Polynomial growth estimates for multilinear singular integral operators

by

MICHAEL CHRIST and JEAN-LIN JOURNÉ

Princeton University
Princeton, NJ, U.S.A.

This paper originated with a question of Yves Meyer: Let T be a convolution Calderón-Zygmund operator on $\mathbf{R}^d$, $d\geq 2$, with kernel $K(x-y)$. Let $b^1, \ldots, b^m$ be m L^∞ functions and let $F\in C^\infty(\mathbf{C}^m)$. Does the kernel

$$L(x,y)=K(x-y)\,F(\ldots,\int_0^1 b_i(tx+(1-t)y)\,dt,\ldots)$$

define an operator bounded on $L^2(\mathbf{R}^d)$? When $d=1$ this is equivalent to asking whether the nth Calderón commutator is bounded on L^2 with polynomial growth of the operator norm, that is, with a bound Cn^M as $n\to\infty$ [2]. The argument in [1] also reduces the higher-dimensional problem to proving the boundedness of a sequence of operators, which we call the d-commutators, with polynomial growth. The kernel of the nth d-commutator is

$$L(x,y)=K(x-y)\left[\int_0^1 a(tx+(1-t)y)\,dt\right]^n$$

where $a\in L^\infty(\mathbf{R}^d)$ is complex-valued, and the question is whether

$$\left\|\int L(x,y)f(y)\,dy\right\|_2\leq Cn^M\|a\|_\infty^n\|f\|_2$$

for all $f\in L^2$, $a\in L^\infty$ and $n\in\mathbf{Z}^+$, the integral being suitably interpreted. This question is motivated in part by work of Leichtnam [3], and in part by the formal analogy with the Calderón commutators. We answer it in the affirmative.

For an arbitrary $a\in L^\infty$, the expresssion $\int_0^1 a(tx+(1-t)y)\,dt$ is a far less regular function of x, y when $d\geq 2$ than when $d=1$. Consequently the kernels of the d-commutators fail to satisfy the "standard estimates" of Calderón-Zygmund theory, and the general boundedness criterion of [5] does not apply. In fact the d-commutators actually fail to map L^∞ to BMO when $d \geq 2$. The first issue here is the boundedness of the individual d-commutors.

The second issue is polynomial growth of the bounds. None of the techniques already known for the Calderón commutators, direct or via the Cauchy integral on Lipschitz curves, seem to generalize to the case $d \geq 2$. This has lead us to formalize the notion of a multilinear singular integral operator (MSIO) used implicitly in [2] and [4]; we regard the d-commutators as multilinear operators in f and a, with f and a placed on an even footing. This formalization permits a transparent generalization of the $T1$-Theorem of [5] to the multilinear context, a generalization which easily yields polynomial growth of bounds for MSIO's in fairly general circumstances. In particular we obtain a new, conceptually simple proof of the boundedness of the Calderón commutators, with a bound $c_\delta(n+1)^{1+\delta}$ for all $\delta>0$. However, when $d \geq 2$ the d-commutators lie slightly outside the scope of this general result, and their analysis involves further considerations.

In Section 1 we review some background material on Calderón-Zygmund theory. In Section 2 we prove a $T1$-Theorem for Carleson measures and apply it to the Kato operator [2] in dimension 1. The notion of MSIO is discussed in Section 3, where a general boundedness criterion is proved. The application to the Calderón commutators follows in Section 4. In Section 5 we indicate some elements of the study of the d-commutators and analyse the smoothness of $m_{x,y}a = \int_0^1 a(tx+(1-t)y)\,dt$ in dimensions $d \geq 2$. It turns out that on the average $m_{x,y}a$ is somewhat smoother than is apparent; it is on this extra smoothness that our proof is based. In Section 6 we split the nth d-commutator into two parts. The first part is treated by applying the general theory of MSIO's of Section 3. The second part, to which the general theory does not apply because its kernel is insufficiently regular, is treated in Section 7. The final section treats the L^p boundedness for $p \neq 2$.

In a forthcoming paper the second author will extend the theory of MSIO's to the product setting to establish polynomial growth for the Calderón-Coifman bicommutators [17].

We are grateful to the referee for the comments which have helped to improve our exposition.

1. Preliminaries

A singular integral operator is initially defined as a mapping from $C_0^\infty(\mathbf{R}^d)$ to its dual. In other words it is defined by a bilinear form on $[C_0^\infty(\mathbf{R}^d)]^2$. In the next definition we emphasize this aspect, which is more suitable for a generalization to the multilinear context.

Definition 1. Let $\delta>0$. A δ-bilinear singular integral form (δ-BSIF) is a mapping T: $[C_0^\infty(\mathbf{R}^d)]^2 \to \mathbf{C}$ with the following property: if f and g have disjoint supports, then

$$T(g,f)=\iint K(x,y)\,g(x)f(y)\,dx\,dy \tag{1.1}$$

where K is a function defined for $x\neq y$ such that, for all x, y, and x' satisfying $|x-x'|\leq|x-y|/2$,

$$|K(x,y)|\leq\frac{c}{|x-y|^d} \tag{1.2}$$

$$|K(x,y)-K(x',y)|\leq\frac{c|x-x'|^\delta}{|x-y|^{d+\delta}} \tag{1.3}$$

$$|K(y,x)-K(y,x')|\leq\frac{c|x-x'|^\delta}{|x-y|^{d+\delta}}. \tag{1.4}$$

The best constant c in (1.2) is denoted as $|K|_0$, and in (1.2), (1.3) and (1.4), $|K|_\delta$ or $|T|_\delta$. Notice that if $\delta'<\delta$

$$|K|_{\delta'}\leq c_{\delta,\delta'}\,|K|_0^{1-\delta'/\delta}\,|K|_\delta^{\delta'/\delta} \tag{1.5}$$

A δ-BSIF can be extended to $C_{00}^\infty(\mathbf{R}^d)\times C_b^\infty(\mathbf{R}^d)$ or $C_b^\infty(\mathbf{R}^d)\times C_{00}^\infty(\mathbf{R}^d)$, where $C_b^\infty(\mathbf{R}^d)$ denotes the space of bounded C^∞ functions and $C_{00}^\infty(\mathbf{R}^d)$, the subspace of $C_0^\infty(\mathbf{R}^d)$ of functions with vanishing integral [5]. We then denote by $T_1(1)$ the element of $[C_{00}^\infty(\mathbf{R}^d)]'$ defined by

$$\langle g, T_1(1)\rangle = T(g,1) \quad \text{for all } g\in C_{00}^\infty(\mathbf{R}^d) \tag{1.6}$$

and define $T_2(1)$ dually.

Definition 2. The δ-BSIF T has the weak-boundedness property (WBP) if for all pairs of $C_0^\infty(\mathbf{R}^d)$ functions f and g whose supports have diameter at most $4t$,

$$|T(g,f)|\leq ct^d(\|g\|_\infty+t\|\nabla g\|_\infty)(\|f\|_\infty+t\|\nabla f\|_\infty). \tag{1.7}$$

The best constant c in (1.7) is denoted $|T|_W$.

Definition 3. The δ-BSIF T is bounded if for all pairs of $C_0^\infty(\mathbf{R}^d)$ functions f and g

$$|T(g,f)| \leq c\|f\|_2\|g\|_2. \tag{1.8}$$

The best constant c in (1.8) is denoted $\|T\|_{2,2}$ and $\|T\|_\delta = |K|_\delta + \|T\|_{2,2}$. The following are well-known.

THEOREM A. *Let T be a δ-BSIF. The following are each equivalent to the boundedness of T:*

$$|T(g,f)| \leq c\|g\|_{H^1}\|f\|_\infty \tag{1.9}$$

$$|T(f,g)| \leq c\|g\|_{H^1}\|f\|_\infty \tag{1.10}$$

for all $g \in C_{00}^\infty(\mathbf{R}^d)$, $f \in C_0^\infty(\mathbf{R}^d)$, or

$$|T(g,f)| \leq ct^d\|f\|_\infty\|g\|_\infty \tag{1.11}$$

for all, $g,f \in C_0^\infty(\mathbf{R}^d)$ whose supports have diameter at most $4t$.

A proof of this theorem can be found in [6].

$T1$-THEOREM [5]: *The form T is bounded if and only if $T_1 1$ and $T_2 1$ lie in* BMO *and T has the* WBP, *and then*

$$\|T\|_{2,2} \leq c(\|T_1 1\|_{\mathrm{BMO}} + \|T_2 1\|_{\mathrm{BMO}} + |T|_W) + c_\delta |T|_\delta. \tag{1.12}$$

The main ingredients in the proof of this theorem are the almost-orthogonality lemma of Cotlar-Knapp-Stein, quadratic estimates and Carleson measures. We shall briefly recall these elements for future reference.

LEMMA CKS [7]. *Let $(R_t)_{t>0}$ be a family of operators on a Hilbert space H. If for some $\alpha>0$, all $s>0$ and all $t>0$,*

$$\|R_{st}^* R_t\| + \|R_t R_{st}^*\| \leq c(s \wedge s^{-1})^\delta \tag{1.13}$$

then $\int_0^{+\infty} R_t dt/t$ defines a bounded operator and is strongly convergent.

An easy corollary of this lemma is that if only $\|R_tR_{st}^*\| \leqslant c(s\wedge s^{-1})^\delta$, then for each $x\in H$

$$\int_0^{+\infty} \|R_tx\|^2\frac{dt}{t} \leqslant c\|x\|^2. \tag{1.14}$$

Definition 4. A function $w:\mathbf{R}_+^{d+1}\to\mathbf{C}$ is a Carleson function if for all balls B of $\mathbf{R}^d$

$$\left[\frac{1}{|B|}\int_{B\times]0,r[} |w(x,t)|^2\,dx\frac{dt}{t}\right]^{1/2} \leqslant c \tag{1.15}$$

where r is the radius of B.

The best constant in (1.15) is denoted $|w|_c$ or $|w_t|_c$.

