ON THE NON-LINEAR COHOMOLOGY OF LIE EQUATIONS. II

BY

HUBERT GOLDSCHMIDT and DONALD SPENCER

Institute for Advanced Study and Princeton University, Princeton, N.J., U.S.A.

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Chapter II. Non-linear cohomology

7. Lie equations and their non-linear cohomology

Let $R_k \subset J_k(T)$ be a differential equation; set $R_{k-1} = J_{k-1}(T)$, $R_{k-2} = J_{k-2}(T)$, $\tilde{R}_{k+l} = \nu^{-1}R_{k+l} \subset \tilde{J}_{k+l}(T)$, $R^0_{k+l} = R_{k+l} \cap J^0_{k+l}(T)$, $\tilde{\mathcal{R}}_{k+l} = \nu^{-1}\mathcal{R}_{k+l} \subset \tilde{J}_{k+l}(\mathfrak{J})$, and set $\tilde{J}_l(R_k) = \nu^{-1}J_l(R_k) \subset \tilde{J}_{(l,k)}(T)$. For $l \geqslant -1$, let $g_{k+l} \subset S^{k+l}J_0(T)^* \otimes J_0(T)$ be the kernel of π_{k+l-1}: $R_{k+l} \to R_{k+l-1}$ or of π_{k+l-1}: $\tilde{R}_{k+l} \to \tilde{R}_{k+l-1}$.

Definition 7.1. A differential equation $R_k \subset J_k(T)$ is a Lie equation if $[\tilde{\mathcal{R}}_k, \tilde{\mathcal{R}}_k] \subset \tilde{\mathcal{R}}_k$.

It follows from (1.15) and (1.16) that

$$[\tilde{\mathcal{R}}_{k+1}, \mathcal{R}_k] \subset \mathcal{R}_k \text{ and } [R_{k+1}, R_{k+1}] \subset R_k. \tag{7.1}$$

On the other hand, we have, for all $l \geqslant 0$,

$$[\tilde{\mathcal{R}}_{k+l}, \tilde{\mathcal{R}}_{k+l}] \subset \tilde{\mathcal{R}}_{k+l} \tag{7.2}$$

(cf. Proposition 4.3 of [19]). In particular, if R_{k+l} is a vector bundle, then R_{k+l} is a Lie equation and

$$\tilde{R}_{k+l} = \tilde{\lambda}_l^{-1}(\tilde{J}_l(R_k)) \tag{7.3}$$

where $\tilde{\lambda}_l$: $\tilde{J}_{k+l}(T) \to \tilde{J}_{(l,k)}(T)$. We remark that the sheaf Sol (R_k) of solutions of R_k is stable under the Lie bracket of vector fields. We say that R_k is *formally transitive* if π_0: $R_k \to J_0(T)$ is surjective. The differential equations $J_k(T; \varrho)$ and $J_k(V)$ considered in § 6 are Lie equations, and $J_k(T; \varrho)$ is formally transitive.

A differentiable sub-groupoid P_k of Q_k is a Lie equation (finite form) if it is a fibered submanifold of π: $Q_k \to X$. For $x \in X$, $I_k(x) \in P_k$ and $V_{I_k(x)}(P_k)$ determines a subspace $\tilde{R}_{k,x}$ of $\tilde{J}_k(T)_x$. The vector sub-bundle $R_k \subset J_k(T)$ such that $R_{k,x} = \nu(\tilde{R}_{k,x})$ is a Lie equation (infinitesimal form); we say that P_k is a finite form of R_k. For example, the sub-groupoids $Q_k(\varrho)$ and $Q_k(V)$ of Q_k are finite forms of $J_k(T; \varrho)$ and $J_k(V)$ respectively. We have $\tilde{R}_k \cdot F =$

$V_F(P_k)$ for $F \in P_k$, and R_k is formally transitive if and only if the restriction to P_k of the projection $\pi_0: Q_k \to X \times X$ is a submersion. We denote by $\mathcal{P}_k$ the sheaf of sections of P_k and by $\tilde{\mathcal{P}}_k = \mathcal{P}_k \cap \tilde{Q}_k$ the sheaf of invertible sections of P_k; we set

$$\tilde{\mathcal{P}}^{\cdot}_{k,a} = \{F \in \tilde{\mathcal{P}}_{k,a} \mid F(a) = I_k(a)\}$$

for $a \in X$.

