$ for $(p_0/2)'<q\le\infty$. By our assumption on h,

$$\sup_{\varkappa\in\mathbf{Z}}\int|h'_\varkappa(s)|\,ds\le CA. \tag{2.3}$$

We denote by $\mathcal{L}_k$ the standard Littlewood–Paley operator, such that

$$\widehat{\mathcal{L}_kf}(\xi)=\beta(2^{k_1}\xi_1)\beta(2^{k_2}\xi_2)\hat f(\xi)$$

and define the operator S_{ks} by

$$\widehat{S_{\varkappa s}f}(\xi)=\begin{cases}\hat f(\xi), & \text{if } 2^\varkappa\xi_1/\xi_2>s,\ \xi_1\ge0,\ \xi_2\ge0,\\ 0, & \text{otherwise.}\end{cases}$$

Then from (2.1) we see that

$$T_k f(x)=\int_{1/8}^{8} \mathcal{L}_k S_{k_1-k_2,s} f(x) h'_{k_1-k_2}(s)\, ds.$$

Then, if $\omega\geq 0$ is a weight we apply the Cauchy–Schwarz inequality to obtain

$$\int |T_k f(x)|^2 \omega(x)\, dx \leq CA \iint |\mathcal{L}_k S_{k_1-k_2,s} f(x)|^2 |h'_{k_1-k_2}(s)|\, ds\, \omega(x)\, dx. \tag{2.4}$$

Let $M_{(1)}$, $M_{(2)}$ be the Hardy–Littlewood maximal functions with respect to the coordinate directions and let $M_{\varkappa,s}$ be the Hardy–Littlewood maximal function with respect to the direction perpendicular to $\{\xi; 2^{\varkappa}\xi_1/\xi_2=s\}$, i.e. in the direction $(1,-2^{-\varkappa}s)$. Then using weighted norm inequalities for singular integral operators due to Córdoba and Fefferman ([13], see also [18]) we see that the expression on the right hand side of (2.4) is dominated by

$$C_\alpha A \int |f(x)|^2 M_{(1)} M_{(2)} \left[\int_{1/8}^{8} (M_{k_1-k_2,s}\omega^\alpha)^{1/\alpha} |h'_{k_1-k_2}(s)|\, ds \right](x)\, dx,$$

where $\alpha>1$. Now the proof of (2.2) is completed by the following

Proposition 2.1. *Let, for $\alpha\geq 1$,*

$$\mathfrak{M}_\alpha \omega(x) = \sup_{\varkappa\in\mathbf{Z}} \int_I (M_{\varkappa,s}\omega^\alpha)^{1/\alpha}(x) |\lambda_\varkappa(s)|\, ds$$

where $I=\left[\frac{1}{8},8\right]$ and

$$\sup_{\varkappa\in\mathbf{Z}} \int_I |\lambda_\varkappa(s)|\, ds \leq B<\infty.$$

Then $\mathfrak{M}_\alpha$ is bounded on $L^p(\mathbf{R}^2)$, $\alpha<p<\infty$, with norm $\leq C_{p,\alpha}B$.

Proof. Since

$$\mathfrak{M}_\alpha(\omega) \leq B^{1-1/\alpha} [\mathfrak{M}_1(\omega^\alpha)]^{1/\alpha}$$

it suffices to prove that $\mathfrak{M}_1$ is bounded on L^p, $1<p<\infty$ with norm C_pB. If $\alpha>1$ then $\mathfrak{M}_\alpha$ will be bounded on L^p, $p>\alpha$, with norm $\leq C_{p/\alpha}^{1/\alpha}B$.

We follow arguments by Nagel, Stein and Wainger [23] as modified by Christ (see [2]). Let $\varphi\colon\mathbf{R}\to\mathbf{R}$ be smooth, even, nonnegative, with $\varphi(0)>0$ such that $\widehat{\varphi}$ has compact support in $\left[-\frac{1}{20},\frac{1}{20}\right]$. Let

$$\psi(\xi_1,\xi_2)=\widehat{\varphi}(\xi_1+\xi_2)$$

and define for $\varkappa \in \mathbf{Z}$

$$\widehat{P^l_{\varkappa s}\omega}(\xi) = \psi(2^l\xi_1, s2^{l-\varkappa}\xi_2)\widehat{\omega}(\xi).$$

It suffices to show that for $1<p<\infty$, N being an arbitrary positive integer

$$\left\| \sup_{-N\le\varkappa\le N} \int_I \sup_{l\in\mathbf{Z}} |P^l_{\varkappa s}\omega|\,|\lambda_\varkappa(s)|\,ds \right\|_p \le C_p B\|\omega\|_p, \tag{2.5}$$

where C_p is independent of N. Then an application of the monotone convergence theorem allows to pass to the limit. We note that for fixed $\varkappa$ and $1<p\le\infty$

$$\left\| \int_I \sup_{l\in\mathbf{Z}} |P^l_{\varkappa s}\omega|\,|\lambda_\varkappa(s)|\,ds \right\|_p \le C\int_I \left\|\sup_{l\in\mathbf{Z}} |P^l_{\varkappa s}\omega|\right\|_p |\lambda_\varkappa(s)|\,ds \le C_p B\|\omega\|_p \tag{2.6}$$

by the L^p estimate for the one-dimensional Hardy–Littlewood maximal function $M_{k,s}$. This means that we know a priori that the left hand side of (2.5) is bounded by $BC_p(N)\|f\|_p$ (with $C_p(N)\le C_p' N$) and it remains to be shown that $C_p(N)$ can be chosen independently of N. In what follows we define $C_p(N)$ to be the best constant in (2.5).

We first consider the case $2\le p<\infty$. Since the L^∞-estimate is trivial it suffices to prove the L^2 inequality. We smoothly split ψ into two parts, $\psi=\psi^0+\psi^1$ with ψ^1 supported in the unit ball and ψ^0 supported in the cone $\{\xi; |\xi_1+\xi_2|/|\xi|\le\frac12\}$. We correspondingly define the operators $P^{l,0}_{\varkappa s}$ and $P^{l,1}_{\varkappa s}$. Note that there is the pointwise inequality

$$|P^{l,1}_{\varkappa s}\omega(x)| \le CM_{(1)}M_{(2)}\omega(x) \tag{2.7}$$

which implies

$$\left\| \sup_{\varkappa} \int_I \sup_{l\in\mathbf{Z}} |P^{l,1}_{\varkappa s}\omega|\,|\lambda_\varkappa(s)|\,ds \right\|_p \le C_p B\|\omega\|_p, \quad 1<p\le\infty. \tag{2.8}$$

Concerning $P^{l,0}_{\varkappa s}$ we have

$$|P^{l,0}_{\varkappa s}\omega(x)| \le C[M_{(1)}M_{(2)}\omega(x) + M_{\varkappa,s}\omega(x)]$$

and therefore

$$\left\| \int_I \sup_{l\in\mathbf{Z}} |P^{l,0}_{\varkappa s}\omega|\,|\lambda_\varkappa(s)|\,ds \right\|_p \le C_{p,0}B\|\omega\|_p \tag{2.9}$$

for $1<p\le\infty$. Note that

$$\psi^0(2^l\cdot, s2^{l-\varkappa}\cdot)\widehat{\omega} = \chi(\cdot, 2^{-\varkappa}\cdot)\psi^0(2^l\cdot, s2^{l-\varkappa}\cdot)\widehat{\omega},$$

where χ is smooth, homogeneous of degree zero, identically 1 on $\{\xi;|\xi_1+\xi_2|/|\xi|\le 4\}$ and zero on $\{\xi;|\xi_1+\xi_2|/|\xi|\ge 8\}$. Define the standard angular Littlewood–Paley operator $R_\varkappa$ by

$$\widehat{R_\varkappa f}(\xi)=\chi(\xi_1,2^{-\varkappa}\xi_2)\hat f(\xi).$$

Then

$$P^{l,0}_{\varkappa s}\omega=P^{l,0}_{\varkappa s}R_\varkappa\omega \tag{2.10}$$

and, as a consequence of multiparameter Littlewood–Paley theory and the Marcinkiewicz multiplier theorem,

$$\Big\|\Big(\sum_\varkappa|R_\varkappa f|^2\Big)^{1/2}\Big\|_p\le C\|f\|_p,\quad 1<p<\infty. \tag{2.11}$$

Now by (2.10)

$$\sup_{\varkappa\in\mathbf{Z}}\left|\int_I\sup_{l\in\mathbf{Z}}|P^{l,0}_{\varkappa s}\omega||\lambda_\varkappa(s)|\,ds\right|\le\left(\sum_\varkappa\left[\int_I\sup_{l\in\mathbf{Z}}|P^{l,0}_{\varkappa s}R_\varkappa\omega||\lambda_\varkappa(s)|\,ds\right]^2\right)^{1/2} \tag{2.12}$$

and using (2.9) we see that the square of the L^2-norm of the right hand side equals

$$\begin{aligned}\sum_\varkappa\left\|\int_I\sup_{l\in\mathbf{Z}}|P^{l,0}_{\varkappa s}R_\varkappa\omega||\lambda_\varkappa(s)|\,ds\right\|_2^2&\le\sum_\varkappa\left[\int_I\Big\|\sup_{l\in\mathbf{Z}}|P^{l,0}_{\varkappa s}R_\varkappa\omega|\Big\|_2|\lambda_\varkappa(s)|\,ds\right]^2\\&\le CB^2\sum_\varkappa\|R_k\omega\|_2^2\le C'B^2\|\omega\|_2^2.\end{aligned}$$

We have proved

$$\left\|\sup_{\varkappa\in\mathbf{Z}}\left[\int_I\sup_{l\in\mathbf{Z}}|P^{l,0}_{\varkappa s}\omega||\lambda_\varkappa(s)|\,ds\right]\right\|_p\le CB\|\omega\|_p,\quad 2\le p\le\infty. \tag{2.13}$$

By (2.8) and (2.13) we see that

$$C_p(N)\le C,\quad 2\le p\le\infty.$$

We now assume $1<p<2$ and begin with the observation that for any sequence $\{\omega_k\}$ of weights we have

(2.14)
$$\left\|\left(\sum_\varkappa\left|\int_I\sup_{l\in\mathbf{Z}}|P^{l,0}_{\varkappa s}\omega_\varkappa||\lambda_\varkappa(s)|\,ds\right|^p\right)^{1/p}\right\|_p\le C_{p,0}B\left\|\left(\sum_\varkappa|\omega_\varkappa|^p\right)^{1/p}\right\|_p,\quad 1<p<\infty.$$

This is immediate from (2.9). Next positivity of $P^l_{\varkappa s}$ implies that

$$\begin{aligned}(2.15)\qquad \Big\|\sup_{\varkappa}\int_I \sup_l |P^l_{\varkappa s}\omega_\varkappa||\lambda_\varkappa(s)|\,ds\Big\|_p &\le \Big\|\sup_{\varkappa}\int_I \sup_l\Big|P^l_{\varkappa s}\Big[\sup_{\mu}|\omega_\mu|\Big]\Big||\lambda_\varkappa(s)|\,ds\Big\|_p\\ &\le BC_p(N)\Big\|\sup_{\varkappa}|\omega_\varkappa|\Big\|_p\end{aligned}$$

by the definition of $C_p(N)$. From (2.15) and (2.8) it follows that for $1<p\le\infty$

$$(2.16)\qquad \Big\|\sup_{\varkappa\in\mathbf{Z}}\int_I \sup_{l\in\mathbf{Z}} |P^{l,0}_{\varkappa s}\omega_\varkappa||\lambda_\varkappa(s)|\,ds\Big\|_p \le C_{p,1}BC_p(N)\Big\|\sup_{\varkappa}[|\omega_\varkappa|]\Big\|_p.$$

Now if we interpolate (2.14) with (2.16) we obtain for $p\le q\le\infty$
(2.17)

$$\Big\|\Big(\sum_{\varkappa\in\mathbf{Z}}\Big|\int_I \sup_{l\in\mathbf{Z}}|P^{l,0}_{\varkappa s}\omega_\varkappa||\lambda_\varkappa(s)|\,ds\Big|^q\Big)^{1/q}\Big\|_p \le C_{p,2}BC_p(N)^{1-p/q}\Big\|\Big(\sum_{\varkappa}|\omega_\varkappa|^q\Big)^{1/q}\Big\|_p.$$

Using (2.12), (2.17) and (2.11) we obtain for $1<p\le 2$

$$\begin{aligned}(2.18)\qquad \Big\|\sup_{\varkappa\in\mathbf{Z}}\Big|\int_I \sup_{l\in\mathbf{Z}}|P^{l,0}_{\varkappa s}\omega|\,|\lambda_\varkappa(s)|\,ds\Big|\Big\|_p &\le \Big\|\Big(\sum_{\varkappa}\Big[\sup_{l\in\mathbf{Z}}|P^{l,0}_{\varkappa s}R_\varkappa\omega||\lambda_\varkappa(s)|\,ds\Big]^2\Big)^{1/2}\Big\|_p\\ &\le C_{p,2}BC_p(N)^{1-p/2}\Big\|\Big(\sum_{\varkappa}|R_\varkappa\omega|^2\Big)^{1/2}\Big\|_p\\ &\le C_{p,3}BC_p(N)^{1-p/2}\|\omega\|_p.\end{aligned}$$

Finally it follows from (2.8) and (2.18) that

$$C_p(N)\le[C'_p+C_{p,3}C_p(N)^{1-p/2}]$$

which implies that $C_p(N)$ is bounded by a constant depending only on p but not on N. This finishes the proof of the proposition. □

3. Examples in higher dimensions

We show in this section that Theorem 1.1 and Corollary 1.4 have no immediate analogue in terms of localized multiparameter Sobolev spaces in higher dimensions. Our examples imply the sharpness of Theorems 1.5 and 1.7.