One interest of Carleson functions lies in the following fact. Let f be an L^2 function on $\mathbf{R}^d$ and $p_tf(x)$ denote its Poisson integral. Then [8]

$$\int\int_{\mathbf{R}_+^{d+1}} |p_tf(x)|^2|w(x,t)|^2\frac{dx\,dt}{t} \leqslant c|w|_c^2\|f\|_2^2. \tag{1.16}$$

This inequality accounts for the wide use of quadratic estimates in [9], [2], [4]. We shall however encounter a slight technical difficulty in reducing our problems to quadratic estimates. Even though this is quite standard, we shall describe why it occurs, and how it is dealt with.

Let ψ be a radial function in $C_{00}^\infty(\mathbf{R}^d)$, and for all $t>0$ let Q_t be the convolution operator with symbol $\hat{\psi}(t\xi)$. We shall have to show that for certain families $(f_t)_{t>0}$ of L^2 functions the integral $\int_0^{+\infty}Q_tf_t\,dt/t$ is weakly convergent and defines an L^2 function. The easiest way to do this is to choose $\hat{\psi}$ as the product of two functions $\hat{\psi}_1$ and $\hat{\psi}_2$ of the same kind, so that Q_t can be written as $Q_t^{(1)}Q_t^{(2)}$. Then, in order to show that for $g\in L^2$, $\int_0^{+\infty}\langle g,Q_tf_t\rangle\,dt/t$ is absolutely convergent, one uses Cauchy-Schwarz to dominate it by

$$\left[\int_0^{+\infty} \|Q_t^{(2)*}g\|_2^2\frac{dt}{t}\right]^{1/2}\left[\int_0^{+\infty} \|Q_t^{(1)}f_t\|_2^2\frac{dt}{t}\right]^{1/2}$$

The first factor is equal to $c_0\|g\|_2$ for some $c_0>0$, by Plancherel's theorem. One is then reduced to estimating

$$\left[\int_0^{+\infty} \|Q_t^{(1)}f_t\|_2^2\frac{dt}{t}\right]^{1/2}.$$

This route is unavailable to us for the following reason. The operator Q_t will arise as $-t(\partial/\partial t)P_t$ where $(P_t)_{t>0}$ is defined as follows. Let φ be a non-negative radial $C_0^\infty(\mathbf{R}^d)$ function with $\int\varphi=1$ and let P_t be the multiplier with symbol $\hat{\varphi}(t\xi)$ for all $t>0$. The condition $\varphi\geqslant 0$ will be needed to ensure that P_t is a contraction on L^∞, which will be essential in our argument, but prevents us from writing $\hat{\psi}(\xi)=-\langle\xi,\nabla\hat{\varphi}(\xi)\rangle$ as a product $\hat{\psi}_1(\xi)\hat{\psi}_2(\xi)$ in a straightforward way. What we do instead is to introduce an auxiliary function $\tilde{\psi}$ and to define $\tilde{Q}_t$ accordingly, so that $\tilde{Q}_t$ is self-adjoint and $\int_0^{+\infty}\tilde{Q}_t^2\,dt/t=I$. To show that $\int_0^{+\infty}Q_tf_t dt/t$ is weakly convergent, it then suffices to show that

$$\int_0^{+\infty}\int_0^{+\infty}\tilde{Q}_{st}^2Q_tf_t\frac{dt}{t}\frac{ds}{s}$$

is weakly convergent. Using Cauchy-Schwarz as before, we see that this reduces to estimating

$$\int_0^{+\infty}\left[\int_0^{+\infty}\|\tilde{Q}_{st}Q_tf_t\|^2\frac{dt}{t}\right]^{1/2}\frac{ds}{s}. \tag{1.17}$$

To show that the integration in s has no harmful effect, it suffices to show that $\tilde{Q}_{st}Q_t$ behaves like a small Q_t when s is very small or large. More precisely we have the following.

LEMMA 1. *Let ψ and $\tilde{\psi}$ be two functions in $C_{00}^\infty(\mathbf{R}^d)$. For all $s>0$ let η_s be defined by*

$$\eta_s(x)=\int_{\mathbf{R}^d}\psi(x-y)\,s^{-d}\tilde{\psi}\left(\frac{y}{s}\right)dy.$$

Then

$$|\eta_s(x)|+|\nabla\eta_s(x)|\leqslant c(s\wedge s^{-1})^{1/2}\frac{1}{1+|x|^{d+1/2}}. \tag{1.18}$$

This lemma is straightforward and we omit its proof.

From the above remarks it follows that if $(f_t)_{t>0}$ are L^2 functions then

$$\left\|\int_0^{+\infty}Q_tf_t\frac{dt}{t}\right\|_2\leqslant c\sup_\theta\left[\int_0^{+\infty}\|\bar{Q}_tf_t\|_2^2\frac{dt}{t}\right]^{1/2} \tag{1.19}$$

where $\tilde{Q}_t$ is the multiplier with symbol $\hat{\theta}(t\xi)$ and the sup ranges over those radial θ such that $\int\theta=0$ and

$$|\theta(x)|+|\nabla\theta(x)|\leq\frac{1}{1+|x|^{d+1/2}}. \tag{1.20}$$

If the right-hand side of (1.19) is finite it follows that the left-hand side converges weakly.

We shall conclude these preliminaries with a lemma of Coifman and Meyer. For every function $\beta\in C_0^\infty(\mathbf{R}^d)$ and $(x,t)\in\mathbf{R}_+^{d+1}$ denote by β_t^x the function such that

$$\beta_t^x(z)=\frac{1}{t^d}\beta\left(\frac{z-x}{t}\right).$$

For all $t>0$ and $\delta\in]0,1[$ we denote by $w_{\delta,t}$ the function defined on $\mathbf{R}^d$ by

$$w_{\delta,t}(z)=\frac{t^\delta}{t^{d+\delta}+|z|^{d+\delta}}.$$

LEMMA 2. *Let T be a δ-BSIF having the WBP. If $\theta\in C_0^\infty(\mathbf{R}^d)$ and $\xi\in C_{00}^\infty(\mathbf{R}^d)$,*

$$|T(\theta_t^x,\xi_t^y)|\leq cw_{\delta,t}(x-y) \tag{1.21}$$

for all $x,y\in\mathbf{R}^d$ and $t>0$.

Conversely let $(T_t)_{t>0}$ be a family of operators whose kernels satisfy

$$|T_t(x,y)|\leq cw_{\delta,t}(x-y) \tag{1.22}$$

$$|\nabla_x T_t(x,y)|+|\nabla_y T_t(x,y)|\leq\frac{c}{t}w_{\delta,t}(x-y). \tag{1.23}$$

Then if $T_t1=0$ for all $t>0$, the integral $\int\langle g,T_tf\rangle dt/t$ is absolutely convergent for f and $g\in C_0^\infty(\mathbf{R}^d)$. The bilinear form T it defines is a δ'-BSIF for all $\delta'<\delta$ and has the WBP.

We shall omit the proof.

2. A *T*1-Theorem for Carleson measures

Carleson measures were used in [2] to efficiently estimate norms of families of multilinear operators. We shall see that this can be done in some generality using the following theorem.

Definition 4. A family $\mathscr{S}=(S_t)_{t>0}$ of operators given by kernels satisfying

$$|S_t(x,y)| \leq c w_{\varepsilon,t}(x-y) \tag{2.1}$$

$$|S_t(x,y)-S_t(x,z)| \leq c\left(\frac{|y-z|}{t+|x-y|}\right)^{\varepsilon} w_{\varepsilon,t}(x-y), \tag{2.2}$$

for all x, y and z such that $|y-z| \leq \frac{1}{2}(t+|x-y|)$, is an ε-family. It is bounded if for all $f \in L^2$

$$\left[\int \|S_t f\|_2^2 \frac{dt}{t}\right]^{1/2} \leq c\,\|f\|_2. \tag{2.3}$$

We denote by $|S_t|_\varepsilon$ or $|\mathscr{S}|_\varepsilon$ the best constant in (2.1) and (2.2) and by $\|\mathscr{S}\|_2$ the best constant in (2.3).

THEOREM 1. *Let $\mathscr{S}$ be an ε-family. It is bounded if and only if $F:\mathbf{R}_+^{d+1}\to\mathbf{C}$ defined by $F(x,t)=S_t1(x)$, is a Carleson function. In this case for all $a \in L^\infty$, $|S_t a|_c < +\infty$ and*

$$|S_t a|_c \leq \|a\|_\infty |S_t 1|_c + C_\varepsilon \|a\|_\infty |S_t|_\varepsilon. \tag{2.4}$$

The essential difference between this theorem and the $T1$-Theorem is the absence of a multiplicative constant in front of $\|a\|_\infty |S_t 1|_c$. This will enable us to apply Theorem 1 repeatedly to obtain polynomial growth in cases where the $T1$-Theorem would yield exponential growth.

To prove the theorem let φ and $(P_t)_{t>0}$ be as before. Let $\{S_t 1\}$ be the operator of pointwise multiplication by the function $S_t 1$. Notice that $\mathscr{S}'=(S_t')_{t>0}=(S_t-\{S_t 1\}P_t)_{t>0}$ is itself an ε-family. Moreover $S_t'1=0$. It follows that for all $s>0$, $t>0$, $\|S_t' S_{st}'^{*}\| \leq c(s \wedge s^{-1})^{\delta}$ for some $\delta>0$. By (1.14), $\mathscr{S}'$ is bounded. Therefore $\mathscr{S}$ is bounded if and only if $(\{S_t 1\}P_t)_{t>0}$ is a bounded ε-family, that is, if $|S_t 1|_c < +\infty$. Notice that in this case, if $a \in L^\infty$,

$$|S_t a|_c \leq |(S_t 1)(P_t a)|_c + |S_t' a|_c. \tag{2.5}$$

Since P_t is a contraction on L^∞,

$$|(S_t 1)(P_t a)|_c \leq \|a\|_\infty |S_t 1|_c. \tag{2.6}$$

Finally, to prove that

$$|S_t'a|_c \leqslant c\|a\|_\infty |S_t|_\varepsilon, \tag{2.7}$$

consider an arbitrary ball B in $\mathbf{R}^d$ of radius r. Set $a=a_1+a_2$ where $a_1=a\chi_{2B}$. Then $a_1 \in L^2$ and its contribution may be treated using the boundedness of $\mathscr{S}'$. The a_2-term is treated using (2.1) for $(S_t')_{t>0}$. We omit the details, which are standard [9]. Clearly (2.5), (2.6) and (2.7) imply (2.4) and Theorem 1 is proved.