For each Lie equation $R_k \subset J_k(T)$, we can construct a corresponding finite form P_k in the manner described in [19]. In fact, the sub-bundle $\{\tilde{R}_k F \mid F \in Q_k\}$ of $V(Q_k)$ is integrable since τ_k (see § 2) is a morphism of Lie algebras from $\Gamma(X, \tilde{J}_k(T))$ to the algebra of vector fields on Q_k; therefore it defines a foliation on Q_k which is transverse to I_k. The set of leaves passing through I_k forms a germ of submanifold of Q_k in the neighborhood of I_k and we can choose a representative P_k of this germ which is a differentiable sub-groupoid of Q_k and hence a finite form of R_k. Since any finite form P_k of R_k is a representative of this germ, the group $\tilde{\mathcal{P}}^{\cdot}_{k,a}$ depends only on R_k and not on the choice of the corresponding finite form P_k of R_k.

Let $R_k \subset J_k(T)$ be a Lie equation and P_k a finite form of R_k. Then the sub-bundle $\{\tilde{J}_l(R_k) \cdot F \mid F \in Q_{(l,k)}\}$ of $V(Q_{(l,k)})$ is integrable and defines a foliation on $Q_{(l,k)}$ which is transverse to $j_l(I_k) = \lambda_l(I_{k+l})$. The set of leaves passing through $j_l(I_k)$ forms a germ of submanifold of $Q_{(l,k)}$ along $j_l(I_k)$; the set $\tilde{J}_l(P_k) = J_l(P_k) \cap Q_{(l,k)}$ of jets of order l of sections of $\tilde{\mathcal{P}}_k$ is a representative of this germ. Suppose that R_{k+l}, the l-th prolongation of R_k, is a vector bundle and hence a Lie equation, and let P_{k+l} be a finite form of R_{k+l}. In view of (7.3) and the commutativity of (2.8), we conclude that $P_{k+l} = (\lambda_l)^{-1}\tilde{J}_l(P_k)$ in a neighborhood of I_{k+l}. Thus P_{k+i} coincides with the l-th prolongation $(P_k)_{+l} = (\lambda_l)^{-1}(J_l(P_k) \cap \lambda_l(Q_{k+l}))$ of P_k in a neighborhood of I_{k+l}; therefore $\pi_k P_{k+l} \subset P_k$ in a neighborhood of I_k and $\pi_k \tilde{\mathcal{P}}^{\cdot}_{k+l} \subset \tilde{\mathcal{P}}^{\cdot}_k$.

Let $R'_m \subset J_m(T)$, where $m \geqslant k$, be a Lie equation such that $\pi_k(R'_m) = R_k$, and let $P'_m \subset Q_m$ be a finite form of R'_m. Then by the implicit-function theorem we have $\pi_k(P'_m) = P_k$ in a neighborhood of I_k. Thus $\pi_k: \tilde{\mathcal{P}}'^{\cdot}_{m,a} \to \tilde{\mathcal{P}}^{\cdot}_{k,a}$ is surjective for all $a \in X$. Assume that R_{k+1} is a vector bundle and that $\pi_k: R_{k+1} \to R_k$ is surjective; then there exists a finite form P_k of R_k such that $\pi_k: (P_k)_{+1} \to P_k$ is surjective (see [19], Proposition 6.1). If moreover g_k is 2-acyclic, the finite form P_k, regarded as a differential equation in $J_k(E)$ where $E = X \times X$ is viewed as a bundle over X via pr_1, is formally integrable by Theorem 8.1 of [4] and Lemma 6.15 of [19] ([19], Theorem 6.16). If R_k is assumed to be formally integrable, we deduce from these remarks the existence of a finite form $P_k \subset Q_k$ of R_k which is formally integrable; for such a finite form P_k, the structure of affine bundle of $(P_k)_{+(l+1)}$ over $(P_k)_{+l}$ gives, by restriction of $\partial: Q^{k+l}_{k+l+1} \to S^{k+l+1}J_0(T)^* \otimes J_0(T)$, an isomorphism of bundles of Lie groups

$$\partial: Q^{k+l}_{k+l+1} \cap (P_k)_{+(l+1)} \to g_{k+l+1} \tag{7.4}$$

(see [19], § 6). We remark that $Q_k(\varrho)$, $Q_k(V)$, with $k \geq 1$, are formally integrable and their l-th prolongations are $Q_{k+l}(\varrho)$, $Q_{k+l}(V)$ respectively.