Let $L^p(L^2)$ be the space of functions f on $\mathbf{R}^d=\mathbf{R}^{d_1}\oplus\mathbf{R}^{d_2}$ such that

$$\|f\|_{L^p(L^2)}=\Big(\int\Big[\int|f(x',x'')|^2dx''\Big]^{p/2}dx'\Big)^{1/p}<\infty.$$

For a bounded function m on $\mathbf{R}^d$ we denote by $\|m\|_{M_p}$ the operator norm of the convolution operator T defined by $\widehat{Tf}=m\hat{f}$ and by $\|m\|_{M_{p2}}$ the norm of T as a bounded operator on $L^p(L^2)$. By a theorem of Herz and Rivière [19]

$$\|m\|_{M_{p,2}} \le C\|m\|_{M_p} \tag{3.1}$$

for $1\le p\le\infty$. We shall use the following

Lemma 3.1. *Let $\{m_\varkappa\}$ be a sequence of bounded functions in $\mathbf{R}^{d_1}$. Let $\chi\in C^\infty(\mathbf{R}^{d_2})$ be supported in $\{\xi''; \frac{1}{2}\le|\xi''|\le 2\}$ and equal to 1 if $1-\varepsilon\le|\xi''|\le 1+\varepsilon$ for some $\varepsilon>0$. Let*

$$m(\xi',\xi'')=\sum_\varkappa \chi(2^{-6\varkappa}\xi'')m_\varkappa(\xi')$$

and define $T_\varkappa$ by $\widehat{T_\varkappa f}(\xi)=m_\varkappa(\xi)\hat{f}(\xi)$. Then for $1<p<\infty$ we have the inequality

$$\Big\|\Big(\sum_\varkappa |T_\varkappa f_\varkappa|^2\Big)^{1/2}\Big\|_{L^p(\mathbf{R}^{d_1})} \le C_p\|m\|_{M_{p2}(\mathbf{R}^d)}\Big\|\Big(\sum_\varkappa |f_\varkappa|^2\Big)^{1/2}\Big\|_{L^p(\mathbf{R}^{d_1})}.$$

Proof. Let $\beta_0\in C^\infty$ be supported in $\{\xi'':1-\varepsilon\le|\xi''|\le 1+\varepsilon\}$ such that

$$\|\beta_0\|_{L^2(\mathbf{R}^{d_2})}=1.$$

Let

$$g_\varkappa(x',x'')=2^{3\varkappa}f_\varkappa(x')\mathcal{F}^{-1}_{\mathbf{R}^{d_2}}[\beta_0(2^{-6\varkappa}\cdot)](x'')$$

then by an application of Plancherel's theorem in the second variable it follows that

$$\Big\|\Big(\sum_\varkappa |g_\varkappa|^2\Big)^{1/2}\Big\|_{L^p(\mathbf{R}^d)} = (2\pi)^{-d_1/2}\Big\|\Big(\sum_\varkappa |f_\varkappa|^2\Big)^{1/2}\Big\|_{L^p(\mathbf{R}^{d_1})}. \tag{3.2}$$

Next let $L_\varkappa$ denote convolution in $\mathbf{R}^{d_2}$ with $\mathcal{F}^{-1}_{\mathbf{R}^{d_2}}[\beta_0(2^{-\varkappa}\cdot)]$. By Littlewood–Paley theory we have for $1<p<\infty$

$$\Big\|\sum_\varkappa L_\varkappa g_\varkappa\Big\|_{L^p(\mathbf{R}^d)} \le C_p\Big\|\Big(\sum_\varkappa |g_\varkappa|^2\Big)^{1/2}\Big\|_{L^p(\mathbf{R}^d)}. \tag{3.3}$$

Now

$$\begin{aligned}
\Big\|\Big(\sum_\varkappa |T_\varkappa f_\varkappa|^2\Big)^{1/2}\Big\|_{L^p(\mathbf{R}^{d_1})} &= \Big\|\Big(\sum_\varkappa \int |\beta_0(2^{-6\varkappa}\xi'')2^{-3\varkappa}T_\varkappa f_\varkappa|^2 d\xi''\Big)^{1/2}\Big\|_{L^p(\mathbf{R}^{d_1})}\\
&= (2\pi)^{d_1/2}\Big\|\Big(\sum_\varkappa |\mathcal{F}^{-1}_{\mathbf{R}^d}[m_\varkappa\mathcal{F}_{\mathbf{R}^d}[L_\varkappa L_\varkappa g_\varkappa]]|^2\Big)^{1/2}\Big\|_{L^p(L^2)}\\
&= (2\pi)^{d_1/2}\Big\|\Big(\sum_\varkappa \Big|L_\varkappa\mathcal{F}^{-1}_{\mathbf{R}^d}\Big[m\mathcal{F}_{\mathbf{R}^d}\Big[\sum_j L_j g_j\Big]\Big]\Big|^2\Big)^{1/2}\Big\|_{L^p(L^2)},
\end{aligned}$$

where the last identity holds in view of the support properties of β_0. By Littlewood–Paley theory

$$\begin{aligned}\Big\|\Big(\sum_{\varkappa}\Big|L_{\varkappa}\mathcal{F}^{-1}_{\mathbf{R}^d}\Big[m\mathcal{F}_{\mathbf{R}^d}\Big[\sum_j L_j g_j\Big]\Big]\Big|^2\Big)^{1/2}\Big\|_{L^p(L^2)} &\\ \le C_p\Big\|\mathcal{F}^{-1}_{\mathbf{R}^d}\Big[m\mathcal{F}_{\mathbf{R}^d}\Big[\sum_j L_j g_j\Big]\Big]\Big\|_{L^p(L^2)} &\\ \le C_p\|m\|_{M_{p2}}\Big\|\sum_j L_j g_j\Big\|_{L^p(L^2)} &\\ \le C_p'\|m\|_{M_{p2}}\Big\|\Big(\sum_j|g_j|^2\Big)^{1/2}\Big\|_{L^p(L^2)} &\\ = C_p'\|m\|_{M_{p2}}(2\pi)^{-d_1/2}\Big\|\Big(\sum_j|f_j|^2\Big)^{1/2}\Big\|_{L^p(\mathbf{R}^{d_1})}. &\quad\square\end{aligned}$$

We now show that the restriction $q\ge2$ (corresponding to $\frac{4}{3}\le p\le4$) in Theorem 1.5 is necessary. In what follows we denote by $L^p(L^2)$ the space of functions in $\mathbf{R}^3$ with

$$\|f\|_{L^p(L^2)}=\Big(\iint\Big[\int|f(x_1,x_2,x_3)|^2dx_2\Big]^{p/2}dx_1dx_3\Big)^{1/p}<\infty$$

and correspondingly define M_{p2}.

Fix $N\gg0$ and let

$$g_N(s_1,s_2)=\sum_{\varkappa=2}^{N}\eta(N(s_1-\alpha_\varkappa))\widetilde{\chi}(2^{-6\varkappa}s_2); \tag{3.4}$$

where

$$\alpha_\varkappa=1+\frac{2N}{\varkappa}, \tag{3.5}$$

and $\eta\in C^\infty$ is nonnegative, equal to 1 in $\left[-\frac{1}{4},\frac{1}{4}\right]$ and supported in $\left[-\frac{1}{2},\frac{1}{2}\right]$. Similarly $\widetilde{\chi}$ is as in Lemma 3.1, supported in $\pm\left(\frac{1}{2},2\right)$ and equal to 1 in $\pm(1/\sqrt{2},\sqrt{2})$. Then

$$\sup_{s_1,s_2>0}\|\beta_{(1)}\otimes\beta_{(2)}g_N(s_1\,\cdot,s_2\,\cdot)\|_{\mathcal{H}^q_\alpha(\mathbf{R}^2)}\le CN^{\alpha-1/q}. \tag{3.6}$$

Lemma 3.2. *Let $m_{(N)}$ be the homogeneous extension of g_N defined in* (3.4), (3.5). *There is a positive constant c such that*

$$\|m_{(N)}\|_{M_{p,2}} \ge \begin{cases} cN^{1/2-2/p}, & 4<p<\infty, \\ c(\log N)^{1/4}, & p=4. \end{cases} \tag{3.7}$$

A comparison of (3.6) and (3.7) shows that in the case $p>4$ the condition

$$\sup_{s_1,s_2>0} \|\beta_{(1)} \otimes \beta_{(2)} g_\pm(s_1\,\cdot\,, s_2\,\cdot\,)\|_{\mathcal{H}^q_\alpha(\mathbf{R}^2)} < \infty$$

does not imply $m\in M_p$ for the homogeneous extension m if $\alpha<\frac{1}{2}+1/q-2/p$ (it does not even imply $m\in M_{p2}$). Similar statements follow by duality for $1<p<\frac{4}{3}$. This yields the sharpness of Theorem 1.5. By interpolation an improvement of the H^p estimates would lead to an improvement of the L^p estimates and this implies the sharpness of Theorem 1.7.