This theorem is nothing but a general version of the commutation lemma of Coifman-McIntosh-Meyer [2]. We are going to see, however, that it permits us to improve the estimates of [2] for the Kato operator $\sqrt{DaD}$ in dimension 1. The problem is to estimate the norm of $\int_0^{+\infty} q_t(M_bp_t)^k dt/t$ where M_b is the operator of multiplication by a function $b \in L^\infty$, and q_t and p_t are for all $t>0$ convolution operators with symbols $t\xi/(1+t^2\xi^2)$ and $1/(1+t^2\xi^2)$ respectively.

Proposition 1. *For all $\delta>0$, there exists a constant $c_\delta>0$ such that for all $k \geqslant 0$, $f \in L^2$, $b \in L^\infty$,*

$$\left\| \int_0^{+\infty} q_t(M_bp_t)^k f \frac{dt}{t} \right\|_2 \leqslant c_\delta (1+k)^{1+\delta} \|f\|_2 \|b\|_\infty^k, \tag{2.8}$$

the integral being weakly convergent.

First we reduce (2.8) to a quadratic estimate. To this end we write

$$\frac{t\xi}{1+t^2\xi^2} = \left(\frac{t|\xi|}{1+t^2\xi^2}\right)^{1/2} \beta(t\xi).$$

If α_t denotes the convolution operator with symbol $(t|\xi|/(1+t^2\xi^2))^{1/2}$, then for all $g \in L^2$, $\int_0^{+\infty} \|\alpha_t g\|_2^2 dt/t = c_0\|g\|_2^2$ for some constant c_0. Let β_t denote the convolution operator with symbol $\beta(t\xi)$, so that $q_t = \alpha_t\beta_t$. For any test function $g \in L^2$,

$$\begin{aligned} \left| \left\langle g, \int_0^\infty q_t(M_bp_t)^k f \frac{dt}{t} \right\rangle \right| &\leqslant \int_0^\infty |\langle \alpha_t g, \beta_t (M_bp_t)^k f \rangle| \frac{dt}{t} \\ &\leqslant \left(\int_0^\infty \|\alpha_t g\|_2^2 \frac{dt}{t} \right)^{1/2} \left(\int_0^\infty \|\beta_t (M_bp_t)^k f\|_2^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

So it suffices to prove

$$\left[\int_0^\infty \|\beta_t(M_bp_t)^k f\|_2^2 \frac{dt}{t} \right]^{1/2} \leqslant c_\delta (1+k)^{1+\delta} \|f\|_2 \|b\|_\infty^k. \tag{2.9}$$

By routine arguments it may be shown that the kernel $\beta_t(x-y)$ of β_t satisfies

$$|\beta_t(x-y)| \leq c\left(\frac{t}{|x-y|}\right)^{1/2} \frac{1}{t+|x-y|};$$

see for instance [10, p. 73]. The kernel of p_t is $(2t)^{-1}e^{-|x-y|/t}$, so p_t is a contraction on L^∞. These estimates on the kernels of β_t and p_t imply immediately that the kernel of $\beta_t(M_b p_t)^k$ satisfies (2.1) and (2.2) for all exponents $\varepsilon \leq 1/2$, with a constant $c_\delta(1+k)^\delta \|b\|_\infty^k$ where $\delta=\delta(\varepsilon)$ may be taken to be arbitrarily small by choosing ε to be sufficiently small. (2.9) follows from Theorem 1 and

$$|\beta_t(M_b p_t)^k 1|_c \leq A_\delta k^{1+\delta} \|b\|_\infty^k \quad \text{for all } k \geq 1. \tag{2.10}$$

We prove (2.10) by induction on k. For $k=0$ it is routine that $|\beta_t b|_c \leq c_0 \|b\|_\infty$ [9]. For $k \geq 1$

$$\begin{aligned}
|\beta_t(M_b p_t)^k 1|_c &= |\beta_t(M_b p_t)^{k-1} b|_c \\
&\leq \|b\|_\infty |\beta_t(M_b p_t)^{k-1} 1|_c + c_\delta \|b\|_\infty^k k^\delta \\
&\leq A_\delta (k-1)^{1+\delta} \|b\|_\infty^k + c_\delta k^\delta \|b\|_\infty^k \\
&\leq A_\delta k^{1+\delta} \|b\|_\infty^k
\end{aligned}$$

provided A_δ is large enough. The first inequality results from Theorem 1 and the second from the induction hypothesis.

3. Multilinear singular integral forms

A Coifman-Meyer multilinear operator T is usually defined, for some $k \geq 1$, on $[L^\infty(\mathbf{R}^d)]^k \times L^2(\mathbf{R}^d)$ or on a subspace of it. It is then determined by a form U defined on $C \times D(T)$ where C is some space of test functions and $D(T)$ is the domain of T. Let $g \in C$ and $(a_1, a_2, \ldots, a_n, f) \in D(T)$. This form U is related to T by

$$U(g, a_1, \ldots, a_n, f) = \langle g, T(a_1, \ldots, a_n, f) \rangle. \tag{3.1}$$

One feature of the expression $U(g, a_1, \ldots, a_n, f)$ is its formal symmetry in all the $n+2$ functions $g, a_1, \ldots, a_n$ and f. In most examples this symmetry is actually more than formal. This suggests the following.

Definition 5. *Let* $\delta \in]0,1]$. A δ-n-linear singular integral form (δ-n SIF) is a mapping $U: [C_0^\infty(\mathbf{R}^d)]^n \to \mathbf{C}$ with the following property (3.3). For each $1 \leq i < j \leq n$, and any $(n-2)$

$C_0^\infty(\mathbf{R}^d)$ functions $h_1,\ldots,h_k,\ldots,h_n$, $k\neq i,j$, define $U_{ij}(h_1,\ldots,h_k,\ldots,h_n)$ as a bilinear form on $[C_0^\infty(\mathbf{R}^d)]^2$ so that for $h_i,h_j \in C_0^\infty(\mathbf{R}^d)$

$$[U_{ij}(h_1,\ldots,h_k,\ldots,h_n)](h_i,h_j) = U(h_1,\ldots,h_n). \tag{3.2}$$

Then $U_{ij}(h_1,\ldots,h_k,\ldots,h_n)$ is a δ-BSIF and

$$|U_{ij}(h_1,\ldots,h_k,\ldots,h_n)|_\delta \leqslant c_{ij} \prod_{k\neq i,j} \|h_k\|_\infty. \tag{3.3}$$

The best constants in (3.3) are denoted $|U_{ij}|_\delta$ and $\sup_{i,j}|U_{ij}|_\delta$ is denoted $|U|_\delta$.

From Theorem A we see that any of the estimates

$$|U(f_1,\ldots,f_n)| \leqslant c_i\left(\prod_{k\neq i}\|f_k\|_\infty\right)\|f_i\|_{H^1} \quad \text{for all } f_1,\ldots,f_n \in C_0^\infty \tag{3.4}$$

is equivalent to any of the estimates

$$|U(f_1,\ldots,f_n)| \leqslant c_{ij}\left(\prod_{k\neq i,j}\|f_k\|_\infty\right)\|f_i\|_2\|f_j\|_2 \quad \text{for all } f_1,\ldots,f_n \in C_0^\infty. \tag{3.5}$$

Let $\|U\|_i$, $1\leqslant i\leqslant n$, and $\|U\|_{i,j}$, $1\leqslant i<j\leqslant n$ denote the best constants in (3.4) and (3.5) and let $\|U\|$ be their maximum. Then

$$\|U\| \leqslant c_\delta(\inf(\|U\|_i, \|U\|_{i,j}, 1\leqslant i<j\leqslant n)+|U|_\delta) \tag{3.6}$$

with a constant c_δ independent of n or U. We say that U is bounded if $\|U\|<+\infty$.

To U and to each integer $m\in\{1,2,\ldots,n\}$ we associate a multilinear operator $\pi_U^{(m)}$, defined by

$$\langle h_m, \pi_U^{(m)}(h_1,\ldots,h_{m-1},h_{m+1},\ldots,h_n)\rangle = U(h_1,\ldots,h_m,\ldots,h_n).$$

$$\pi_U^{(m)} \text{ maps } (C_0^\infty)^{n-1} \text{ to } (C_0^\infty)'.$$

We are going to use Theorem 1 to give a boundedness criterion for δ-n SIF's. First observe that as in the bilinear case [5], $U(f_1,\ldots,f_n)$ can be given a precise meaning when one function is in $C_{00}^\infty(\mathbf{R}^d)$ and all the others are in $C_b^\infty(\mathbf{R}^d)$. We then define for all $i\in[1,n]$ $U_i 1$ to be the element of $[C_{00}^\infty(\mathbf{R}^d)]'$ such that for all $g\in C_{00}^\infty(\mathbf{R}^d)$,

$$\langle g, U_i 1\rangle = U(1,\ldots,1,g,1,\ldots,1) \tag{3.7}$$

where g is at the ith place. By (3.4) it is necessary that $U_i 1$ be in BMO for all i, for U to be bounded. We next turn to an analogue of the WBP. Let $(P_t)_{t>0}$ be as before.