We summarize and amplify some of the above considerations as a proposition.

PROPOSITION 7.1. *Let $R_k \subset J_k(T)$ be a Lie equation and assume that, for all $l \geq 0$, R_{k+l} is a vector bundle and that $P_{k+l} \subset Q_{k+l}$ is a finite form of R_{k+l}. Then:*

(i) *P_{k+l} is equal to the l-th prolongation $(P_k)_{+l}$ of P_k in a neighborhood of I_{k+l}, and $\pi_{k+l} P_{k+l+m} \subset P_{k+l}$ in a neighborhood of I_{k+l}, for all l, $m \geq 0$.*

(ii) *For $m \geq k$ and $a \in X$, the groups $\tilde{\mathcal{D}}^{\cdot}_{m,a}$ depend only on R_k, and the mapping π_m: $Q_{m+l} \to Q_m$ induces a mapping π_m: $\tilde{\mathcal{D}}^{\cdot}_{m+l,a} \to \tilde{\mathcal{D}}^{\cdot}_{m,a}$ for $l \geq 0$.*

(iii) *Let $R'_m \subset J_m(T)$ be a Lie equation with $m \geq k$ and $\pi_k(R'_m) = R_k$, and let P'_m be a finite form of R'_m. Then $\pi_k(P'_m) = P_k$ in a neighborhood of I_k and π_k: $\tilde{\mathcal{D}}'^{\cdot}_{m,a} \to \tilde{\mathcal{D}}^{\cdot}_k$ is surjective for all $a \in X$. Moreover, if $F \in \mathcal{D}_{k,a}$ with $F(a) = I_k(a)$, $a \in X$, and $G \in J_1(P'_m)$ with $J_1(\pi_k) G = j_1(F)(a)$ and $\pi_0 G = I_m(a)$, then there exists $F' \in \mathcal{D}'_{m,a}$ satisfying $\pi_k F' = F$ and $j_1(F')(a) = G$.*

(iv) *If R_k is formally integrable, then it possesses a formally integrable finite form P_k and the mappings π_{k+l}: $\tilde{\mathcal{D}}^{\cdot}_{k+l+1,a} \to \tilde{\mathcal{D}}^{\cdot}_{k+l,a}$ are surjective, where $P_{k+l} = (P_k)_{+l}$, for all $l \geq 0$ and $a \in X$.*

(v) *Let $R^{\#}_k \subset R_k$ be a Lie equation and $P^{\#}_k$ a finite form of $R^{\#}_k$. Then $P^{\#}_k \subset P_k$ in a neighborhood of I_k.*

Let $R_k \subset J_k(T)$ be a Lie equation and $P_k \subset Q_k$ a finite form of R_k. Since P_k is a groupoid, if $F \in \tilde{\mathcal{D}}_{k,a}$, $a \in X$, by (2.5) the mapping (2.2) restricts to give a mapping

$$F\colon \tilde{R}_{k,a} \to \tilde{R}_{k,b} \tag{7.5}$$

where $b =$ target $F(a)$. If $F \in (P_k)_{+1}$, by (2.4) the mapping (2.1) restricts to give a mapping

$$F\colon R_{k,a} \to R_{k,b} \tag{7.6}$$

where $a =$ source F, $b =$ target F.

We have the following proposition:

PROPOSITION 7.2. *Let $R_k \subset J_k(T)$ be a Lie equation and $P_k \subset Q_k$ a finite form of R_k. Let $F \in \tilde{Q}_{k+1}$; then the following two assertions are equivalent:*

(i) $$\mathcal{D}F \in \mathcal{T}^* \otimes \mathcal{R}_k;$$

(ii) $$\tilde{\mathcal{D}}F \in J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_k.$$

If $\pi_k F \in \tilde{\mathcal{D}}_k$, then (i) *and* (ii) *are equivalent to:*

(iii) $$F \in (\tilde{\mathcal{D}}_k)_{+1} = \lambda_1^{-1}(J_1(\mathcal{D}_k) \cap \lambda_1(\tilde{Q}_{k+1})).$$

This proposition is a consequence of Lemma 2.2, (i) and (iii), and Proposition 6.9 of [19].