Proof of Lemma 3.2. Let $\beta_1 \in C_0^\infty$ be supported in $(\frac{3}{4},\frac{5}{4})$ and equal to 1 in $(\frac{7}{8},\frac{9}{8})$. Let χ be as in Lemma 3.1 supported in $\{|\xi_2|\in(\frac{4}{5},\frac{6}{5})\}$ and equal to 1 in $\{|\xi_2|\in(\frac{9}{10},\frac{11}{10})\}$. Let

$$m_\varkappa(\xi_1,\xi_3) = \beta(\xi_3)\eta(4N(\xi_1/\xi_3-\alpha_\varkappa))$$

and

$$\mu_{(N)}(\xi) = \sum_{\varkappa=2}^{N} \chi(2^{-6\varkappa}\xi_2) m_\varkappa(\xi_1,\xi_3).$$

In view of the properties of η, χ, $\tilde{\chi}$ and the Marcinkiewicz multiplier theorem

$$\|\mu_{(N)}\|_{M_{p2}} \le C_p \|m_{(N)}\|_{M_{p2}}, \quad 1<p<\infty.$$

Now assume $4\le p<\infty$. Let

$$R_\varkappa = \{(x_1,x_3); |x_1-\alpha_\varkappa x_3| \le 10^{-3}N, |\alpha_\varkappa x_1 - x_3| \le 10^{-3}\}.$$

For $\xi\in\operatorname{supp} m_\varkappa$, $x\in R_\varkappa$ we have $|x_1\xi_1+x_3\xi_3|\le\pi/4$ and therefore

$$\left|\int m_\varkappa(\xi_1,\xi_3) e^{i(x_1\xi_1+x_3\xi_3)} d\xi_1 d\xi_3\right| \ge \left|\int m_\varkappa(\xi_1,\xi_3)\cos(x_1\xi_1+x_3\xi_3) d\xi_1 d\xi_3\right| \ge cN^{-1}$$

for some fixed positive constant c. Let

$$\begin{aligned} \widetilde{R}_\varkappa &= \{(x_1,x_3); 10^{-4}N/2 \le |x_1-\alpha_\varkappa x_3| \le 10^{-4}N, 10^{-4}/2 \le |\alpha_\varkappa x_1 - x_3| \le 10^{-4}\}, \\ R^*_\varkappa &= \{(x_1,x_3); |x_1-\alpha_\varkappa x_3| \le 10^{-4}N, |\alpha_\varkappa x_1 - x_3| \le 10^{-4}\} \end{aligned}$$

and let $\chi_\varkappa$ be the characteristic function of $\widetilde{R}_\varkappa$. Then

$$\mathcal{F}^{-1}[m_\varkappa \mathcal{F}\chi_\varkappa] \geq c', \quad x \in R^*_\varkappa.$$

By Lemma 3.1

$$\left\| \left(\sum_{\varkappa=2}^{N} |\mathcal{F}^{-1}_{\mathbf{R}^2}[m_\varkappa \mathcal{F}_{\mathbf{R}^2}\chi_\varkappa]|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^2)} \leq C_p \|\mu_{(N)}\|_{M_{p2}} \left\| \left(\sum_{\varkappa=2}^{N} |\chi_\varkappa|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^2)}.$$

Now one verifies that

$$\left\| \left(\sum_{\varkappa} |\chi_\varkappa|^2 \right)^{1/2} \right\|_p \approx N^{2/p}.$$

In view of the overlap of the rectangles $R^*_\varkappa$ we have for some small constant $c_1>0$, and for $|x|\leq c_1 N$ and for $|x|\leq cN$ we have

$$\left(\sum_{\varkappa} |\mathcal{F}^{-1}[m_\varkappa \mathcal{F}\chi_\varkappa](x)|^2 \right)^{1/2} \approx N^{1/2}(1+|x|)^{-1/2}$$

and consequently

$$\left\| \left(\sum_{\varkappa} |\mathcal{F}^{-1}[m_\varkappa \mathcal{F}\chi_\varkappa]|^2 \right)^{1/2} \right\|_p \approx \begin{cases} N^{1/2} & \text{if } p>4, \\ N^{1/2}(\log N)^{1/4} & \text{if } p=4. \end{cases}$$

This implies the assertion. □

Next we consider the class of homogeneous functions m in $\mathbf{R}^3$ with the property that the restrictions $h_\pm$ to the hyperplanes $\{\xi; \xi_3=\pm 1\}$ satisfy the hypotheses of the Marcinkiewicz multiplier theorem in the plane; that is

$$\begin{aligned} \|h\|_\infty &\leq A, \\ \sup_{j_1\in\mathbf{N}} \sup_{s_2} \int_{I_{j_1}} \left| \frac{\partial h}{\partial s_1}(s_1, s_2) \right| ds_1 &\leq A, \\ \sup_{j_2\in\mathbf{N}} \sup_{s_1} \int_{I_{j_2}} \left| \frac{\partial h}{\partial s_2}(s_1, s_2) \right| ds_2 &\leq A, \\ \sup_{j\in\mathbf{N}^2} \iint_{I_{j_1}\times I_{j_2}} \left| \frac{\partial^2 h}{\partial s_1 \partial s_2}(s_1, s_2) \right| ds_1 ds_2 &\leq A, \end{aligned} \tag{3.8}$$

where I_{j_1} etc. is as in (1.1). We show that (3.8) is not sufficient to guarantee $m\in M_p$, for any $p\neq 2$. The argument here follows Fefferman's solution [16] of the multiplier problem for the ball (see also [14], [21]).

Let $\mathfrak{a}=\{\alpha_\varkappa\}$ be an arbitrary sequence of numbers in $[1,2)$ and let $m_\varkappa$ be defined in the first quadrant such that

$$m_\varkappa(\xi_1,\xi_3)=\begin{cases} 1, & 1\le \xi_1/\xi_3\le \alpha_\varkappa, \\ 0, & \alpha_\varkappa<\xi_1/\xi_3\le 2. \end{cases} \tag{3.9}$$

Let

$$m_{\mathfrak{a}}=\sum_\varkappa \beta(2^\varkappa \xi_2/\xi_3)m_\varkappa(\xi_1,\xi_3).$$

Suppose the assumptions (3.8) imply $m\in M_p$ for some $p\neq 2$. Then a limiting argument as in [25, p. 109] would imply that $m_{\mathfrak{a}}$ is an L^p multiplier with norm independent of the choice of $\{\alpha_\varkappa\}_{k\in\mathbf{Z}}$ and by (3.1) a corresponding statement on $L^p(L^2)$ would follow. However we have

Lemma 3.3. *The inequality*

$$\|\mathcal{F}^{-1}[m_{\mathfrak{a}}\mathcal{F}f]\|_{L^p(L^2)}\le C\|f\|_{L^p(L^2)}$$

does not hold independently of $\mathfrak{a}$ *if* $p\neq 2$.

For example if we take for $\mathfrak{a}$ an enumeration of the rationals in $[1,2)$ then $m_{\mathfrak{a}}\in M_{p2}$ if and only if $p=2$.

Proof. Arguing as above the assumption $m_{\mathfrak{a}}\in M_p$ implies a vector-valued estimate for directional Hilbert transforms, namely

$$\bigg\|\Big(\sum_\varkappa |H_\varkappa f_\varkappa|^2\Big)^{1/2}\bigg\|_{L^p(\mathbf{R}^2)}\le C\bigg\|\Big(\sum_\varkappa |f_\varkappa|^2\Big)^{1/2}\bigg\|_{L^p(\mathbf{R}^2)},$$

where $H_\varkappa$ is the Hilbert transform in the direction $(1,-\alpha_\varkappa)$. But as in [16] the existence of the Kakeya set prohibits such inequalities for $p\neq 2$ (unless further restrictions on the family of directions $(1,-\alpha_\varkappa)$ are made). $\square$

4. Weighted norm inequalities in higher dimensions

We deduce Theorem 1.3 from a weighted norm inequality; the procedure is analogous to Stein's proof of the Hörmander multiplier theorem (see [25, Ch. IV]). Here, however, the positive operator which controls the problem is not the Hardy–Littlewood maximal operator but a multiple iteration of variants of Kakeya-type maximal operators. The main step of the argument is contained in Lemma 4.7; one proves a weighted inequality for a variant of Córdoba's sectorial square-function.

For $i=1,\dots,d-1$ let $\mathcal{R}_{n_i}^{(i,d)}$ be the family of all rectangles with the dimensions 1×2^{n_i}, centered at the origin in the x_i-x_d plane and let $\widetilde{\mathcal{R}}_{n_i}^{(i,d)}$ be the family of all parallelograms of the form $\{(x_i,x_d);(2^{k_i}x_i,2^{k_d}x_d)\in R_0\}$ where $R_0\in\mathcal{R}_{n_i}^{(i,d)}$ and k_i,k_d are integers. Let

$$\begin{aligned}M_{n_i}^{(i,d)}\omega(x_1,\dots,x_d)\\ &=\sup_{R\in\widetilde{\mathcal{R}}_{n_i}^{(i,d)}}\frac{1}{|R|}\int_R|\omega(x_1,\dots,x_{i-1},x_i-y_i,x_{i+1},\dots,x_{d-1},x_d-y_d)|\,dy_i\,dy_d.\end{aligned}$$

$M_{n_i}^{(i,d)}$ is a variant of the Kakeya maximal function, invariant under the dilations $(x_i,x_d)\mapsto(2^{k_i}x_i,2^{k_d}x_d)$. The proof of the L^2-estimate in [10] can be easily modified to yield

$$\|M_{n_i}^{(i,d)}\omega\|_2\le Cn_i\|\omega\|_2;$$

for a more singular variant see also [11].

Next, for $n=(n_1,\dots,n_{d-1})$ define

$$\mathcal{M}_n=M_{n_{d-1}}^{(d-1,d)}\circ M_{n_{d-2}}^{(d-2,d)}\circ\dots\circ M_{n_1}^{(1,d)}$$

and, for $N\in\mathbf{N}$, let $\mathcal{M}_n^N=\mathcal{M}_n\circ\dots\circ\mathcal{M}_n$ be the N-fold application of the operator $\mathcal{M}_n$. Finally, if $M_{(i)}$ denotes the Hardy–Littlewood maximal operator with respect to the variable x_i let

$$\widetilde{\mathcal{M}}_n^N=M_{(1)}\circ\dots\circ M_{(d)}\circ\mathcal{M}_n^N\circ M_{(1)}\circ\dots\circ M_{(d)}.$$

Theorem 4.1. *Let $\gamma>\frac{1}{2}$ and suppose that*

$$\sup_{t\in(\mathbf{R}_+)^{d-1}}\|\beta_{(1)}\otimes\dots\otimes\beta_{(d-1)}g_\pm(t_1\,\cdot,\dots,t_{d-1}\,\cdot)\|_{\mathcal{H}_\gamma^2(\mathbf{R}^{d-1})}\le B_\gamma<\infty. \tag{4.1}$$

Let m be the homogeneous extension of $g_\pm$ and define T by $\widehat{Tf}(\xi)=m(\xi)\hat{f}(\xi)$. Let $0<\varepsilon<\gamma-\frac{1}{2}$, let $N(\varepsilon)$ be the smallest positive integer $>3+2/\varepsilon$ and define $\mathfrak{M}_\varepsilon$ by

$$\mathfrak{M}_\varepsilon\omega=\sum_{n\in\mathbf{N}_0^{d-1}}2^{-\varepsilon(n_1+\dots+n_{d-1})}\widetilde{\mathcal{M}}_n^{N(\varepsilon)}\omega.$$

Then for $s>1$

$$\int|Tf(x)|^2\omega(x)\,dx\le C_{\varepsilon,s}B_\gamma\int|f(x)|^2(\mathfrak{M}_\varepsilon[\omega^s])^{1/s}dx. \tag{4.2}$$

Proof of Theorem 1.5. Since the operator $\omega\mapsto(\mathfrak{M}_\varepsilon(|\omega|^s))^{1/s}$ is bounded on L^q, $q>2s/(1-\varepsilon)$, the weighted norm inequality (4.2) and duality imply under the assumption (4.1) that T is bounded on L^p, for $2 \le p \le 4$. The general result of Theorem 1.5 follows then by interpolation, using the technique in [9]. $\square$

Before we prove Theorem 4.1 we recall a few facts about vector-valued weighted norm inequalities. First if H_1, H_2 are Hilbert spaces and $\mathcal{K}$ is a convolution kernel in $\mathbf{R}$, with values in the space $\mathcal{B}(H_1, H_2)$ of bounded operators, then $\mathcal{K}$ is called a regular singular integral operator if

$$\begin{aligned} |\widehat{\mathcal{K}}(\xi)|_{\mathcal{B}(H_1,H_2)} &\le C, \\ |\mathcal{K}(x)|_{\mathcal{B}(H_1,H_2)} &\le C|x|^{-1}, \\ |\mathcal{K}(x-y)-\mathcal{K}(x)|_{\mathcal{B}(H_1,H_2)} &\le C|y|^{\delta}|x|^{-1-\delta}, \quad |x|>2|y|>0; \end{aligned}$$

here $0<\delta\le 1$ is fixed. By a vector-valued version of a theorem of Córdoba and Fefferman ([13], see also [18, Ch. IV.3]) there is an inequality

$$\int |\mathcal{K}*f(x)|^p_{H_2}\omega(x)\,dx \le C_{\sigma,p}\int |f(x)|^p_{H_1}(M(|\omega|^{\sigma}))^{1/\sigma}(x)\,dx \tag{4.3}$$

where $1<p<\infty$, $\sigma>1$.