Definition 6. The δ-n SIF U has the WBP if for all $1 \leq i < j \leq n$ and all $t > 0$, f_i, f_j in $C_0^\infty(\mathbf{R}^d)$ with supports of diameter less than $4t$, and f_k, $k \neq i,j$, in $C_0^\infty(\mathbf{R}^d)$,

$$\begin{aligned} &|U(P_t f_1, \ldots, P_t f_{i-1}, f_i, P_t f_{i+1}, \ldots, P_t f_{j-1}, f_j, P_t f_{j+1}, \ldots, P_t f_n)| \\ &\quad \leq c_{ij} \left(\prod_{k \neq i,j} \|f_k\|_\infty \right) t^d (\|f_i\|_\infty + t\|\nabla f_i\|_\infty)(\|f_j\|_\infty + t\|\nabla f_j\|_\infty). \end{aligned} \tag{3.8}$$

The best constants in (3.8) are denoted $|U_{ij}|_w$ and their maximum is denoted $|U|_w$. Notice that all these constants depend implicitly on the function φ defining the P_t's. Since φ is fixed throughout the paper we omit this dependence. Note that because of the presence of the operators P_t, $|U_{ij}|_w$ is slightly different from $|U_{ij}|_w$ as defined in Definition 2.

THEOREM 2. *A δ-n SIF U is bounded if and only if it has the WBP and all the $U_i 1$'s, $i \in [1, n]$, lie in BMO.*

$$\|U\| \leq c_\delta \left(\sum_{i=1}^{n} \|U_i 1\|_{\mathrm{BMO}} + n^2(|U|_\delta + |U|_w) \right), \tag{3.9}$$

where c_δ does not depend on n.

The fact that c_δ is independent of n will yield polynomial growth for families of multilinear operators.

The proof of Theorem 2 is very much in the spirit of the proof of the $T1$-Theorem given by Coifman and Meyer in [11]. Therefore we shall merely outline it. First observe that, by the WBP of U, if f_i, $1 \leq i \leq n$, are in $C_0^\infty(\mathbf{R}^d)$,

$$\lim_{t \to 0} U(P_t f_1, \ldots, P_t f_n) = U(f_1, \ldots, f_n).$$

and

$$\lim_{t \to +\infty} U(P_t f_1, \ldots, P_t f_n) = 0.$$

It follows that if $Q_t=-t(\partial/\partial t)P_t$, the integral

$$\int_0^{+\infty}\sum_{m=1}^{n} U(P_tf_1,\dots,P_tf_{m-1},Q_tf_m,P_tf_{m+1},\dots,P_tf_n)\frac{dt}{t}$$

is convergent and is equal to $U(f_1,\dots,f_n)$. Actually, the WBP of U and the proof of Lemma 2 imply that for each m the integral

$$\int_0^{+\infty} U(P_tf_1,\dots,P_tf_{m-1},Q_tf_m,P_tf_{m+1},\dots,P_tf_n)\frac{dt}{t} \tag{3.9}$$

is absolutely convergent. We have therefore decomposed U as the sum of m n-linear forms $V^{(m)}$, $1\leq m\leq n$. While it is not clear that the $V^{(m)}$ are themselves δ'-n SIF's for some $\delta'\in]0,1[$, Lemma 2 shows that $|V^{(m)}_{ij}|_{\delta'}$, as defined by (3.2) and (3.3) with $V^{(m)}$ in place of U, is finite for $\delta'<\delta$ if $m\in\{i,j\}$ and in this case $|V^{(m)}_{ij}|_{\delta'}\leq c_{\delta,\delta'}|U_{ij}|_\delta$. Hence if we have

$$|V^{(m)}(f_1,\dots,f_n)|\leq K_1\|f_i\|_2\|f_m\|_2\prod_{\substack{k\neq i\\ k\neq m}}\|f_k\|_\infty \tag{3.10}$$

for some $i\neq m$, it follows from Theorem A that

$$|V^{(m)}(f_1,\dots,f_n)|\leq K_2\|f_m\|_{H^1}\prod_{k\neq m}\|f_k\|_\infty$$

with $K_2\leq c_\delta(K_1+|U|_\delta)$. For all $j\neq m$, another application of Theorem A to $V^{(m)}_{mj}$ gives

$$|V^{(m)}(f_1,\dots,f_n)|\leq K_3\|f_j\|_{H^1}\prod_{k\neq j}\|f_k\|_\infty$$

as well. Therefore if we prove (3.10) with a bound

$$c(\|U_m1\|_{\mathrm{BMO}}+n(|U_\delta|+|U_w|))=c_{(3.11)} \tag{3.11}$$

for $1\leq m\leq n$, we obtain

$$|V^{(m)}(f_1,\dots,f_n)|\leq c_{(3.11)}\|f_1\|_{H^1}\prod_{k>1}\|f_k\|_\infty$$

for all m, and Theorem 2 follows from (3.6).

Let

$$I=\int_0^{+\infty} U(P_tf_1,\ldots,P_tf_{m-1},Q_tf_m,P_tf_{m+1},\ldots,P_tf_n)\frac{dt}{t}.$$

In the notation introduced above,

$$\begin{aligned} I&=\int_0^{+\infty} \langle Q_tf_m,\pi_U^{(m)}(P_tf_1,\ldots,P_tf_n)\rangle\frac{dt}{t} \\ &=\left\langle f_m,\int_0^{+\infty} Q_t\pi_U^{(m)}(P_tf_1,\ldots,P_tf_n)\frac{dt}{t}\right\rangle. \end{aligned}$$

An application of (1.19) and Cauchy-Schwarz gives

$$|I|\leqslant\sup_\theta\left(\int_0^\infty \|\bar{Q}_tf_t\|_2^2\frac{dt}{t}\right)^{1/2}\|f_m\|_2$$

where $f_t=\pi_U^{(m)}(P_tf_1,\ldots,P_tf_n)$ and θ, $\bar{Q}_t$ are as in (1.19). By definition

$$\begin{aligned} \bar{Q}_tf_t(x)&=\int\theta_t(x-u)f_t(u)\,du=\int\theta_t(u-x)f_t(u)\,du \\ &=\langle\theta_t^x,f_t\rangle=U(\ldots,P_tf_{m-1}\,\theta_t^x,P_tf_{m+1},\ldots). \end{aligned}$$

Therefore, assuming without loss of generality that $m=1$ and $i=n$, it suffices to demonstrate

$$\left[\int_{\mathbf{R}_+^{n+1}}|U(\theta_t^x,P_tf_2,\ldots,P_tf_n)|^2\frac{dx\,dt}{t}\right]^{1/2}\leqslant c_{(3.11)}\|f_n\|_2\prod_{j=2}^{n-1}\|f_j\|_\infty \tag{3.12}$$

when θ satisfies (1.20). By Theorem 1 the left-hand side of (3.12) is dominated by

$$(|U(\theta_t^\cdot,P_tf_2,\ldots,P_tf_{n-1},1)|_c+(|U|_w+|U|_\delta)\prod_{j=2}^{n-1}\|f_j\|_\infty)\|f_n\|_2. \tag{3.13}$$

Repeated applications of Theorem 1 immediately yield a domination of (3.13) by

$$(|U(\theta_t^\cdot,1,\ldots,1)|_c+n(|U|_w+|U|_\delta))\left[\prod_{j=2}^{n-1}\|f_j\|_\infty\right]\|f_n\|_2. \tag{3.14}$$

Finally observe that $|U(\theta_t^\cdot,1,\ldots,1)|_c\leqslant c\|U_11\|_{\mathrm{BMO}}$ by the characterization of BMO in terms of Carleson measures and the definition (3.7) of U_11. Therefore (3.10) holds with the bound (3.11) and Theorem 2 is proved.

Notice that in replacing the f_i's by 1 one after the other to go from (3.13) to (3.14), one can proceed in any order. Moreover if $U_1 1=0$ it may happen that replacing only one or two f_i's by 1 suffices to annihilate $U(\theta_t^*, P_t f_2, \dots, P_t f_{n-1}, 1)$, in which case it would be a big waste to continue applying Theorem 1 to estimate 0! To make this precise we define $N(1)$ to be the minimum number of f_i's, $2 \leqslant i \leqslant n$, whose replacement by 1 annihilates $U(Q_t f_1, P_t f_2, \dots, P_t f_n)$. We define $N(m)$ similarly for all $m \in [1,n]$.

THEOREM 2'. *In Theorem* 2 (3.9) *holds with* $\Sigma_m N(m)$ *instead of* n^2.

This refinement is clear from the proof of Theorem 2. We shall see that for the multilinear forms associated to the Calderón commutators $\Sigma_{m=1}^n N(m) \leqslant Cn$.

4. The Calderón commutators

Recall that the Calderón commutators $T_n[a]$, where $a \in L_{\mathbf{C}}^\infty(\mathbf{R})$, are initially defined as bilinear forms on $[C_0^\infty(\mathbf{R})]^2$. Let A be an anti-derivative of a and $f, g \in C_0^\infty(\mathbf{R})$. Then,

$$\langle g, T_n[a]f\rangle = \lim_{\varepsilon\to 0} \int\int_{|x-y|>\varepsilon} \left(\frac{A(x)-A(y)}{x-y}\right)^n \frac{f(y)g(x)}{x-y}\, dx\, dy. \tag{4.1}$$

The existence of the limit is an easy consequence of the smoothness of f and g and of the size and antisymmetry of

$$\left(\frac{A(x)-A(y)}{x-y}\right)^n \frac{1}{x-y}.$$

The theorem of Coifman-McIntosh-Meyer [2] says:

$$\|T_n[a]\|_{2,2} \leqslant c(n+1)^4 \|a\|_\infty^n.$$

PROPOSITION 2. *For all* $\delta>0$ *there exists* $c_\delta>0$ *such that*

$$\|T_n[a]\|_{2,2} \leqslant c_\delta (n+1)^{1+\delta} \|a\|_\infty^n. \tag{4.2}$$

We present this estimate purely as an illustration of Theorem 2 and claim neither sharpness nor novelty. Indeed it is conceivable that the growth rate in (4.2) can be or has been obtained, or even improved, from the work of Murai on the Cauchy kernel [12].

To prove Proposition 2 we consider the $(n+2)$-linear form on $[C_0^\infty(\mathbf{R}^d)]^{n+2}$ defined by

$$U^{(n)}(f_1,\dots,f_n,f_{n+1},f_{n+2})=\lim_{\varepsilon\to 0}\int\int_{|x-y|>\varepsilon}\left[\prod_{i=1}^{n} m_{x,y}f_i\right]f_{n+1}(x)f_{n+2}(y)\frac{dx\,dy}{x-y}$$

where $m_{x,y}f=\int_0^1 f(tx+(1-t)y)\,dt$.