Let $R_k \subset J_k(T)$ be a Lie equation; assume that, for all $l \geq 0$, R_{k+l} is a vector bundle and let P_{k+l} be a finite form of R_{k+l}. For $l \geq 0$ and $a \in X$, we define the group

$$H^0(P_k)_{k+l,a} = \{f \in (\mathrm{Aut}\,(X))_a \mid j_{k+l}(f) \in \tilde{\mathcal{P}}^{\cdot}_{k+l,a}\};$$

we note that it does not depend on the choice of P_{k+l} and therefore depends only on R_k. Let

$$(T^* \otimes R_{k+l})^\wedge = (T^* \otimes R_{k+l}) \cap (T^* \otimes J_{k+l}(T))^\wedge,$$

$$(J_0(T)^* \otimes \tilde{R}_{k+l})^\wedge = (J_0(T)^* \otimes \tilde{R}_{k+l}) \cap (J_0(T)^* \otimes \tilde{J}_{k+l}(T))^\wedge,$$

and

$$Z^1(R_{k+l}) = \{u \in (\mathcal{T}^* \otimes \mathcal{R}_{k+l})^\wedge \mid \mathcal{D}_1 u = 0\},$$

$$\bar{Z}^1(R_{k+l}) = \{u \in (J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_{k+l})^\wedge \mid \bar{\mathcal{D}}_1 u = 0\}.$$

By Proposition 7.1, (i) and Proposition 7.2, we obtain, for $l \geq 0$ and $a \in X$, the following two non-linear Spencer complexes

$$H^0(P_k)_{k+l+1,a} \xrightarrow{j_{k+l+1}} \tilde{\mathcal{P}}^{\cdot}_{k+l+1,a} \xrightarrow{\mathcal{D}} (\mathcal{T}^* \otimes \mathcal{R}_{k+l})^\wedge_a \xrightarrow{\mathcal{D}_1} (\wedge^2 \mathcal{T}^* \otimes \mathcal{R}_{k+l-1})_a,$$

$$H^0(P_k)_{k+l+1,a} \xrightarrow{j_{k+l+1}} \tilde{\mathcal{P}}^{\cdot}_{k+l+1,a} \xrightarrow{\bar{\mathcal{D}}} (J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_{k+l})^\wedge_a \xrightarrow{\bar{\mathcal{D}}_1} (\wedge^2 J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_{k+l-1})_a.$$

According to (7.5), (7.6) and (2.43) the group $\tilde{\mathcal{P}}^{\cdot}_{k+l+1,a}$ operates on the right on $Z^1(R_{k+l})_a$ and $\bar{Z}^1(R_{k+l})_a$ in the manner of (2.31) and (2.40). Set

$$H^1(P_k)_{k+l,a} = Z^1(R_{k+l})_a / \tilde{\mathcal{P}}^{\cdot}_{k+l+1,a},$$

$$\bar{H}^1(P_k)_{k+l,a} = \bar{Z}^1(R_{k+l})_a / \tilde{\mathcal{P}}^{\cdot}_{k+l+1,a};$$

these non-linear Spencer cohomologies of P_k are the sets of orbits under the right operations of $\tilde{\mathcal{P}}^{\cdot}_{k+l+1,a}$. We shall say that the orbit $[u]$ of $u \in Z^1(R_{k+l})_a$ (resp. $\bar{Z}^1(R_{k+l})_a$) is the cohomology class of u in $H^1(P_k)_{k+l,a}$ (resp. $\bar{H}^1(P_k)_{k+l,a}$). Then $u, v \in Z^1(R_{k+l})_a$ (resp. $\bar{Z}^1(R_{k+l})_a$) are cohomologous if and only if there exists $F \in \tilde{\mathcal{P}}^{\cdot}_{k+l+1,a}$ such that $u^F = v$, and u is cohomologous to zero if and only if $u = \mathcal{D}F$ (resp. $u = \bar{\mathcal{D}}F$) for some $F \in \tilde{\mathcal{P}}^{\cdot}_{k+l+1,a}$. We denote by 0 the orbit of $0 \in Z^1(R_{k+l})_a$ (resp. $\bar{Z}^1(R_{k+l})_a$) in $H^1(P_k)_{k+l,a}$ (resp. $\bar{H}^1(P_k)_{k+l,a}$), and so these cohomologies are sets with distinguished elements 0. Since the groups $\tilde{\mathcal{P}}^{\cdot}_{k+l+1,a}$ depend only on R_k, these cohomologies depend only on R_k and not on the choice of the finite forms P_{k+l}. Finally we remark that the vanishing of the cohomology $H^1(P_k)_{k+l,a}$ or $\bar{H}^1(P_k)_{k+l,a}$ is equivalent to the exactness of the first of the above complexes at $(\mathcal{T}^* \otimes \mathcal{R}_{k+l})^\wedge_a$ or of the second at $(J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_{k+l})^\wedge_a$, respectively.