Littlewood–Paley functions can be associated with regular singular integral operators. Let $\beta \in C_0^\infty\left(\frac{1}{2}, 2\right)$ then it is straightforward to check that the operator $\{f_\varkappa\}_{\varkappa\in\mathbf{Z}} \mapsto \sum \mathcal{F}^{-1}[\beta(2^\varkappa|\cdot|)\mathcal{F}f]$ is a $\mathcal{B}(l^2,\mathbf{R})$-valued regular singular integral operator. Likewise the adjoint operator $f\mapsto\{\mathcal{F}^{-1}[\beta(2^\varkappa|\cdot|)\mathcal{F}f]\}_{\varkappa\in\mathbf{Z}}$ is a $\mathcal{B}(\mathbf{R},l^2)$-valued regular singular integral operator. Here l^2 may refer to a space of sequences with values in a Hilbert space.

Next let $k\in\mathbf{Z}^d$ and denote by $\mathcal{L}_k$ be the standard Littlewood–Paley operator with multiplier $\prod_{i=1}^d \beta(2^{k_i}|\xi_i|)$. Then a repeated application of (4.3) yields

Lemma 4.2. *For $1<p<\infty$, $s>1$ we have the inequalities*

$$\int \Big|\sum_{k\in\mathbf{Z}^d}\mathcal{L}_k f_k\Big|^p \omega(x)\,dx \le C_{s,p}\int\Big(\sum_{k\in\mathbf{Z}^d}|f_k(x)|^2\Big)^{p/2}(M_{(1)}\circ\dots\circ M_{(d)}[\omega^s])^{1/s}(x)\,dx,$$
$$\int\Big(\sum_{k\in\mathbf{Z}^d}|\mathcal{L}_k f|^2\Big)^{p/2}\omega(x)\,dx \le C_{s,p}\int |f(x)|^p(M_{(1)}\circ\dots\circ M_{(d)}[\omega^s])^{1/s}(x)\,dx.$$

We need also the following pointwise estimate concerning a square-function involving translates of a fixed Schwartz-function η. It implies L^p-boundedness for $p>2$, a result which is due to Carleson. A proof of the pointwise estimate can be found in [24], see also [12], [18].

Lemma 4.3. *Let η be a Schwartz function in $\mathbf{R}^d$ and let $A\in GL(d,\mathbf{R})$. Then*

$$\sum_{k\in\mathbf{Z}^d} |\mathcal{F}^{-1}[\eta(A\cdot -k)\mathcal{F}f](x)|^2 \le C_N \int \frac{|f(x-A^t y)|^2}{(1+|y|)^N}\,dy. \tag{4.4}$$

Proof of Theorem 4.1

There is no loss of generality in assuming that m is supported in $\{\xi;\xi_i\ge 0, i=1,\dots,d\}$. Setting

$$\phi(\xi)=\prod_{i=1}^d \beta(\xi_i) \tag{4.5}$$

we decompose

$$m(\xi)=\sum_{k\in\mathbf{Z}^d} m_k(2^{k_1}\xi_1,\dots,2^{k_d}\xi_d)$$

where

$$m_k(\xi)=\phi(\xi)g_k(\xi_1/\xi_d,\dots,\xi_{d-1}/\xi_d)$$

and g_k has compact support in $\left(\frac{1}{2},2\right)^{d-1}$. Note that $g_k=g_{k'}$ if $k_i-k_d=k_i'-k_d'$, $i=1,\dots,d-1$. We introduce a further decomposition using the dyadic smooth cutoff functions $\psi_n=\psi_{n_1}\otimes\dots\otimes\psi_{n_{d-1}}$ (cf. the second definition of the space $\mathcal{H}^q_\alpha$ in the introduction). We decompose

$$m_k(\xi)=\sum_{n\in(\mathbf{N}_0)^{d-1}} \phi(\xi)g_k*\widehat{\psi_n}(\xi_1/\xi_d,\dots,\xi_{d-1}/\xi_d)=\sum_{n\in(\mathbf{N}_0)^{d-1}}\phi(\xi)m_k^n(\xi). \tag{4.6}$$

We may write

$$g_k*\widehat{\psi_n}=g_k^n*\widehat{\psi_n} \tag{4.7}$$

where

$$g_k^n=g_k*\widehat{\widetilde{\psi}_n}$$

and $\widetilde{\psi}_n=\widetilde{\psi}_{n_1}\otimes\dots\otimes\widetilde{\psi}_{n_{d-1}}$ is similarly defined as ψ_n (say, with $\widetilde{\psi}_{n_i}$ supported in $\pm[2^{n_i-2},2^{n_i+2}]$, equal to 1 in supp ψ_{n_i}). Let us note in passing that in view of the support properties of the Fourier transform of g_k^n we have the following version of Sobolev's imbedding theorem

$$\sup_{s_{d_1+1},\dots,s_{d-1}} \|g_k^n(\cdot,s_{d_1+1},\dots,s_{d-1})\|_{L^p(\mathbf{R}^{d_1})}\le C2^{(n_{d_1+1}+\dots+n_{d-1})/p}\|g_k^n\|_{L^p(\mathbf{R}^{d-1})}, \tag{4.8}$$

see the argument in [27, p. 18].

Let T_k^n be defined by

$$\widehat{T_k^n f}(\xi) = [\phi m_k^n](2^{k_1}\xi_1, \dots, 2^{k_d}\xi_d)\hat{f}(\xi). \tag{4.9}$$

Let $0<\varepsilon'<\varepsilon$, say $\varepsilon'=\varepsilon/2$. An application of Lemma 4.2 shows that it suffices to prove the inequality

$$\begin{aligned}(4.10)\quad &\sum_{k\in\mathbf{Z}^d}\int |T_k^n \mathcal{L}_k f(x)|^2 \omega(x)\,dx \\ &\le C_N 2^{(n_1+\dots+n_{d-1})(1/2+\varepsilon')}\|g_k^n\|_2 \sum_{k\in\mathbf{Z}^d}\int |\mathcal{L}_k f(x)|^2 \mathcal{M}_n^N \omega(x)\,dx, \quad N>2+\frac{1}{\varepsilon'}.\end{aligned}$$

In order to avoid complicated notation we shall assume $d=3$ in what follows. This case is entirely typical of the general situation in higher dimensions.

In order to use the homogeneity of the multipliers we have to introduce finer decompositions of g_n^k. For $\nu_1=2^{n_1-3}, 2^{n_1-3}+1, \dots, 2^{n_1+3}$ and $\nu_2=2^{n_2-3}, 2^{n_2-3}+1, \dots, 2^{n_2+3}$ let

$$u_\nu = (u_{\nu_1}^1, u_{\nu_2}^2) = (2^{-n_1}\nu_1, 2^{-n_2}\nu_2) \tag{4.11}$$

and

$$I_\nu = I_{\nu_1}^1 \times I_{\nu_2}^2 = [u_{\nu_1}^1 - 2^{-n_1-1}, u_{\nu_1}^1 + 2^{-n_1-1}] \times [u_{\nu_2}^2 - 2^{-n_2-1}, u_{\nu_2}^2 + 2^{-n_2-1}].$$

Furthermore let

$$\begin{aligned}{}^c\!I^1 &= \mathbf{R}\setminus \bigcup_{\nu_1=2^{n_1-3}}^{2^{n_1+3}} I_{\nu_1}^1 = \mathbf{R}\setminus\left[\tfrac{1}{8}-2^{-n_1-1}, 8+2^{-n_1-1}\right],\\ {}^c\!I^2 &= \mathbf{R}\setminus \bigcup_{\nu_2=2^{n_2-3}}^{2^{n_2+3}} I_{\nu_2}^2 = \mathbf{R}\setminus\left[\tfrac{1}{8}-2^{-n_2-1}, 8+2^{-n_2-1}\right].\end{aligned}$$

Setting

$$\begin{aligned} g_{k\nu}^n(s) &= \int_{I_\nu} g_k^n(u)\widehat{\psi_n}(s-u)\,du,\\ g_{k\nu_1}^{n,1}(s) &= \int_{I_{\nu_1}^1\times {}^c\!I^2} g_k^n(u)\widehat{\psi_n}(s-u)\,du,\\ g_{k\nu_2}^{n,2}(s) &= \int_{{}^c\!I^1\times I_{\nu_2}^2} g_k^n(u)\widehat{\psi_n}(s-u)\,du,\\ \varrho_k^n(s) &= \int_{{}^c\!I^1\times {}^c\!I^2} g_k^n(u)\widehat{\psi_n}(s-u)\,du,\end{aligned} \tag{4.12}$$

we split

$$T_k^n = \sum_{\nu_1=2^{n_1-3}}^{2^{n_1+3}} \sum_{\nu_2=2^{n_2-3}}^{2^{n_2+3}} T_{k\nu}^n + \sum_{\nu_1=2^{n_1-3}}^{2^{n_1+3}} T_{k\nu_1}^{n,1} + \sum_{\nu_2=2^{n_2-3}}^{2^{n_2+3}} T_{k\nu_2}^{n,2} + T_k^{n,0}, \tag{4.13}$$

where

$$\begin{aligned}
\widehat{T_{k\nu}^n f}(\xi) &= g_{k\nu}^n(2^{k_1-k_3}\xi_1/\xi_3, 2^{k_2-k_3}\xi_2/\xi_3)\phi(2^{k_1}\xi_1, 2^{k_2}\xi_2, 2^{k_3}\xi_3)\hat{f}(\xi),\\
\widehat{T_{k\nu_1}^{n,1} f}(\xi) &= g_{k\nu_1}^{n,1}(2^{k_1-k_3}\xi_1/\xi_3, 2^{k_2-k_3}\xi_2/\xi_3)\phi(2^{k_1}\xi_1, 2^{k_2}\xi_2, 2^{k_3}\xi_3)\hat{f}(\xi),\\
\widehat{T_{k\nu_2}^{n,2} f}(\xi) &= g_{k\nu_2}^{n,2}(2^{k_1-k_3}\xi_1/\xi_3, 2^{k_2-k_3}\xi_2/\xi_3)\phi(2^{k_1}\xi_1, 2^{k_2}\xi_2, 2^{k_3}\xi_3)\hat{f}(\xi),\\
\widehat{T_k^{n,0} f}(\xi) &= \varrho_k^n(2^{k_1-k_3}\xi_1/\xi_3, 2^{k_2-k_3}\xi_2/\xi_3)\phi(2^{k_1}\xi_1, 2^{k_2}\xi_2, 2^{k_3}\xi_3)\hat{f}(\xi).
\end{aligned} \tag{4.14}$$

We set

$$\begin{aligned}
b_{k\nu}^n &= \sup_{u\in I_\nu} |g_n^k(u)|,\\
b_{k\nu_1}^{n,1} &= \sup_{u\in I_{\nu_1}^1 \times {}^cI^2} |g_n^k(u)|,\\
b_{k\nu_2}^{n,2} &= \sup_{u\in {}^cI^1 \times I_{\nu_2}^2} |g_n^k(u)|,\\
b_k^n &= \sup_{u\in {}^cI^1 \times {}^cI^2} |g_n^k(u)|.
\end{aligned} \tag{4.15}$$

Since the Fourier transform of g_k^n is supported in $[-2^{n_1+3}, 2^{n_1+3}] \times [-2^{n_2+3}, 2^{n_2+3}]$, suitable variants of the Plancherel–Polya theorem (see [27, p. 19]) and the Sobolev embedding theorem imply

$$\left(\sum_{\nu_1=2^{n_1-3}}^{2^{n_1+3}} \sum_{\nu_2=2^{n_2-3}}^{2^{n_2+3}} [b_{k\nu}^n]^r \right)^{1/r} \le C_r 2^{(n_1+n_2)/r} \|g_k^n\|_r, \quad 0 < r \le \infty \tag{4.16}$$

with the appropriate interpretation for $r=\infty$; moreover we have

$$\begin{aligned}
\left(\sum_{\nu_1=2^{n_1-3}}^{2^{n_1+3}} [b_{k\nu_1}^{n,1}]^r \right)^{1/r} &\le C 2^{n_1/r} \sup_{s_2} \|g_k^n(\,\cdot\,, s_2)\|_{L^r(\mathbf{R})}\\
&\le C 2^{(n_1+n_2)/r} \|g_k^n\|_{L^r(\mathbf{R}^2)}
\end{aligned} \tag{4.17}$$

and a similar statement with the s_1 and s_2 variables interchanged. Also by (4.8) b_k^n is bounded by $C2^{(n_1+n_2)/r}\|g_k^n\|_r$.