Observe first that if $1\leq i\leq n$, then $N(i)=2$. Indeed, for all $f_1,\dots,f_n\in C_0^\infty(\mathbf{R})$,

$$U^{(n)}(P_tf_1,\dots,P_tf_{i-1},Q_tf_i,P_tf_{i+1},\dots,P_tf_n,1,1)=0$$

because of the antisymmetry of $1/(x-y)$. Hence

$$\sum_{i=1}^{n+2}N(i)\leq 4n+2. \tag{4.3}$$

Also $U_i^{(n)}1=0$ for all $i\in[1,n+2]$ because $U^{(n)}$ is invariant under simultaneous translations of all the f_j's, $j\in[1,n+2]$. By Theorem 2′,

$$\|T_n[a]\|_{2,2}\leq c_\delta\|a\|_\infty^n(n+1)\left[|U^{(n)}|_\delta+|U^{(n)}|_w\right]. \tag{4.4}$$

We are going to show

$$|U^{(n)}|_0\leq 2\quad\text{and}\quad |U^{(n)}|_1\leq 2(n+1) \tag{4.5}$$

and

$$|U^{(n)}|_w\leq C. \tag{4.6}$$

By (1.5), (4.5) implies $|U^{(n)}|_\delta\leq c_\delta(n+1)^\delta$ so that (4.2) follows from (4.4), (4.5) and (4.6). To check (4.5) and (4.6) we shall limit ourselves to the case where $i=1$ and $j=n+2$ in definitions 5 and 6, the other cases being similar or simpler.

Let $f_2,\dots,f_{n+1}\in C_0^\infty(\mathbf{R}^d)$. The kernel $K(v,y)$ of $U_{1,n+2}^{(n)}(f_2,\dots,f_{n+1})$ is given by

$$K(v,y)=\int_{x:\,[x\wedge y,\,x\vee y]\ni v} f_{n+1}(x)\left[\prod_{i=2}^{n} m_{x,y}f_i\right]\frac{1}{|x-y|}\frac{dx}{x-y}.$$

From this expression we see that

$$|K|_0\leq\prod_{i=2}^{n+1}\|f_i\|_\infty\quad\text{and}\quad |K|_1\leq(2n+2)\prod_{i=2}^{n+1}\|f_i\|_\infty.$$

We turn to (4.6). We want to estimate $U(f_1, P_t f_2, \ldots, P_t f_{n+1}, f_{n+2})$ when f_1 and f_{n+2} have supports of diameter at most $4t$. Assume by scale-invariance that $t=1$ and let f_{n+2} be supported in the interval of length 4 centered at y_0. Let $g(x)=f_{n+1}(x)$ for $|x-y_0|\leq 100$ and $=0$ for $|x-y_0|>100$, and let $h=f_{n+1}-g$. Then

$$|m_{x,y}f_1| \leq 4\|f_1\|_\infty |x-y|^{-1} \quad \text{for all} \quad x \in \operatorname{support}(P_1 h), y \in \operatorname{support}(f_{n+2})$$

since $P_1 f_1$ is supported in an interval of length 4. Hence

$$|U^{(n)}(f_1, P_1 f_2, \ldots, P_1 f_n, P_1 h, f_{n+2})| \leq c\prod_{i=1}^{n} \|f_i\|_\infty \int\int_{\substack{|y-y_0|\leq 4 \\ |x-y_0|\geq 90}} |x-y|^{-2} |P_1 h(x)|\,|f_{n+2}(y)|\,dx\,dy$$
$$\leq c\prod_{i=1}^{n+2} \|f_i\|_\infty .$$

For the contribution of g note that $P_1 g$ is supported in an interval of fixed length, and $\|P_1 g\|_{C^1}\leq c\|f_{n+1}\|_\infty$. If the distance between the supports of $P_1 g$ and f_{n+2} is at least one then the desired bound for $U^{(n)}$ follows by direct size estimates. Otherwise for arbitrary L^∞ functions $g_1, \ldots, g_n$ consider the kernel

$$L(x,y) = (x-y)^{-1} \prod_{i=1}^{n} m_{x,y}\, g_i;$$

L is antisymmetric and satisfies

$$|L(x,y)| \leq c|x-y|^{-1} \prod \|g_i\|_\infty .$$

Thus

$$\left|U^{(n)}(g_1, \ldots, g_n, P_1 g, f_{n+2})\right| = \left| \int\int L(x,y)\, P_1 g(x) f_{n+2}(y)\,dx\,dy \right|$$
$$= \frac{1}{2}\left| \int\int L(x,y)\,[P_1 g(x) f_{n+2}(y) - P_1 g(y) f_{n+2}(x)]\,dx\,dy \right|$$
$$\leq c\|f_{n+2}\|_{C^1} \|P_1 g\|_{C^1} \prod \|g_i\|_\infty \int\int_{\substack{|x-y_0|\leq c \\ |y-y_0|\leq c}} |x-y|^{-1}|x-y|\,dx\,dy$$

and the desired estimate follows.

5. The d-commutators

Let T be a Calderón-Zygmund convolution operator on $\mathbf{R}^d$, assumed to be bounded on L^2. It is associated to a kernel $K(x-y)$ satisfying (1.2), (1.3) and (1.4), in the sense of (1.1). We shall also denote by $K(x-y)$ its distribution-kernel, in the sense of the Schwartz kernel theorem. Let $f_1,\ldots,f_{n+2}$ be $n+2$ functions in $C_0^\infty(\mathbf{R}^d)$ and for each $a\in C_b^\infty(\mathbf{R}^d)$ and $x\neq y$ let

$$m_{x,y}\,a=\int_0^1 a(tx+(1-t)\,y)\,dt.$$

Then the integral

$$\iint K(x-y)\left[\prod_{1\le i\le n} m_{x,y}f_i\right]f_{n+1}(x)f_{n+2}(y)\,dx\,dy \tag{5.1}$$

is well-defined and determines an $(n+2)$-linear form W.

THEOREM 3. *For each $\delta>0$ there exists c_δ such that for all $n>0$*

$$|W(f_1,\ldots,f_{n+2})|\le c_\delta n^{2+\delta}\left(\prod_{i=1}^n \|f_i\|_\infty\right)\|f_{n+1}\|_2\,\|f_{n+2}\|_2. \tag{5.2}$$

In order to see the difference between the d-commutators and the Calderón commutators, we shall indicate some elements of the proof when $n=1$. We want an a priori estimate, valid for $a, f, g\in C_0^\infty(\mathbf{R}^d)$:

$$\left|\iint K(x-y)\,m_{x,y}\,a\,g(x)f(y)\,dx\,dy\right|\le c\|a\|_\infty\,\|g\|_2\,\|f\|_2. \tag{5.3}$$

Observe that the kernel $K_a(x,y)$ defined for $x\neq y$ by $K_a(x,y)=K(x-y)m_{x,y}a$ does satisfy (1.2) with a constant $\|a\|_\infty|K|_0$ but satisfies (1.3) or (1.4) with a bound which depends not only on $\|a\|_\infty$ but also on $\|\nabla a\|_\infty$ and on the size of the support of a when $d>1$. Therefore a straightforward application of Calderón-Zygmund theory will not provide an estimate like (5.3) depending only on $\|a\|_\infty$. We shall rely on some weaker kind of smoothness for K_a, which the following lemma expresses.

LEMMA 3. *For all $x_0\in\mathbf{R}^d$, $0<r<R$ and $a\in C_0^\infty(\mathbf{R}^d)$*

$$\iiint_{\substack{x,y,y' \in B(x_0,R) \\ |y-y'|<r}} |m_{x,y}\,a - m_{x,y'}\,a|^2\, dy\, dy'\, dx \leqslant c\left(\frac{r}{R}\right)^{2/3} r^d R^{2d} \|a\|_\infty^2. \tag{5.4}$$

Similar inequalities appear in [13] and [14]. Notice that it is the positive exponent in the factor $(r/R)^{2/3}$ which expresses the smoothness of $m_{x,y}\,a$.

Since (5.4) is dilation- and translation-invariant we may assume that $R=1$ and $x_0=0$, and also that a is supported in $\{z, |z| \leqslant 1\}$. Then it is enough to show that

$$\iint_{\substack{|z-z'|<r \\ |z|,|z'|\leqslant 2}} |m_{x,x+z}\,a - m_{x,x+z'}\,a|^2\, dz\, dz'\, dx \leqslant c r^{d+2/3} \|a\|_2^2. \tag{5.5}$$

The left-hand side of (5.5) is translation invariant. Therefore, by Plancherel, (5.5) is equivalent to

$$\sup_{\xi \in \mathbf{R}^d} \iint_{\substack{|z-z'|\leqslant r \\ |z|,|z'|\leqslant 2}} |m_{0,z}\, e^{i\langle \cdot,\xi\rangle} - m_{0,z'}\, e^{i\langle \cdot,\xi\rangle}|^2 dz\, dz' \leqslant c r^{d+2/3}. \tag{5.6}$$

Let ξ be fixed. Clearly

$$|m_{0,z}\, e^{i\langle \cdot,\xi\rangle} - m_{0,z'}\, e^{i\langle \cdot,\xi\rangle}| \leqslant |\xi|\, r,$$

which gives a majorant $cr^{d+2}|\xi|^2$ for the left-hand side of (5.6). This is sufficient as long as $|\xi| \leqslant r^{-2/3}$. When $|\xi| \geqslant r^{-2/3}$ an immediate calculation shows that

$$\int_{|z|\leqslant 2} |m_{0,z}\, e^{i\langle \cdot,\xi\rangle}|^2\, dz \leqslant \frac{c}{|\xi|} \leqslant c r^{2/3}.$$

This implies (5.6) and the lemma is proved.