All the cohomologies we shall consider are sets with distinguished elements 0. By a mapping of cohomology, we shall mean a mapping between two cohomologies sending 0 into 0 and, by an isomorphism of cohomology, we shall mean a bijective mapping between two cohomologies sending 0 into 0. However, in the latter part of this section (namely, in Propositions 7.9, 7.10, 7.11 and Corollary 7.1) and in § 10 (namely, in Theorems 10.3 and 10.4) mappings connecting cohomologies occur which are only bijective and do not necessarily send 0 into 0.

Using Proposition 7.1, (ii), we see that, for $l, m \geq 0$, $a \in X$, the mappings $\pi_{k+l}\colon R_{k+l+m} \to R_{k+l}$ induce mappings of cohomology

$$\pi_{k+l}\colon H^1(P_k)_{k+l+m,a} \to H^1(P_k)_{k+l,a},$$

$$\pi_{k+l}\colon \bar{H}^1(P_k)_{k+l+m,a} \to \bar{H}^1(P_k)_{k+l,a},$$

and we define the non-linear Spencer cohomology of P_k to be the projective limits

$$H^1(P_k)_a = \varprojlim H^1(P_k)_{k+l,a},$$

$$\bar{H}^1(P_k)_a = \varprojlim \bar{H}^1(P_k)_{k+l,a}$$

for $a \in X$. These cohomologies are also sets with distinguished elements 0, and they depend only on R_k and not on the choice of the finite forms.

According to Lemma 2.2, (i) and (iv), the mapping (2.44) restricts to give, for $l \geq 0$, bijections

$$(T^* \otimes R_{k+l})^\wedge \to (J_0(T)^* \otimes \tilde{R}_{k+l})^\wedge, \tag{7.7}$$

$$Z^1(R_{k+l}) \to \bar{Z}^1(R_{k+l}). \tag{7.8}$$

According to Lemma 2.2, (ii), (7.8) induces for $a \in X$ an isomorphism of cohomology

$$H^1(P_k)_{k+l,a} \to \bar{H}^1(P_k)_{k+l,a}.$$

Thus:

PROPOSITION 7.3. *Let $R_k \subset J_k(T)$ be a Lie equation; assume that, for all $l \geq 0$, R_{k+l} is a vector bundle and let P_{k+l} be a finite form of R_{k+l}. Then the mapping* (2.44) *induces isomorphisms of cohomology, for all $l \geq 0$ and $a \in X$,*

$$H^1(P_k)_{k+l,a} \to \bar{H}^1(P_k)_{k+l,a},$$

$$H^1(P_k)_a \to \bar{H}^1(P_k)_a.$$

According to Proposition 7.3, we may identify $H^1(P_k)_a$ and $\bar{H}^1(P_k)_a$ and define the non-linear Spencer cohomology of R_k to be

$$\tilde{H}^1(R_k)_a = H^1(P_k)_a = \bar{H}^1(P_k)_a$$

for $a \in X$. We set

$$\tilde{H}^1(R_k) = \bigcup_{a \in X} \tilde{H}^1(R_k)_a.$$

Definition 7.2. We say that the second fundamental theorem holds for R_k if $\tilde{H}^1(R_k) = 0$.