We need pointwise estimates for the convolution kernels $K_{k\nu}^n$, $K_{k\nu_1}^{n,1}$, $K_{k\nu_2}^{n,2}$, $K_k^{n,0}$ of the operators $T_{k\nu}^n$, $T_{k\nu_1}^{n,1}$, $T_{k\nu_2}^{n,2}$, $T_k^{n,0}$, respectively.

Lemma 4.4. *Let* $e_\nu=(u^1_{\nu_1},u^2_{\nu_2},1)$, $e^1_{\nu,1}=(u^1_{\nu_1},0,1)$ *and* $e^2_{\nu,2}=(0,u^2_{\nu_2},1)$. *Let*

$$\begin{aligned}
W^n_{\nu N}(x)&=2^{-n_1-n_2}(1+|\langle e_\nu,x\rangle|)^{-N}(1+2^{-n_1}|x_1|)^{-N}(1+2^{-n_2}|x_2|)^{-N},\\
W^{n,1}_{\nu_1 N}(x)&=2^{-n_1}(1+|\langle e^1_{\nu_1},x\rangle|)^{-N}(1+2^{-n_1}|x_1|)^{-N}(1+|x_2|)^{-N},\\
W^{n,2}_{\nu_2 N}(x)&=2^{-n_2}(1+|\langle e^2_{\nu_2},x\rangle|)^{-N}(1+|x_1|)^{-N}(1+2^{-n_2}|x_2|)^{-N},\\
W^{n,0}_{N}(x)&=(1+|x_1|)^{-N}(1+|x_2|)^{-N}(1+|x_3|)^{-N}.
\end{aligned}$$

Let

$$U^n_{k\nu,N}(x)=2^{-k_1-k_2-k_3}W^n_{k\nu}(2^{-k_1}x_1,2^{-k_2}x_2,2^{-k_3}x_3)$$

and similarly define $U^{n,1}_{k\nu_1,N}$, $U^{n,2}_{k\nu_2,N}$, $U^{n,0}_{k,N}$. *Then*

$$\begin{aligned}
&|\partial^\gamma_x K^n_{k\nu}(x)|\le C_{\gamma N}b^n_{k\nu}2^{-k_1\gamma_1-k_2\gamma_2-k_3\gamma_3}U^n_{k\nu,N}(x),\\
&|\partial^\gamma_x K^{n,1}_{k\nu_1}(x)|\le C_{\gamma N}b^{n,1}_{k\nu_1}2^{-n_2N}2^{-k_1\gamma_1-k_2\gamma_2-k_3\gamma_3}U^{n,1}_{k\nu_1,N}(x),\\
&|\partial^\gamma_x K^{n,2}_{k\nu_2}(x)|\le C_{\gamma N}b^{n,2}_{k\nu_2}2^{-n_1N}2^{-k_1\gamma_1-k_2\gamma_2-k_3\gamma_3}U^{n,2}_{k\nu_2,N}(x),\\
&|\partial^\gamma_x K^{n,0}_{k}(x)|\le C_{\gamma N}b^{n}_{k}2^{-n_1N}2^{-n_2N}2^{-k_1\gamma_1-k_2\gamma_2-k_3\gamma_3}U^{n,0}_{k,N}(x).
\end{aligned}\tag{4.18}$$

Proof. First consider $K^n_{k\nu}$. Using the homogeneity of the multiplier and the decay properties of $\widehat{\psi_n}$ we see that

$$\begin{aligned}
&\left|\partial^{N_1}_{\xi_1}\partial^{N_2}_{\xi_2}\langle e_\nu,\nabla_\xi\rangle^{N_3}\left[\phi(\xi)\int_{I_\nu}g^n_k(u)\widehat{\psi_n}\left(\frac{\xi_1}{\xi_3}-u_1,\frac{\xi_2}{\xi_3}-u_2\right)du\right]\right|\\
&\quad\le C(N_1,N_2,N_3,M)b^n_{k\nu}\frac{2^{N_1n_1}}{(1+2^{n_1}|\xi_1/\xi_3-u^1_{\nu_1}|)^M}\,\frac{2^{N_2n_2}}{(1+2^{n_2}|\xi_2/\xi_3-u^2_{\nu_2}|)^M}.
\end{aligned}\tag{4.19}$$

Using integration by parts we obtain

$$\begin{aligned}
&2^{k_1+k_2+k_3}|K^n_{k\nu}(2^{k_1}x_1,2^{k_2}x_2,2^{k_3}x_3)|\\
&\le C_{NM}\int_{[1/2,2]^3}(1+2^{n_1}|\xi_1/\xi_3-u^1_{\nu_1}|)^{-M}(1+2^{n_2}|\xi_2/\xi_3-u^2_{\nu_2}|)^{-M}d\xi\,2^{n_1+n_2}W^n_{\nu N}(x)\\
&\le C_N W^n_{\nu N}(x).
\end{aligned}$$

In view of the compact support of ϕ we get the same estimates for the derivatives of the left hand side and the desired estimates for $K^n_{k\nu}$ and its derivatives follow.

The estimate for $K^{n,0}_k$ has nothing to do with homogeneity: By the decay properties of $\widehat{\psi_n}$ we have

$$|\partial^{\gamma_1}_{s_1}\partial^{\gamma_2}_{s_2}\varrho^n_k(s)|\le C_{\gamma N}2^{-(n_1+n_2)N}(1+|s_1|)^{-N}(1+|s_2|)^{-N}$$

and hence

$$\left|\partial_{\xi_1}^{\gamma_1}\partial_{\xi_2}^{\gamma_2}\partial_{\xi_3}^{\gamma_3}[\phi(\xi)\varrho_k^n(\xi_1/\xi_3,\xi_2/\xi_3)]\right|\le C_{\gamma N}2^{-(n_1+n_2)N}.$$

The desired estimate for $K_k^{n,0}$ follows by integration by parts. In the proof of the estimate for $K_k^{n,1}$ we replace (4.19) by

$$\begin{aligned}&\left|\partial_{\xi_1}^{N_1}\partial_{\xi_2}^{N_2}\langle e_\nu^1,\nabla_\xi\rangle^{N_3}[\phi(\xi)g_{k\nu_1}^{n,1}(\xi_1/\xi_3,\xi_2/\xi_3)]\right|\\ &\qquad\le C(N_1,N_2,N_3,M)b_{k\nu}^n\frac{2^{N_1n_1}2^{(N_2+N_3-M)n_2}}{(1+2^{n_1}|\xi_1/\xi_3-u_{\nu_1}^1|)^M}\end{aligned}$$

and argue as above. □

In what follows we shall denote by $\widetilde{\beta}$ a function which is similar to β but equals 1 on the support of β. Next let $\chi\in C_0^\infty(\mathbf{R})$ be supported in $\left(-\frac{3}{4},\frac{3}{4}\right)$ such that $\sum_{\varkappa\in\mathbf{Z}}\chi^2(\cdot-\varkappa)\equiv1$. Again let $\widetilde{\chi}\in C_0^\infty$ be defined similarly to χ but equal to 1 on the support of χ. We define the operator $A_{k\nu}^n$ by

$$\widehat{A_{k\nu}^nf}(\xi)=\chi(2^{n_1}(2^{k_1}\xi_1-u_{\nu_1}^12^{k_3}\xi_3))\chi(2^{n_2}(2^{k_2}\xi_2-u_{\nu_2}^22^{k_3}\xi_3))\widetilde{\beta}^2(2^{k_3}\xi_3)\hat{f}(\xi).$$

Lemma 4.5. *There is a weighted norm inequality*

(4.20)
$$\int\sum_k|T_k^n\mathcal{L}_kf(x)|^2\omega(x)\,dx\le C2^{n_1+n_2}\|g_k^n\|_2^2\int\sum_{k,\nu}|A_{k\nu}^n\mathcal{L}_kf(y)|^2\mathcal{M}_n\omega(y)\,dy.$$

Proof. Set $S_{k\nu\mu}^n=T_{k\mu}^nA_{k\nu}^n$, $S_{k\nu\mu_1}^{n,1}=T_{k\mu_1}^{n,1}A_{k\nu}^n$, $S_{k\nu\mu_2}^{n,2}=T_{k\mu_2}^{n,2}A_{k\nu}^n$ and $S_{k\nu}^{n,0}=T_k^{n,0}A_{k\nu}^n$. Then

$$T_k^n\mathcal{L}_kf=\sum_\nu\left[\sum_\mu S_{k\nu\mu}^n+\sum_{\mu_1}S_{k\nu\mu_1}^{n,1}+\sum_{\mu_2}S_{k\nu\mu_2}^{n,2}+S_{k\nu}^{n,0}\right]A_{k\nu}^n\mathcal{L}_kf. \tag{4.21}$$

Let $H_{k\nu\mu}^n$, $H_{k\nu\mu_1}^{n,1}$, $H_{k\nu\mu_2}^{n,2}$ and $H_{k\nu}^{n,0}$ be the convolution kernels of the operators $S_{k\nu\mu}^n$, $S_{k\nu\mu_1}^{n,1}$, $S_{k\nu\mu_2}^{n,2}$ and $S_{k\nu}^{n,0}$, respectively. Fix N (say equal to 100) and let $U_{k\nu}^n\equiv U_{k\nu,100}^n$ etc. The proof of Lemma 4.4 shows that

$$\begin{aligned}|H_{k\nu\mu}^n(x)|&\le C\frac{b_{k\mu}^n}{(1+|\mu_1-\nu_1|^2)(1+|\mu_2-\nu_2|^2)}U_{k\nu}^n(x),\\ |H_{k\nu\mu_1}^{n,1}(x)|&\le C_N2^{-n_2N}\frac{b_{k\mu_1}^{n,1}}{(1+|\mu_1-\nu_1|^2)}U_{k\nu}^n(x),\\ |H_{k\nu\mu_2}^{n,2}(x)|&\le C_N2^{-n_2N}\frac{b_{k\mu_2}^{n,2}}{(1+|\mu_2-\nu_2|^2)}U_{k\nu}^n(x),\\ |H_{k\nu}^{n,0}(x)|&\le C_N2^{-n_1N}2^{-n_2N}b_k^nU_{k\nu}^n(x).\end{aligned}$$

We observe that $\|U^n_{k\nu}\|_1 \le C$ is bounded uniformly in ν, n, k. Therefore

$$\begin{aligned}
&\Big|\sum_\nu\sum_\mu S^n_{k\nu\mu}A^n_{k\nu}\mathcal{L}_k f(x)\Big| \\
&\quad\le C\sum_\nu\sum_\mu \frac{|b^n_{k\mu}|}{(1+|\mu_1-\nu_1|^2)(1+|\mu_2-\nu_2|)^2}\Big(\int U^n_{k\nu}(x-z)dz\Big)^{1/2} \\
&\qquad\times\Big(\int |A^n_{k\nu}\mathcal{L}_k f(y)|^2 U^n_{k\nu}(x-y)\,dy\Big)^{1/2} \\
&\quad\le C\Big(\sum_\nu\Big|\sum_\mu \frac{|b^n_{k\mu}|}{(1+|\mu_1-\nu_1|^2)(1+|\mu_2-\nu_2|)^2}\Big|^2\Big)^{1/2} \\
&\qquad\times\Big(\sum_\nu\int |A^n_{k\nu}\mathcal{L}_k f(y)|^2 U^n_{k\nu}(x-y)\,dy\Big)^{1/2} \\
&\quad\le C\Big(\sum_\mu |b^n_{k\mu}|^2\Big)^{1/2}\Big(\sum_\nu\int |A^n_{k\nu}\mathcal{L}_k f(y)|^2 U^n_{k\nu}(x-y)dy\Big)^{1/2} \\
&\quad\le C2^{n_1+n_2}\|g^n_k\|_2\Big(\sum_\nu\int |A^n_{k\nu}\mathcal{L}_k f(y)|^2 U^n_{k\nu}(x-y)dy\Big)^{1/2},
\end{aligned}$$

where for the last inequality we have used (4.16). Using also (4.17) we derive the same inequality for the other three remaining terms in (4.21) and obtain
(4.22)

$$\Big(\sum_k |T^n_k\mathcal{L}_k f(x)|^2\Big)^{1/2} \le C2^{(n_1+n_2)/2}\|g^n_k\|_2\Big(\int\sum_{k,\nu}|A^n_{k\nu}\mathcal{L}_k f(y)|^2 U^n_{k\nu}(x-y)\,dy\Big)^{1/2}.$$