Recall that in Calderón-Zygmund theory, smoothness assumptions such as (1.3) and (1.4) are used in particular to show almost-orthogonality of certain families of operators [7]. We are going to see that (5.4) expresses enough smoothness to permit the same thing.

Lemma 4. *Let $(S_t)_{t>0}$ be a family of operators given by kernels $S_t(x,y)$ satisfying* (1.22) *and* (1.23). *Let $S_t[a]$ be the operator given by the kernel $S_t(x,y)m_{x,y}a$. Then the family of operators $((I-P_t)S_t[a](I-P_t))_{t>0}$ satisfies the assumptions of Lemma* CKS *for some $\delta>0$ with a constant depending only on $\|a\|_\infty$.*

Decomposing $I-P_t$ as $\int_0^1 Q_{st}\,ds/s$ we see that it is enough to show for all $s<1$ and some $\varepsilon>0$

$$\|Q_{st}S_t[a]\|_{2,2}\leq cs^{\varepsilon}\|a\|_{\infty}. \tag{5.7}$$

Let $\theta\in C_0^\infty(\mathbf{R}_+)$ be supported in [1/2,2] and satisfy, for $x>0$, $\Sigma_{k\in\mathbf{Z}}\theta(2^kx)=1$ and let $\theta_0=\Sigma_{k\geq 1}\theta(2^k\cdot)$. We write the kernel of $Q_{st}S_t[a]$ as

$$L(x,y)=L(x,y)\,\theta_0\left(\frac{|x-y|}{t}\right)+\sum_{k\leq 0}L(x,y)\,\theta\left(2^k\left(\frac{|x-y|}{t}\right)\right).$$

On the operator side this gives a decomposition of $Q_{st}S_t[a]$ as $V(s,t,a)+\Sigma_{k\leq 0}V(s,t,a,k)$. It suffices to establish

$$\|V(s,t,a)\|_{2,2}\leq cs^{\varepsilon} \tag{5.8}$$

and

$$\|V(s,t,a,k)\|_{2,2}\leq cs^{\varepsilon}2^{k\varepsilon'} \tag{5.9}$$

for some $\varepsilon'>0$. To this effect we need to recall a basic fact: if an operator V has a kernel $V(x,y)$ supported in a strip $|x-y|\leq\tau$, then

$$\|V\|_{2,2}^2\leq c\sup_z\int_{\substack{|x-z|<\tau\\|y-z|<\tau}}|V(x,y)|^2\,dx\,dy. \tag{5.10}$$

Writing out the kernels of $V(s,t,a)$ or $V(s,t,a,k)$, $k\leq 0$, and using (5.10) and (5.4), we obtain (5.8) and (5.9).

It is easy to verify that if T is our original convolution operator, then $(Q_tT)_{t>0}$ satisfies (1.22) and (1.23) with $\delta=1$. By Lemma 4 and Lemma 1 the integral

$$z_t^a=\int_0^{+\infty}(I-P_t)\,[(Q_tT)\,[a]](I-P_t)\frac{dt}{t}$$

converges strongly and determines a bounded operator on L^2 of norm dominated by $c\|a\|_\infty$. If $T[a]$ denotes the first d-commutator, the integral $\int_0^{+\infty}(Q_tT)[a]dt/t$ is weakly convergent and is equal to $T[a]$, since a is assumed to lie in C_0^∞. Thus we have a representation of $T[a]-z_t^a$ as

$$\int_0^{+\infty}P_t((Q_tT)\,[a])(I-P_t)+(I-P_t)((Q_tT)\,[a])\,P_t\frac{dt}{t}. \tag{5.11}$$

Notice that each of these three pieces is already smoother than the original operator, because of the factor P_t, and is closer to being a Calderón-Zygmund operator. The point is now to take advantage of the formal symmetry of the expression $\langle g, T[a]f\rangle$ in a, g and f and to do for the couples (a,f) and (a,g) what we just did for (f,g). Rather than pursuing the case of the first d-commutator, we next present the outline of the proof in the general case.

6. Outline of the proof of Theorem 3 and treatment of the Calderón-Zygmund part

For $t>0$ we denote by W_t the $(n+2)$-linear form derived from W by replacing T by Q_tT, so that for $f_1,\ldots,f_{n+2}\in C_0^\infty(\mathbf{R}^d)$, the integral $\int_0^{+\infty} W_t(f_1,\ldots,f_{n+2})dt/t$ is absolutely convergent and equals $W(f_1,\ldots,f_{n+2})$. From now on we shall implicitly assume that our integrals are truncated in order to ensure convergence.

For each f_i, $1\leq i\leq n+2$, we write $f_i=P_tf_i+(I-P_t)f_i$ inside $W_t(\ldots)$. By developing we obtain 2^{n+2} integrals, out of which exactly one has only P_t's and $n+2$ have one $(I-P_t)$ and $(n+1)$ P_t's. We shall treat the $2^{n+2}-n-3$ remaining terms in the next section using variants of Lemma 4. For the $n+3$ first terms we are going to see that Theorems 2 and 2′ apply.

We denote by $U^{(0)}$ the $(n+2)$-linear form $\int_0^{+\infty} W_t(P_tf_1,\ldots,P_tf_{n+2})dt/t$ and for each $i\in[1,n+2]$ we denote by $U^{(i)}$ the $(n+2)$-linear form

$$\int_0^{+\infty} W_t(P_tf_1,\ldots,P_tf_{i-1},(I-P_t)f_i,P_tf_{i+1},\ldots,P_tf_{n+2})\frac{dt}{t}\equiv\int_0^{\infty} U_t^{(i)}(f_1,\ldots,f_{n+2})\,dt/t.$$

LEMMA 5. *For each $i\in[0,n+2]$, $U^{(i)}$ is a δ-$(n+2)$ SIF for some $\delta>0$. Moreover given $\varepsilon>0$, there exist $\delta>0$ and $c>0$ such that $|U^{(i)}|_\delta\leq c(n+1)^\varepsilon$, where c does not depend on i or n.*

To fix ideas let us write for instance the definition of $U^{(0)}$. If $f_1,\ldots,f_{n+2}\in C_0^\infty(\mathbf{R}^d)$ then

$$U^{(0)}(f_1,\ldots,f_{n+2})=\int_0^{+\infty}\iint (Q_tT)(x,y)\left[\prod_{i=1}^{n} m_{x,y}P_tf_i\right]P_tf_{n+1}(x)P_tf_{n+2}(y)\,dx\,dy\,\frac{dt}{t}.$$

It is clear that the size estimates (1.2) involved in computing $|U^{(0)}|_\delta$ hold with a constant independent of n. And for any fixed $\delta<1$, since Q_tT satisfies the bound in (1.21), the constants in (1.3) and (1.4) grow at most like n. Using (1.5) we see that for δ small enough they grow at most like n^ε for any $\varepsilon>0$. This remains true uniformly for $|U^{(i)}|_\delta$, $i\in[1,n+2]$.

LEMMA 6. *For each $i\in[0,n+2]$ $U^{(i)}$ has the* WBP *and $|U^{(i)}|_w\leqslant c_\varepsilon(n+1)^\varepsilon$ where $\varepsilon>0$ is arbitrary and c_ε is independent of i and n.*

Notice that if $f_1,\dots,f_{n+2}\in C_0^\infty(\mathbf{R}^d)$.

$$|W_t(f_1,\dots,f_{n+2})|\leqslant c\|f_j\|_2\|f_k\|_2\prod_{l\neq j,k}\|f_l\|_\infty \tag{6.1}$$

where c is independent of j, k or n. Indeed only size estimates are involved in proving (6.1) and these are uniform in n.

Suppose we want to verify the WBP of $U^{(i)}$ at scale s. Then by (6.1), if $t\geqslant s$ the integrand

$$U_t^{(i)}(P_sf_1,\dots,P_sf_{j-1},f_j,P_sf_{j+1},\dots,f_k,\dots,P_sf_{n+2})$$

will be dominated by

$$C\prod_{l\neq j,k}\|f_l\|_\infty\|f_j\|_2\|P_tf_k\|_2 \quad \text{if} \quad k\neq i.$$

Otherwise $j\neq i$. In both cases we obtain a majorant

$$\left(\prod_{m=1}^{n+2}\|f_m\|_\infty\right)s^d\left(\frac{s}{t}\right)^{d/2},$$

which is integrable on $[s,+\infty]$. When t is less than s the gain comes from the fact that $(Q_tT)1=0$. More precisely if X_t denotes either P_t or $I-P_t$, not necessarily the same at each occurrence,

$$\begin{aligned}|W_t(&P_sP_tf_1,\dots,X_tf_j,\dots,X_tf_k,\dots,P_sP_tf_{n+2})|\\ &\leqslant cn\left(\frac{t}{s}\right)^\delta s^d\prod_{l\neq j,k}\|f_l\|_\infty(\|f_j\|_\infty+s\|\nabla f_j\|_\infty)\times(\|f_k\|_\infty+s\|\nabla f_k\|_\infty),\end{aligned} \tag{6.2}$$

where c depends only on $\delta<1$. In order to get a growth rate of n^ε it suffices to use (6.2) when $t\leqslant s2^{-n^\varepsilon}$ and to use the trivial bound given by (6.1) for t between $s2^{-n^\varepsilon}$ and s.

Since $U^{(i)}$, $0\leqslant i\leqslant n+2$, are δ-$(n+2)$ SIF's we can apply Theorems 2 and 2′. Notice that $U_j^{(i)}1=0$ for all i and j by translation invariance. When $i=0$, this plus Theorem 2 imply that $\|U^{(0)}\|\leq c_\varepsilon(n+1)^{2+\varepsilon}$. When $i\geq 1$, it is easy to see that $N^{(i)}(k)=1$ if $k\neq i$ since $(I-P_t)1=0$. Theorem 2′ yields $\|U^{(i)}\|\leq c_\varepsilon(n+1)^{1+\varepsilon}$. Therefore the total contribution of the Calderón-Zygmund part is majorized by $c_\varepsilon(1+n)^{2+\varepsilon}$.