Finally there is the pointwise estimate

$$\sup_{k\nu} U^n_{k\nu}*|\omega|(x)\le C\mathcal{M}_n\omega(x) \tag{4.23}$$

and (4.22) and (4.23) imply (4.20). □

Proposition 4.6. *There is the weighted norm inequality*

$$\begin{aligned}
\sum_{k,\nu}\int |A^n_{k\nu}f_k(x)|^2\omega(x)\,dx \hspace{6em}& \\
\le C_{N_0}2^{(n_1+n_2)2\varepsilon'}\sum_k\int |f_k(x)|^2\mathcal{M}_n^{N_0}\omega(x)\,dx,&\quad N_0>2+1/\varepsilon'.
\end{aligned}\tag{4.24}$$

Proof. It is convenient to introduce a decomposition in the ξ_3-variable which will give the factor of $2^{2\varepsilon'(n_1+n_2)}$. We define for $(\lambda_1,\lambda_2)\in\mathbf{Z}^2$ operators $V_{k\lambda}^{n\varepsilon'}$ by

$$\widehat{V_{k\lambda}^{n\varepsilon'}f}(\xi)=\chi^2(2^{k_3}\xi_3-2^{n_1\varepsilon'}\lambda_1)\chi^2(2^{k_3}\xi_3-2^{n_2\varepsilon'}\lambda_2)\hat{f}(\xi).$$

Observe that $A_{k\nu}^n$ is a sum of no more then $O(2^{\varepsilon'(n_1+n_2)})$ operators $V_\lambda^{n\varepsilon'}A_{k\nu}^n$ where $2^{n_1\varepsilon'}\lambda_1\in\left(\frac{1}{20},20\right)$ and $2^{n_2\varepsilon'}\lambda_2\in\left(\frac{1}{20},20\right)$. Therefore it suffices to show that for those λ the inequality

$$\begin{aligned}(4.25)\quad \sum_{k,\nu}\int|V_{k\lambda}^{n\varepsilon'}A_{k\nu}^n f_k(x)|^2\omega(x)\,dx& \\ \le C_{N_0}\sum_k\int|f_k(x)|^2\mathcal{M}_n^{N_0}\omega(x)\,dx,&\quad N_0>2+1/\varepsilon',\end{aligned}$$

holds. In order to show (4.25) we first prove an inequality for an analogous problem in two dimensions.

Lemma 4.7. *Let $\delta\ll1$ and let m,μ,ϱ be integers such that $m>0$, $2^{-m}\mu\in\left(\frac{1}{20},20\right)$ and $2^{-m\delta}\varrho\in\left(\frac{1}{20},20\right)$. Let B_μ^m, $C_\varrho^{m\delta}$ be the operators acting on functions in $\mathbf{R}^2$ defined by*

$$\widehat{B_\mu^m f}(\xi)=\chi(2^m(\xi_1-2^{-m}\mu\xi_2))\hat{f}(\xi),$$
$$\widehat{C_\varrho^{m\delta}f}(\xi)=\chi^2(2^{m\delta}(\xi_2-2^{-m\delta}\varrho))\hat{f}(\xi).$$

Let $l\le\max\{1,m\delta\}$. Then

$$\sum_\mu\int|B_\mu^m C_\varrho^{m\delta}f(x)|^2\omega(x)\,dx\le C\sum_{\mu'}\int|B_{\mu'}^{m-l}C_\varrho^{m\delta}f(x)|^2M_m^{(1,2)}\omega(x)\,dx.$$

Proof. Let

$$R_{\mu\varrho}=\{\xi;|\xi_1-2^{-m}\mu\xi_2|\le2^{-m+1};|\xi_2-2^{-l}\varrho|\le2^{-l+1}\},$$
$$\widetilde{R}_{\mu\varrho}=\{\xi;|\xi_1-2^{-m}\mu\xi_2|\le2^{-m+2};|\xi_2-2^{-l}\varrho|\le2^{-l+4}\}.$$

Let $\xi'\in R_{\mu'\varrho}$ and suppose that $|\mu-\mu'|\le2^{-l+2}$. Let $a_{\mu-\mu'}=(2^{-m}(\mu-\mu'),0)$. Then $\xi'-a_{\mu-\mu'}\in\widetilde{R}_{\mu\varrho}$. Thus

$$R_{\mu'\varrho}\subset a_{\mu-\mu'}+\widetilde{R}_{\mu\varrho}.$$

Define

$$\widehat{\Gamma_{\mu\mu'\varrho}^{ml}f}(\xi)=\chi(2^{m-10}(\xi_1-2^{-m}(\mu-\mu')-2^{-m}\mu\xi_2))\widetilde{\chi}(2^{m\delta}(\xi_2-2^{-m\delta}\varrho))\hat{f}(\xi).$$

and define $\widetilde{C}_{\varrho}^{m\delta}$ similarly as $C_{\varrho}^{m\delta}$ (with χ replaced by $\widetilde{\chi}$). Then

$$\sum_{\mu} |B_{\mu}^{m} C_{\varrho}^{m\delta} f(x)|^2 \le C \sum_{\mu'} \sum_{\mu} |\widetilde{C}_{\varrho}^{m\delta} B_{\mu}^{m} \Gamma_{\mu\mu'\varrho}^{ml} C_{\varrho}^{m\delta} B_{\mu'}^{m-l} f(x)|^2.$$

An integration by parts argument shows that the convolution kernel of $\widetilde{C}_{\varrho}^{m\delta} B_{\mu}^{m}$ is bounded by C_N times

$$w_{\mu N}^{m,\delta}(x) = \frac{2^m}{(1+2^m|\langle x, e_\mu\rangle|)^N} \frac{2^{m\delta}}{(1+2^{m\delta}|\langle x, e_\mu^{\perp}\rangle|)^N}$$

if $e_\mu=(1,-2^{-m}\mu)$, $e_\mu^{\perp}=(2^{-m}\mu,1)$ and if $2^{-m}\mu\approx 1$. Now the argument which led to (4.22) and Lemma 4.3 show that for fixed μ'

$$\begin{aligned} &\sum_{|\mu-\mu'|\le l} \int |\widetilde{C}_{\varrho}^{m\delta} B_{\mu}^{m} \Gamma_{\mu\mu'\varrho}^{ml} C_{\varrho}^{m\delta} B_{\mu'}^{m-l} f(x)|^2 \omega(x)\, dx \\ &\qquad \le C_N \sum_{|\mu-\mu'|\le l} \int |\Gamma_{\mu\mu'\varrho}^{ml} C_{\varrho}^{m\delta} B_{\mu'}^{m-l} f(x)|^2 w_{\mu\varrho}^{m,\delta} * |\omega|(x)\, dx \\ &\qquad \le C_N \int |C_{\varrho}^{m\delta} B_{\mu'}^{m-l} f(x)|^2 \sup_{|\mu-\mu'|\le l} w_{\mu\varrho}^{m,\delta} * w_{\mu\varrho}^{m,\delta} * |\omega|(x)\, dx. \end{aligned}$$

The asserted inequality is an immediate consequence. □

We now conclude the proof of Proposition 4.6. First, since the maximal operator $M_m^{(1,2)}$ is invariant under two-parameter dilations there is a scaled variant of Lemma 4.7. Also we can apply Lemma 4.7 twice, in the x_1-x_3 and in the x_2-x_3 plane; the same applies to the scaled variant. We obtain the inequality

$$\sum_{k,\nu} \int |V_{k\lambda}^{n\varepsilon'} A_{k\nu}^{n} f_k(x)|^2 \omega(x)\, dx \le C \sum_{k,\nu'} \int |V_{k\lambda}^{n\varepsilon'} A_{k\nu'}^{n-l} f_k(x)|^2 \mathcal{M}_n \omega(x)\, dx$$

if $l=(l_1,l_2)$ and $l_1\le n_1\varepsilon'$, $l_2\le n_2\varepsilon'$. We iterate and apply this inequality N times; here $N\le 1+1/\varepsilon'$. The result is an estimate of the left hand side of (4.25) by an expression involving a scaled version of the square-function in Lemma 4.3 (with $A=\operatorname{diag}(2^{k_1},2^{k_2})$). Namely if $\Gamma_{k\nu\lambda}^{n,\delta}$ is defined by

$$\widehat{\Gamma_{k\nu\lambda}^{n,\delta} f}(\xi) = \prod_{i=1}^{2} [\chi^2(2^{m\delta}(2^{k_3}\xi_3-2^{-m\delta}\lambda_i))\chi^2(2^{m\delta}(2^{k_i}\xi_i-2^{-m\delta}\nu_i))]\hat{f}(\xi)$$

we obtain the inequality

$$\sum_{k,\nu}\int |V_{k\lambda}^{n\varepsilon'}A_{k\nu}^{n}f_k(x)|^2\omega(x)\,dx\le C_N\sum_{k,\nu,\lambda'}\int |\Gamma_{k\nu\lambda'}^{n,\varepsilon'}f_k(x)|^2\mathcal{M}_n^N\omega(x)\,dx,$$

$$N\ge 1+\frac{1}{\varepsilon'},$$

from which (4.25) follows by an application of Lemma 4.3. □

The asserted weighted norm inequality (4.10) now follows by an application of Lemma 4.5 and Proposition 4.6. This concludes the proof of Theorem 4.1.

Remark. The weighted inequality in Proposition 3.6 implies

$$\left\|\left(\sum_{k\nu}|A_{k\nu}^n\mathcal{L}_kf|^2\right)^{1/2}\right\|_4\le C_\varepsilon 2^{(n_1+n_2)\varepsilon}\|f\|_4$$

with $C_\varepsilon=O(A^{1/\varepsilon})$ as $\varepsilon\to 0$, some $A>1$. The geometrical arguments by Córdoba [12] show that in fact $C_\varepsilon=O(\varepsilon^{-a})$ for some $a>0$. It would be interesting to find positive operators $\mathcal{N}_\varepsilon$, being uniformly bounded on L^2 such that

$$\sum_{k,\nu}\int |A_{k\nu}^n\mathcal{L}_kf(x)|^2\omega(x)\,dx\le C\varepsilon^{-2a}2^{(n_1+n_2)2\varepsilon}\int |f(x)|^2\mathcal{N}_\varepsilon[\omega](x)\,dx.$$

An analogous problem is to find weighted norm inequalities for radial multipliers and associated maximal functions in $\mathbf{R}^2$, with a positive operator $\mathcal{N}$. In this context weighted inequalities with a nonpositive $\mathcal{N}$ have been proved in [1].

5. H^p-estimates

The purpose of this section is to prove Theorem 1.7. The proof relies on a result on multiparameter Calderón–Zygmund theory obtained by the authors in [4] (extending earlier results by Journé [20] and Fefferman [17]). There it is shown for a large class of singular integral operators T that the boundedness of T on certain scalar and vector-valued rectangle atoms implies the boundedness on H^p.

To be precise let R be an interval in $\mathbf{R}^d$ (i.e. a rectangle parallel to the coordinate axes), and let Q be a nonnegative integer. In what follows, Q will always be $\ge[1/p-1]$ (the largest integer $\le 1/p-1$). Then a is called a (p,Q,R) *rectangle atom* if a is supported in R, if

$$\int_R |a(x)|^2dx\le |R|^{1-2/p}$$

and if for $m=1,\dots,d$

$$\int_{\mathbf{R}^m} a(x_1,\dots,x_m,x_{m+1},\dots,x_d)x_1^{r_1}\dots x_m^{r_m}\,dx_1\dots dx_m=0, \quad r_1,\dots,r_m=0,\dots,Q$$

for almost all $(x_{m+1},\dots,x_d)$; furthermore assume that the analogous cancellation properties hold for all permutations of the variables $x_1,\dots,x_d$.

Now let $\mathbf{R}^d=\mathbf{R}^{d_1}\oplus\mathbf{R}^{d_2}$, and let I be an interval in $\mathbf{R}^{d_1}$. Then we need the notion of an $L^2(\mathbf{R}^{d_2})$-valued (p,Q,I) rectangle atom. This is simply a function a supported on $I\times\mathbf{R}^{d_2}$ such that

$$\iint |a(x',x'')|^2 dx''dx' \le |I|^{1-2/p}$$

and such that for $m=1,\dots,d_1$

$$\int_{\mathbf{R}^m} a(x_1,\dots,x_m,x_{m+1},\dots,x_{d_1+1},\dots,x_d)x_1^{r_1}\dots x_m^{r_m}\,dx_1\dots dx_m=0,$$
$$r_1,\dots,r_m=0,\dots,Q$$

for almost all $(x_{m+1},\dots,x_d)$; furthermore assume that the analogous cancellation properties hold for all permutations of the variables $x_1,\dots,x_{d_1}$.

Now let $T\colon C_0^\infty(\mathbf{R}^d)\to(C_0^\infty(\mathbf{R}^d))'$ be an operator with Schwartz kernel K, with the property that $K(x,y)$ is locally integrable in $\{(x,y);x_i\neq y_i, i=1,\dots,d\}$. Let Φ be a smooth bump function on $\mathbf{R}$ supported in $[1,4]$ such that $\sum_{l=-\infty}^{\infty}\Phi(2^{-l}s)=1$ for $s\neq 0$. For $l=(l_1,\dots,l_{d_1})$, $1\le d_1\le d$, define the operator T^l by

$$T^l f(x)=\int K(x,y)\prod_{i=1}^{d_1}\Phi(2^{-l_i}|x_i-y_i|)f(y)\,dy.$$

Theorem 5.1 [4]. *Let $0<p\le 1$, $s>d(d+1)/2$ and $Q\ge[1/p-1]$, $M>2$. Suppose that*

(1) *T is bounded on $L^2(\mathbf{R}^d)$ with operator norm $\le A$.*

(2) *For all $d_1\in\{1,\dots,d-1\}$, for all $L\in\mathbf{Z}^{d_1}$, for all intervals I in $\mathbf{R}^{d_1}$ with sidelengths $2^{L_1},\dots,2^{L_{d_1}}$, for all $L^2(\mathbf{R}^{d-d_1})$ valued (p,Q,I) rectangle atoms a and for all $l=(l_1,\dots,l_{d_1})$, $l_i>1$, $i=1,\dots,d_1$*

$$\|T^{L+l}a\|_{L^p(\mathbf{R}^{d_1},L^2(\mathbf{R}^{d_2}))}\le A\left(\sum_{i=1}^{d_1} l_i\right)^{-s/p}. \tag{5.1}$$

(3) *The condition analogous to (5.1) is valid for every permutation of the variables $x_1,\dots,x_d$.*

(4) *For all* $L\in\mathbf{Z}^d$, *for all intervals* R *in* $\mathbf{R}^d$ *with sidelengths* $2^{L_1},\dots,2^{L_d}$, *for all* (p,Q,R) *rectangle atoms* a *and for all* $l=(l_1,\dots,l_d)$, $l_i>1$, $i=1,\dots,d-1$

$$\|T^{L+l}a\|_{L^p(\mathbf{R}^d)}\leq A\left(\sum_{i=1}^d l_i\right)^{-s/p}. \tag{5.2}$$

Then T *extends to a bounded operator from the multiparameter Hardy space* $H^p(\mathbf{R}^d)$ *to* $L^p(\mathbf{R}^d)$ *and the operator norm is bounded by* CA. *Here* C *depends only on* p, d *and* s. *If* T *is translation invariant then* T *is bounded on* $H^p(\mathbf{R}^d)$.

We now consider convolution operators T given by Fourier multipliers m via $\widehat{Tf}(\xi)=m(\xi)\hat{f}(\xi)$. For $k\in\mathbf{Z}^{d_1}$ let T_k be the operator with Fourier multiplier $m(\xi)\prod_{i=1}^{d_1}\beta(2^{k_i}\xi_i)$. Variants of the standard Marcinkiewicz multiplier theorem on H^p-spaces follow from Theorem 5.1 and

Proposition 5.2. *Suppose that* $0<p\leq 1$, $\alpha>1/p-\frac{1}{2}$ *and let* Q, ε *be such that* $Q\geq[1/p-1]$ *and* $0<2\varepsilon<\min\{\alpha-1/p+\frac{1}{2},Q-1/p+2,1\}$.

(1) *Suppose that* $1\leq d_1\leq d-1$ *and*

$$\sup_{t\in(\mathbf{R}_+)^{d_1}}\ \sup_{(\xi_{d_1+1},\dots,\xi_d)\in\mathbf{R}^{d-d_1}}\|\beta_{(1)}\otimes\dots\otimes\beta_{(d_1)}m(t_1\cdot,\dots,t_{d_1}\cdot,\xi_{d_i+1},\dots,\xi_d)\|_{\mathcal{H}^2_\alpha(\mathbf{R}^{d_1})}<\infty. \tag{5.3}$$

Then for all $L\in\mathbf{Z}^{d_1}$, *for all intervals* I *in* $\mathbf{R}^{d_1}$ *with sidelengths* $2^{L_1},\dots,2^{L_{d_1}}$, *for all* $L^2(\mathbf{R}^{d-d_1})$*-valued* (p,Q,I) *rectangle atoms* a, *for all* $l=(l_1,\dots,l_{d_1})$, $l_i>1$, $i=1,\dots,d_1$ *and for all* $k\in\mathbf{Z}^{d_1}$

$$\|(T_k)^{L+l}a\|_{L^p(\mathbf{R}^{d_1},L^2(\mathbf{R}^{d-d_1}))}\leq CA\prod_{i=1}^{d_1}2^{-\varepsilon(l_i+|k_i|)}. \tag{5.4}$$

(2) *The inequality analogous to* (5.4) *holds for every permutation of the variables* $x_1,\dots,x_d$.

(3) *Suppose that*

$$\sup_{t\in(\mathbf{R}_+)^d}\|\beta_{(1)}\otimes\dots\otimes\beta_{(d)}m(t_1\cdot,\dots,t_d\cdot)\|_{\mathcal{H}^2_\alpha(\mathbf{R}^d)}<\infty. \tag{5.5}$$

Then for all $L\in\mathbf{Z}^d$, *for all intervals* R *in* $\mathbf{R}^d$ *with sidelengths* $2^{L_1},\dots,2^{L_d}$, *for all* (p,Q,R) *rectangle atoms* a, *for all* $l=(l_1,\dots,l_d)$, $l_i>1$, $i=1,\dots,d$, *for all* $k\in\mathbf{Z}^d$

$$\|(T_k)^{L+l}a\|_{L^p(\mathbf{R}^d)}\leq CA\prod_{i=1}^{d}2^{-\varepsilon(l_i+|k_i|)}. \tag{5.6}$$

If (5.5) *is valid then* m *is bounded and* (5.3) *and the analogous conditions obtained by permuting variables are also satisfied. In particular* (5.5) *implies that* T *is bounded on the multiparameter Hardy space* $H^p(\mathbf{R}^d)$ *and the operator norm is bounded by* CA.

Proposition 5.2 is proved by standard arguments, see for example the proof of [4, Proposition 5.1]. The last conclusion of Proposition 5.2 follows of course by Theorem 5.1. The reader should note that the multipliers in Theorem 1.7 generally do not satisfy the assumption (5.5), even in the two-dimensional case.

Proof of Theorem 1.7. We may clearly assume that $p \le 1$. Again since characteristic functions of half spaces with boundaries parallel to the coordinate axes are Fourier multipliers of multiparameter Hardy spaces there is no loss of generality in assuming that m is supported in $\{\xi; \xi_i \ge 0, i=1,\dots,d\}$. We use the notation introduced in the proof of Theorem 4.1. Let T_k^n be as in (4.9) and set $T^n = \sum_{k\in\mathbf{Z}^d} T_k^n$. We shall show that T^n is bounded on $H^p(\mathbf{R}^d)$ with operator norm bounded by

$$C_s \sup_{k\in\mathbf{Z}^d} \|g_k^n\|_{L^p(\mathbf{R}^{d-1})} 2^{(n_1+\dots+n_{d-1})(2/p-1)} (1+n_1+\dots+n_{d-1})^{(s+d)/p}. \tag{5.7}$$

Since

$$\sum_{n\in(\mathbf{N}_0)^{d-1}} \sup_{k\in\mathbf{Z}^d} \|g_k^n\|_{L^p(\mathbf{R}^{d-1})} 2^{(n_1+\dots+n_{d-1})(2/p-1)} (1+n_1+\dots+n_{d-1})^{(s+d)/p}$$
$$\le C_\varepsilon \sup_{k\in\mathbf{Z}^d} \|g_k\|_{\mathcal{H}_\alpha^p} \quad \text{if } \alpha > 2/p-1$$

the conclusion of Theorem 1.7 follows.

We have to verify the hypotheses (5.1) and (5.2) of Theorem 5.1 for the operator T^n. The mixed norm inequalities are a straightforward consequence of Proposition 5.2. In order to see this let

$$F_h(\xi) = h(\xi_1/\xi_d, \dots, \xi_{d-1}/\xi_d)$$

where h is compactly supported in $\left[\frac{1}{2}, 2\right]^{d-1}$. Then for $\alpha \ge 0$ one has the inequalities

$$\sup_{\xi_d} \|\beta_{(1)} \otimes \dots \otimes \beta_{(d)} F_h(\,\cdot\,, \xi_d)\|_{\mathcal{H}_\alpha^2(\mathbf{R}^{d-1})} \le C \|h\|_{\mathcal{H}_\alpha^2(\mathbf{R}^{d-1})} \tag{5.8}$$

and

$$\sup_{\xi_1} \|\beta_{(1)} \otimes \dots \otimes \beta_{(d)} F_h(\xi_1, \,\cdot\,)\|_{\mathcal{H}_\alpha^2(\mathbf{R}^{d-1})} \tag{5.9}$$
$$\le C \begin{cases} \|h\|_{\mathcal{H}_\alpha^2(\mathbf{R})}, & \text{if } d=2, \\ \|h\|_{\mathcal{H}_\alpha^2(\mathbf{R}^{d-1})} + \sup_{s_1} \sum_{k=2}^{d-1} \|\mathcal{D}_2^\alpha \dots \mathcal{D}_{k-1}^\alpha \mathcal{D}_k^{2\alpha} \mathcal{D}_{k+1}^\alpha \dots \mathcal{D}_{d-1}^\alpha h(s_1, \,\cdot\,)\|_{L^2(\mathbf{R}^{d-2})}, \\ \quad \text{if } d \ge 3. \end{cases}$$

It is straightforward to verify (5.8) and (5.9) if α is a nonnegative even integer and the general case follows by analytic interpolation. Note also that by a version of Sobolev's imbedding theorem $\mathcal{H}^p_\beta(\mathbf{R}^{d-1})\subset\mathcal{H}^2_\alpha(\mathbf{R}^{d-1})$ if $p\le 2$ and $\beta\ge\alpha+\left(1/p-\frac{1}{2}\right)$. Using this and (5.8), (5.9) we see that (5.3) is verified for the case $d_1=d-1$. The other cases follow similarly. An application of Proposition 5.2 implies (5.2).