7. The rough part

We are left with the $2^{n+2}-n-3$ terms for which at least two $(I-P_t)$'s occur. In order to prove polynomial rather than exponential growth we shall group them according to the last two indices for which $(I-P_t)$ occurs. This gives us $(n+2)(n+1)/2$ packets of the following form: for each $1\leq j<k\leq n+2$,

$$W_{j,k}(f_1,\dots,f_{n+2})$$
$$=\int_0^{+\infty} W_t(f_1,\dots,f_{j-1},(I-P_t)f_j,P_tf_{j+1},\dots,P_tf_{k-1},(I-P_t)f_k,P_tf_{k+1},\dots,P_tf_{n+2})\frac{dt}{t}.$$

We claim that, when $f_1,\dots,f_{n+2}\in C_0^\infty(\mathbf{R}^d)$,

$$|W_{j,k}(f_1,\dots,f_{n+2})|\leq c_\varepsilon(n+1)^\varepsilon\left(\prod_{i=1}^n\|f_i\|_\infty\right)\|f_{n+1}\|_2\|f_{n+2}\|_2 \tag{7.1}$$

where c_ε is independent of j, k and n.

We first consider the case where $(j,k)=(n+1,n+2)$. Observe that in this case Lemma 4 applies immediately with the following change. The kernel $S_t(x,y)\,m_{x,y}a$ is replaced by $S_t(x,y)\prod_{i=1}^n m_{x,y}f_i$. By what amounts to Leibniz's rule, an analogue of (5.4) holds for $\prod_{i=1}^n m_{x,y}f_i$ in the following form, for all x_0 in $\mathbf{R}^d$ and $R>0$:

$$\int\int_{\substack{x,y,y'\in B(x_0,R)\\ |y-y'|\leq r}}\left|\prod_{i=1}^n m_{x,y}f_i-\prod_{i=1}^n m_{x,y'}f_i\right|^2 dx\,dy\,dy'\leq cn^2\left(\frac{r}{R}\right)^{2/3}r^dR^{2d}\prod_{i=1}^n\|f_i\|_\infty^2. \tag{7.2}$$

Moreover as in (1.5), $cn^2(r/R)^{2/3}$ may be replaced by $c_\varepsilon n^\varepsilon(r/R)^{\varepsilon/3}$ for any $\varepsilon>0$. Lemma CKS may then be applied as in Lemma 4 to establish (7.1).

For all other pairs (j,k) we shall reduce (7.1) to an L^2 estimate, where the L^2 functions are f_j and f_k. There are two cases, according to whether $k\leq n$ or $k\in\{n+1,n+2\}$. We consider first the case $j<k\leq n$, and as is readily seen, we may then restrict out attention to the case $(j,k)=(n-1,n)$.

If we fix $f_1, \dots, f_n$ in $C_0^\infty(\mathbf{R}^d)$ then the bilinear form which to $(f_{n+1}, f_{n+2}) \in [C_0^\infty(\mathbf{R}^d)]^2$ associates $W_{n-1,n}(f_1, \dots, f_{n+2})$ is a δ-BSIF with norm bounded by $C_\delta \prod_{i=1}^n \|f_i\|_\infty$. By Theorem A, plus dilation and translation invariance of $W_{n-1,n}$, it will suffice to show the following:

For any $f_{n+1}, f_{n+2} \in C_0^\infty(\mathbf{R}^d)$ supported in $B(0,1)$

$$|W_{n-1,n}(f_1, \dots, f_{n+2})| \leq c \prod_{i=1}^{n+2} \|f_i\|_\infty. \tag{7.3}$$

To prove (7.3) we decompose the integral $\int_0^{+\infty} \dots dt/t$ defining $W_{n-1,n}$ as $\int_1^{+\infty} \dots dt/t + \int_0^1 \dots dt/t$. For the first part we observe that if $t \geq 1$

$$\|P_t f_{n+1}\|_2 \leq ct^{-d/2}\|f_{n+1}\|_\infty \quad \text{and} \quad \|P_t f_{n+2}\|_2 \leq ct^{-d/2}\|f_{n+2}\|_\infty$$

because of the restriction on the supports of f_{n+1} and f_{n+2}. It follows by (6.1) that for all $t>0$

$$\begin{aligned} &|W_t(f_1, \dots, f_{n-2}, (I-P_t)f_{n-1}, (I-P_t)f_n, P_t f_{n+1}, P_t f_{n+2})| \\ &\quad \leq c\left(\prod_{i=1}^{n+2} \|f_i\|_\infty\right)(t^{-d} \wedge 1), \end{aligned} \tag{7.4}$$

which yields (7.3) for the part $\int_1^{+\infty} \dots dt/t$. When $t \leq 1$ notice that $P_t f_{n+1}$ and $P_t f_{n+2}$ are supported in $B(0,2)$ if we assume φ to be supported in $B(0,1)$. It follows from the definition of the d-commutators that only the values of f_{n-1} and f_n in $B(0,3)$ will affect the left-hand side of (7.4) when $t \leq 1$. So we may assume that f_{n-1} and f_n are supported in $B(0,3)$. To handle $\int_0^1 \dots dt/t$ we just have to prove

$$\begin{aligned} &\left|\int_0^1 W_t(f_1, \dots, f_{n-2}, (I-P_t)f_{n-1}, (I-P_t)f_n, P_t f_{n+1}, P_t f_{n+2})\frac{dt}{t}\right| \\ &\quad \leq c\left(\prod_{i \neq n-1, n} \|f_i\|_\infty\right)\|f_{n-1}\|_2\|f_n\|_2. \end{aligned} \tag{7.5}$$

Using Lemma CKS we see that it is enough to prove that for $s \in]0,1[$ and for some $\alpha_0 > 0$

$$|W_t(f_1, \dots, f_{n-2}, Q_{st}f, g, P_t f_{n+1}, P_t f_{n+2})| \leq cs^{\alpha_0}\left(\prod_{i \neq n-1, n} \|f_i\|_\infty\right)\|f\|_2\|g\|_2. \tag{7.6}$$

The left-hand side of (7.6) is dominated by

$$c\left(\int\int w_t(x-y)|m_{x,y}\,Q_{st}f|\,|m_{x,y}\,g|dx\,dy\right)\prod_{i\neq n-1,n}\|f_i\|_\infty$$

where $w_t \equiv w_{\delta,t}$ is as in Lemma 2 and δ is the exponent in the estimate (1.3) for the kernel K of T. By Cauchy-Schwarz it is enough to prove

$$\int\int w_t(x-y)|m_{x,y}\,g|^2dx\,dy \leqslant c\|g\|_2^2 \tag{7.7}$$

and

$$\int\int w_t(x-y)|m_{x,y}\,Q_{s,t}f|^2dx\,dy \leqslant cs^{2\alpha_0}\|f\|_2^2. \tag{7.8}$$

Both inequalities are translation invariant and are therefore equivalent to

$$\int w_t(u)|m_{0,u}\,e^{i\langle\xi,\cdot\rangle}|^2du \leqslant c \tag{7.9}$$

and

$$\left(\int w_t(u)|m_{0,u}\,e^{i\langle\xi,\cdot\rangle}|^2du\right)|\hat{\psi}(st\xi)|^2 \leqslant cs^{2\alpha_0}. \tag{7.10}$$

The inequality (7.9) is obvious. To prove (7.10) we may assume $|\hat{\psi}(st\xi)| \geqslant s^{\alpha_0}$, which implies $|\xi| \geqslant cs^{\alpha_0-1}/t$, since $|\psi(\eta)| \leqslant c|\eta|$ for all $\eta\in\mathbf{R}^d$. There exists $\beta=\beta(\delta)>0$ such that

$$\int w_t(u)|m_{0,u}\,e^{i\langle\xi,\cdot\rangle}|^2du \leqslant c(t|\xi|)^{-\beta}$$

for all t, ξ. If α_0 is chosen to be sufficiently small then $|\xi| \geqslant cs^{\alpha_0-1}t^{-1}$ implies

$$(t|\xi|)^{-\beta} \leqslant cs^{\beta(1-\alpha_0)} \leqslant cs^{\alpha_0}.$$

This concludes the study of the case where $j\leqslant k\leqslant n$.

We turn to the case where $j\leqslant n<k$, and restrict our attention to the representative case $j=n$ and $k=n+1$. It will suffice to prove

$$\left|\int_0^\infty W_t(f_1,\ldots,f_{n-1},(I-P_t)f_n,Q_{st}f_{n+1},P_tf_{n+2})\frac{dt}{t}\right| \leqslant cs^{\alpha}n\left|\prod_{i=1}^{n}\|f_i\|_\infty\right|\,\|f_{n+1}\|_2\|f_{n+2}\|_2 \tag{7.11}$$

for some $\alpha>0$, for $0<s\leq 1$. Actually (7.11) holds with a factor of n^{ε} on the right, but since there are only $2n$ terms of this type, the bound n suffices for our purpose.

With $f_1, \ldots, f_n \in C_0^\infty(\mathbf{R}^d)$ fixed define a linear operator U_t by

$$\langle g, U_t f\rangle = W_t(f_1, \ldots, f_{n-1}, (I-P_t)f_n, g, f),$$

so that

$$\int_0^\infty W_t(f_1, \ldots, f_{n-1}, (I-P_t)f_n, (I-P_t)f_{n+1}, P_t f_n)\frac{dt}{t} = \int_0^\infty \langle f_{n+1}, (I-P_t)U_t P_t f_{n+2}\rangle \frac{dt}{t}$$
$$= \int_0^\infty \langle f_{n+1}, \int_0^1 Q_{st} U_t P_t f_{n+2}\frac{ds}{s}\rangle\frac{dt}{t}.$$

By (1.19) it will suffice to show that for $s\in(0,1]$

$$\int_0^\infty \|\bar{Q}_{st} U_t P_t f\|_2^2 \leq cs^{\alpha}n^2\|f\|_2^2,$$

assuming that $\|f_i\|_\infty\leq 1$ for all $i<n$. The left-hand side is majorized by twice the sum of

$$\int_0^\infty \int_{\mathbf{R}^d} |\{\bar{Q}_{st} U_t 1\} P_t f(x)|^2 dx \frac{dt}{t} \tag{7.12}$$

and

$$\int_0^\infty \int_{\mathbf{R}^d} |[\bar{Q}_{st} U_t - \{\bar{Q}_{st} U_t 1\}] P_t f(x)|^2 dx \frac{dt}{t}. \tag{7.13}$$

We first treat (7.13). The first claim is that

$$\|\bar{Q}_{st} U_t P_t f\|_2 + \|\{Q_{st} U_t 1\} P_t f\|_2 \leq cns^{\alpha}\|f\|_2. \tag{7.14}$$

Indeed the kernel of the linear operator $\bar{Q}_{st} U_t$ satisfies (1.22) uniformly in n and s, which suffices to give a bound of $c\|f\|_2$. An application of Lemma 3 and of (5.10), as in the proof of (5.7), yields the extra factor of s^{α} for the L^2 operator norm of $\bar{Q}_{st} U_t f$, at the expense of a factor of n. To bound $\{\bar{Q}_{st} U_t 1\} P_t f$ assume by scale-invariance that $t=1$. Since the convolution kernel for P_1 has compact support, f may be assumed to be supported in a ball B of radius 1, in which event $\|P_1 f\|_\infty\leq c\|f\|_2$. Therefore it suffices to show that

$$\|\bar{Q}_s U_1 1\|_{L^2(B)} \leq cns^{\alpha}.$$

This follows readily from Lemma 3 and (1.20). The operator $[\bar{Q}_{st}U_t-\{\bar{Q}_{st}U_t1\}]P_t$ annihilates constants, and its kernel satisifies the second bound in (1.23) uniformly in s and n, so Lemma CKS plus (7.14) imply (7.13).

To derive (7.12) we must show that

$$\|\bar{Q}_{st}U_t1\|_c\leq Cs^\alpha n\|f_n\|_\infty,$$

assuming henceforth that $\|f_i\|_\infty\leq 1$ for all $i<n$ but allowing f_n to vary freely. By scale-invariance it suffices to show that for each ball B of radius 1,

$$\int_0^1\int_B|(\bar{Q}_{st}U_t1)(x)|^2dx\frac{dt}{t}\leq cs^\alpha n^2\|f_n\|_\infty^2.$$

Consider the linear operator V_t defined by $U_t1\equiv V_t(I-P_t)f_n$. Its kernel $k_t(x,y)$ takes the form $\int_0^1 k_{t,\tau}(x,y)\,d\tau$ where

$$k_{t,\tau}(x,y)=\tau^{-d}l_t(\tau^{-1}(x-y))\prod_{i<n}m_{x,w}f_i \tag{7.15}$$

with $w=w(x,y,\tau)=x-\tau^{-1}(x-y)$, l_t denoting the convolution kernel of Q_tT.

Let $\varepsilon>0$ be a small exponent and let g be the restriction of f_n to the ball of radius $s^{-\varepsilon}$ concentric with B. From (7.15) follows easily

$$|k_t(x,y)|\leq c|x-y|^{-d}(t^{-1}|x-y|\wedge(t^{-1}|x-y|)^{-\delta}) \tag{7.16}$$

where $c<\infty$ and $\delta>0$ depend only on T. Therefore for all $t\leq 1$

$$\|\bar{Q}_{st}V_t(I-P_t)(f_n-g)\|_{L^\infty(B)}\leq cs^\gamma t^\gamma\|f_n\|_\infty$$

for some $\gamma(\varepsilon,\delta)>0$, and so

$$\int_0^1\int_B|\bar{Q}_{st}V_t(I-P_t)(f_n-g)|^2dx\frac{dt}{t}\leq cs^{2\gamma}\|f_n\|_\infty^2.$$

Therefore it suffices to show the existence of $\beta>0$ such that for all $g\in L^2$,

$$\int_0^\infty\int_{\mathbf{R}^d}|\bar{Q}_{st}V_t(I-P_t)g(x)|^2dx\frac{dt}{t}\leq cn^2s^\beta\|g\|_2^2,$$

since in the present situation $\|g\|_2\leq cs^{-\varepsilon d/2}\|f_n\|_\infty$ and we may choose $\varepsilon=\beta/2d$. To simplify the discussion let us suppose temporarily that each l_t is supported in $\{|x-y|\leq ct\}$ and still satisfies (1.22) and (1.23). Then $k_{t,\tau}$ is supported in $\{|x-y|\leq ct\tau\}$. It follows from

(7.2) that for any $\eta>0$ there exists $\varrho>0$ such that for all $t\in(0,\infty)$, $\tau\in(0,1]$, $r\leqslant t\tau$ and $x_0\in\mathbf{R}^d$,

$$\iiint_{\substack{x,y,y'\in B(x_0,t\tau)\\ |y-y'|\leqslant r}}\left|\prod_{i<n} m_{x,w}f_i-\prod_{i<n} m_{x,w'}f_i\right|^2 dx\,dy\,dy'\leqslant cn^2(r/t\tau)^{\varrho}\tau^{-\eta}r^d(t\tau)^{2d}$$

where $w(x,y,\tau)$ and $w'(x,y',\tau)$ are as in (7.15). Moreover the same bound holds if the roles of the x and y variables are interchanged, even though $m_{x,w}f_i$ is not a symmetric function of (x,y). Let $V_{t,\tau}$ have kernel $k_{t,\tau}$. Combining (7.16) and (7.17) with (5.10) yields

$$\|\bar{Q}_{st}V_{t,\tau}(I-P_{t'})f\|_2\leqslant cn(s/\tau\wedge t'/t\tau\wedge 1)^{\varepsilon}\|f\|_2$$

for some $\varepsilon>0$. From Lemma CKS we obtain, for $s\leqslant\tau$,

$$\left\|\left(\int_0^\infty|\bar{Q}_{st}V_{t,\tau}(I-P_t)f|^2\frac{dt}{t}\right)^{1/2}\right\|_2\leqslant cn\left(\frac{s}{\tau}\right)^{\varepsilon}(1+\log(\tau^{-1}))\|f\|_2.$$

Then by Minkowski's integral inequality

$$\left\|\left(\int_0^\infty\left|\int_{s^{1/2}}^1\bar{Q}_{st}V_{t,\tau}(I-P_t)f\,d\tau\right|^2\frac{dt}{t}\right)^{1/2}\right\|_2\leqslant cns^{\varepsilon}\|f\|_2$$

with a smaller value of ε. On the other hand

$$\begin{aligned}\left\|\left(\int_0^\infty\left|\int_0^{s^{1/2}}\bar{Q}_{st}V_{t,\tau}(I-P_t)f\,d\tau\right|^2\frac{dt}{t}\right)^{1/2}\right\|_2&\leqslant\int_0^{s^{1/2}}\left\|\left(\int_0^\infty|\bar{Q}_{st}V_{t,\tau}(I-P_t)f|^2\frac{dt}{t}\right)^{1/2}\right\|_2 d\tau\\&\leqslant\int_0^{s^{1/2}}cn(1+\log(\tau^{-1}))\|f\|_2\,d\tau\\&\leqslant cns^{1/4}\|f\|_2,\end{aligned}$$

concluding the proof under our assumption on the support of l_t. To treat the general case it suffices to decompose each l_t as in (5.8) and (5.9) and to apply the above argument to each term individually, as in the proof of (5.7).

8. L^p-boundedness for the d-commutators

In classical Calderón-Zygmund theory, L^p-boundedness for $p\in]1,+\infty[$, $p\neq2$, usually follows by interpolation between $L^2\to L^2$ and $L^\infty\to\mathrm{BMO}$ or $H^1\to L^1$ or $L^1\to$weak-L^1

estimates. However the d-commutators actually fail to map L^∞ to BMO for general $a \in L^\infty$, and it is not presently known whether they are of weak type $(1, 1)$. Nonetheless L^p-boundedness can still be proved for $p \in]1, +\infty[$ as a consequence of the following lemma, already used elsewhere [15], [16] to show L^p-boundedness of operators slightly outside classical Calderón-Zygmund theory.

LEMMA 7. *Let $(z_t)_{t>0}$ be a family of operators whose kernels satisfy* (2.1). *Suppose that the integral $\int z_t \, dt/t$ defines weakly a bounded operator on L^2, and that for all $s \in]0, 1]$, $\int z_t Q_{st} \, dt/t$ defines weakly a bounded operator on L^2, of norm dominated by cs^ε for some $\varepsilon > 0$. Then $\int z_t \, dt/t$ is bounded on L^p, $p \in]1, 2]$.*

To prove this lemma, observe that $\int z_t \, dt/t - \int_0^1 [\int z_t Q_{st} \, dt/t] \, ds/s$ is a bounded operator $\int z_t P_t \, dt/t$. Its kernel satisfies (1.4) and therefore $\int z_t P_t \, dt/t$ is of weak-type $(1, 1)$ and bounded on all L^p's, $p \in]1, 2]$. Observe also that the kernel of $\int z_t Q_{st} \, dt/t$ satisfies (1.2) uniformly in s and (1.4) with a constant $c_\delta s^{-\delta}$. Hence $\int z_t Q_{st} \, dt/t$ is of weak-type $(1, 1)$ with a constant $c_\delta s^{-\delta}$ for all $\delta > 0$ and bounded on L^p, $1 < p \leq 2$, with a norm majorized by s^{ε_p} for some $\varepsilon_p > 0$, by interpolation. Lemma 7 is proved.

THEOREM 4. *The d-commutators are bounded on L^p, $p \in]1, +\infty[$ with a norm $c_{p,\delta}(n+1)^{2+\delta} \|a\|_\infty^n$ for all $\delta > 0$.*

We apply Lemma 7 with z_t defined by $\langle g, z_t f \rangle = W_t(a, \ldots, a, g, f)$. A careful examination of the proof of Theorem 3 shows that the assumptions of Lemma 7 are satisfied. This proves Theorem 4.

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Received April 14, 1986
Received in revised form November 26, 1986