The main work in the proof consists in the verification of (5.1). Assume that a is a (p,Q,R) rectangle atom and R is an interval of dimensions $2^{L_1}\times\ldots\times 2^{L_d}$. Then we shall prove that

$$\|(T^n_k)^{L+l}a\|_p\le C2^{(n_1+\ldots+n_{d-1})N}\prod_{i=1}^d 2^{-\varepsilon(l_i+|k_i-L_i|)}\|g^n_k\|_\infty,\quad N>2\left(\frac{1}{p}-\frac{1}{2}\right) \tag{5.10}$$

for some $\varepsilon>0$ and also

$$\|(T^n_k)^{L+l}a\|_p\le C2^{(n_1+\ldots+n_{d-1})(2/p-1)}\|g^n_k\|_p. \tag{5.11}$$

We shall use (5.11) only if $\max_j\{k_j-L_j\},\max_j\{l_j\}\le C_p(1+\sum_i n_i)$ where C_p is a large fixed constant while (5.10) is a remainder estimate. In fact applying the Sobolev inequality (4.8) with $d_1=0$ we see that (5.10) and (5.11) imply

$$\begin{aligned}
&\Big\|\sum_{k\in\mathbf{Z}^d}(T^n_k)^{L+l}a\Big\|_p\\
&\le C\Bigg(\sum_{\substack{\max\{|k_i-L_i|,i=1,\ldots,d\}\ge\\ \varepsilon^{-1}(2N+2/p)(n_1+\ldots+n_{d-1})}} 2^{(n_1+\ldots+n_{d-1})(Np+1)}\prod_{i=1}^d 2^{-\varepsilon p(l_i+|k_i-L_i|)}\|g^n_k\|_p^p\\
&\quad+\sum_{\substack{\max\{|k_i-L_i|,i=1,\ldots,d\}<\\ \varepsilon^{-1}(2N+2/p)(n_1+\ldots+n_{d-1})}}\\
&\quad\times\min\Big\{2^{(n_1+\ldots+n_{d-1})(2-p)};2^{(n_1+\ldots+n_{d-1})(Np+1)}\prod_{i=1}^d 2^{-\varepsilon p(l_i+|k_i-L_i|)}\Big\}\|g^n_k\|_p^p\Bigg)^{1/p}\\
&\le C2^{(n_1+\ldots+n_{d-1})(2/p-1)}\frac{(1+n_1+\ldots+n_{d-1})^{(s+d)/p}}{(l_1+\ldots+l_d)^{s/p}}\|g^n_k\|_p
\end{aligned}$$

and it follows that T^n is bounded on H^p with norm not exceeding (5.7).

The verification of (5.10) is easy. Simply observe that

$$|\partial^\gamma_\xi[\phi(\xi)m_{k,n}(\xi)]|\le C_\gamma 2^{n_1\gamma_1+\ldots+n_{d-1}\gamma_{d-1}}(2^{n_1\gamma_d}+\ldots+2^{n_{d-1}\gamma_d})$$

and an application of Proposition 5.2 yields (5.10).

We now verify (5.11) and assume for convenience $d=3$. We show that (using the notation introduced in (4.15))

$$\|(T^n_{k\nu})^{L+l}a\|_p \le Cb^n_{k\nu}2^{(n_1+n_2)(1/p-1)}\|g^n_k\|_p, \tag{5.12}$$

$$\|(T^{n,1}_{k\nu_1})^{L+l}a\|_p \le C_N b^{n,1}_{k\nu_1}2^{n_1(1/p-1)}2^{-n_2N}\|g^n_k\|_p, \tag{5.13}$$

$$\|(T^{n,2}_{k\nu_2})^{L+l}a\|_p \le C_N b^{n,2}_{k\nu_2}2^{-n_1N}2^{n_2(1/p-1)}\|g^n_k\|_p, \tag{5.14}$$

$$\|(T^{n,0}_{k})^{L+l}a\|_p \le C_N b^{n}_{k}2^{-(n_1+n_2)N}\|g^n_k\|_p. \tag{5.15}$$

Using (4.16) and (4.17) with $r=p$ we see that (5.11) follows from (5.12–15). We shall only verify (5.12); the remaining cases are similar or simpler.

We divide the rectangle R (which has dimensions $2^{L_1}\times 2^{L_2}\times 2^{L_3}$) into $\prod_{i=1}^3 \max\{1,2^{L_i-k_i}\}$ congruent intervals R^μ_k of dimensions

$$\min\{2^{L_1},2^{k_1}\}\times\min\{2^{L_2},2^{k_2}\}\times\min\{2^{L_3},2^{k_3}\}$$

and centers y^μ_k. Let be $a^\mu_k = a\chi_{R^\mu_k}$ and let

$$\mathcal{R}^{L+l}_\mu = \{x; 2^{L_i+l_i-2} \le |x_i-(y^\mu_k)_i| \le 2^{L_i+l_i+2}, i=1,2,3\}.$$

Then it is easy to check that if $y\in \operatorname{supp} a^\mu_k$, $x\in\mathcal{R}^{L+l}_\mu$ then for $U^n_{k\nu,N}$ as in Lemma 4.4

$$U^n_{k\nu,N}(x-y) \approx U^n_{k\nu,N}(x-y^\mu_k)$$

and therefore by Lemma 4.4

(5.16)

$$\begin{aligned}
&\|(K^n_{k\nu}\Phi_{L+l})*a\|_p \le C\Big(\sum_\mu \|(K^n_{k\nu}\Phi_{L+l})*a^\mu_k\|_p^p\Big)^{1/p}\\
&\le C_N b^n_{k\nu}\Big(\sum_\mu\int\Big[\int U^n_{k\nu,N}(x-y)|\Phi_{L+l}(x-y)||a^\mu_k(y)|\,dy\Big]^p dx\Big)^{1/p}\\
&\le C_N 2^{(n_1+n_2)(1/p-1)}b^n_{k\nu}\Big(\sum_\mu\int_{\mathcal{R}^{L+l}_\mu}2^{-(n_1+n_2+k_1+k_2+k_3)(p-1)}\big[U^n_{k\nu N}(x-y^\mu_k)\big]^p dx\\
&\quad\times\Big[\int|a^\mu_k(y)|\,dy\,2^{(k_1+k_2+k_3)(1/p-1)}\Big]^p\Big)^{1/p}.
\end{aligned}$$

Using Hölder's inequality we see that

$$\begin{aligned}
2^{(L_1+L_2+L_3)(1/p-1)}&\Big(\sum_\mu\prod_{i=1}^3[\min\{1,2^{(k_i-L_i)(1-p)}\}]\|a^\mu_k\|_1^p\Big)^{1/p}\\
&\le C2^{(L_1+L_2+L_3)(1/p-1)}\|a\|_1 \le C|R|^{1/p-1/2}\|a\|_2\le C'.
\end{aligned} \tag{5.17}$$

We perform the linear volume preserving change of variables

$$(v_1,v_2,v_3)=(x_1,x_2,2^{k_1-k_3}u^1_{\nu_1}x_1+2^{k_2-k_3}u^2_{\nu_2}x_2+x_3)$$

and see that for $N>1/p$

$$\int_{\mathcal{R}_\mu^{L+l}} 2^{-(n_1+n_2+k_1+k_2+k_3)(p-1)}[U^n_{k\nu,N}(x-y^\mu_k)]^p\,dx$$
$$\leq C\int \frac{2^{-k_1-n_1}}{(1+|2^{-k_1-n_1}v_1|)^{Np}}\frac{2^{-k_2-n_2}}{(1+|2^{-k_2-n_2}v_2|)^{Np}}\frac{2^{-k_3}}{(1+|2^{-k_3}v_3|)^{Np}}dv\leq C'.$$

Therefore if $k_i-L_i\leq 0$, $i=1,2,3$, the desired estimate (5.12) follows from (5.16) and (5.17).

In all other cases we use similar arguments together with the cancellation properties of the atom. For example assume $k_1\leq L_1$, $k_2\leq L_2$, $k_3\leq L_3$. Since

$$\iint a^\mu_k(y_1,y_2,y_3)y_1^{r_1}y_2^{r_2}\,dy_1dy_2=0$$

for almost all y_3 for $0\leq r_1,r_2\leq Q$ we see, using Taylor's formula, that

$$(K^n_{k\nu}\Phi_{L+l})*a^\mu_k(x_1,x_2,x_3)=\int_0^1\frac{(1-s)^Q}{Q!}\int\left(\frac{\partial}{\partial x_3}\right)^{Q+1}$$
$$\times(K^n_{k\nu}\Phi_{L+l})(x_1-y_1,x_2-y_2,x_3-(y^\mu_k)_3+s((y^\mu_k)_3-y_3))((y^\mu_k)_3-y_3)^{Q+1}a^\mu_k(y)\,dy\,ds$$

and using Leibniz' rule and Lemma 4.4 we see that

$$|(K^n_{k\nu}\Phi_{L+l})*a^\mu_k(x_1,x_2,x_3)|$$
$$\leq C2^{L_3(Q+1)}\max\{2^{-k_3(Q+1)},2^{-(L_3+l_3)(Q+1)}\}U^n_{k\nu,N}(x-y^\mu_k)b^n_{k\nu}\|a^\mu_k\|_1.$$

Similar considerations in the other cases (where we use that a^μ_k has cancellation in the y_i variable whenever $k_i\geq L_i$) lead to

$$\|(K^n_{k\nu}\Phi_{L+l})*a\|_p\leq C_N\prod_{i=1}^3[\min\{1,(2^{-l_i}+2^{L_i-k_i})\}]^{Q+1}$$
$$\times b^n_{k\nu}\left(\sum_\mu\int_{\mathcal{R}_\mu^{L+l}}2^{(n_1+n_2+k_1+k_2+k_3)(p-1)}[U^n_{k\nu,N}(x-y^\mu_k)]^p\,dx\|a^\mu_k\|_1^p\right)^{1/p}.$$

As above it is easy to check that for $N>1/p$

$$\int_{\mathcal{R}_\mu^{L+l}} 2^{-(n_1+n_2+k_1+k_2+k_3)(p-1)}[U^n_{k\nu,N}(x-y^\mu_k)]^p\,dx$$
$$\leq C\min\{1,2^{L_1+l_1-k_1-n_1}\}\min\{1,2^{L_2+l_2-k_2-n_2}\}\min\{1,2^{L_3+l_3-k_3}\}.$$

Therefore

$$
\begin{aligned}
&\|(K^n_{k\nu}\Phi_{L+l})*a\|_p \\
&\quad \le C2^{(n_1+n_2)(1/p-1)}b^n_{k\nu}\prod_{i=1}^{3}[\min\{1,2^{(L_i+l_i-k_i)/p}\}\min\{1,(2^{-l_i}+2^{L_i-k_i})^{Q+1}\}] \\
&\qquad \times 2^{(k_1+k_2+k_3)(1/p-1)}\Big(\sum_\mu \|a^\mu_k\|_1^p\Big)^{1/p} \\
&\quad \le C2^{(n_1+n_2)(1/p-1)}b^n_{k\nu}2^{(L_1+L_2+L_3)(1/p-1)} \\
&\qquad \times \Big(\sum_\mu \|a^\mu_k\|_1^p\prod_{i=1}^{3}[\min\{1,2^{(L_i-k_i)(1/p-1)}\}]^p\Big)^{1/p} \\
&\quad \le C2^{(n_1+n_2)(1/p-1)}b^n_{k\nu}|R|^{1/p-1}\|a\|_1 \\
&\quad \le C2^{(n_1+n_2)(1/p-1)}b^n_{k\nu}.
\end{aligned}
$$

This proves (5.12) and concludes the proof of Theorem 1.7. □

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Anthony Carbery
Centre for Mathematical Analysis and its Applications
University of Sussex
Falmer, Brighton BN19QH
United Kingdom

Andreas Seeger
Department of Mathematics
University of Wisconsin
Madison, WI 53706
U.S.A.