If $R'_k \subset R_k$ is a Lie equation all of whose prolongations R'_{k+l} are vector bundles and P'_{k+l} is a finite form of R'_{k+l} for $l \geq 0$, the inclusions $R'_{k+l} \subset R_{k+l}$ and Proposition 7.1, (v) induce mappings of cohomology

$$H^1(P'_k)_{k+l,a} \to H^1(P_k)_{k+l,a},$$
$$\bar{H}^1(P'_k)_{k+l,a} \to \bar{H}^1(P_k)_{k+l,a},$$

for all $l \geq 0$, and hence mappings of cohomology

$$\tilde{H}^1(R'_k)_a \to \tilde{H}^1(R_k)_a$$

for $a \in X$.

LEMMA 7.1. *Let $R_k \subset J_k(T)$ be a Lie equation; assume that R_{k+1} is a vector bundle and that π_k: $R_{k+1} \to R_k$ is surjective. Let P_{k+1} be a finite form of R_{k+1} and $u \in (\mathcal{T}^* \otimes \mathcal{R}_k)^\wedge_a$, $a \in X$. Then there exists $F \in \tilde{\mathcal{D}}^{\cdot}_{k+1,a}$ satisfying $u^F(a) = 0$ or $\mathcal{D}F^{-1} = u$ at a.*

Proof. Let $v \in (T^* \otimes R_{k+1})^\wedge_a$ with $\pi_k v = u(a)$. Since $J_1(P_{k+1})$ is an affine sub-bundle of $J_1(Q_{k+1})_{|P_{k+1}}$ over P_{k+1}, there exists $G \in \mathcal{D}_{k+1,a}$ such that $G(a) = I_{k+1}(a)$ and

$$j_1(G)(a) = j_1(I_{k+1})(a) + (\mathrm{id} \otimes \nu^{-1})v.$$

By Proposition 2.2, (i), $j_1(G)(a)$ belongs to $Q_{(1,k+1)}$ and hence $G \in \tilde{\mathcal{D}}^{\cdot}_{k+1,a}$. By (2.27) we have

$$(\mathrm{id} \otimes \nu^{-1})\mathcal{D}G(a) = \partial[j_1(\pi_k G)(a)] = j_1(\pi_k G)(a) - j_1(I_k)(a) = (\mathrm{id} \otimes \nu^{-1})\pi_k v,$$

and so $\mathcal{D}G(a) = u(a)$. Taking $F^{-1} = G$, we obtain the assertion of the lemma.

Now assume that $R_k \subset J_k(T)$ is formally integrable, that P_k is a formally integrable finite form of R_k (which exists by Proposition 7.1, (iv)) and that P_{k+l} is the l-th prolongation $(P_k)_{+l}$ of P_k. Denote by Sol (P_k) the sub-sheaf of Aut (X) composed of the f satisfying $j_k(f) \in \tilde{\mathcal{D}}_k$; it is the sheaf of solutions of the non-linear differential equation $P_k \subset J_k(E)$, where $E = X \times X$. By Proposition 7.2 we have, for $l \geq 0$, the following two non-linear Spencer complexes:

$$\mathrm{Sol}\,(P_k) \xrightarrow{j_{k+l+1}} \tilde{\mathcal{D}}_{k+l+1} \xrightarrow{\mathcal{D}} (\mathcal{T}^* \otimes \mathcal{R}_{k+l})^\wedge \xrightarrow{\mathcal{D}_1} \wedge^2 \mathcal{T}^* \otimes \mathcal{R}_{k+l-1},$$

$$\mathrm{Sol}\,(P_k) \xrightarrow{j_{k+l+1}} \tilde{\mathcal{D}}_{k+l+1} \xrightarrow{\bar{\mathcal{D}}} (J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_{k+l})^\wedge \xrightarrow{\bar{\mathcal{D}}_1} \wedge^2 J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_{k+l-1},$$

which are finite forms of the linear Spencer complexes

$$0 \longrightarrow \mathrm{Sol}\,(R_k) \xrightarrow{j_{k+l+1}} \mathcal{R}_{k+l+1} \xrightarrow{D} \mathcal{T}^* \otimes \mathcal{R}_{k+l} \xrightarrow{D} \wedge^2 \mathcal{T}^* \otimes \mathcal{R}_{k+l-1},$$

$$0 \longrightarrow \mathrm{Sol}\,(R_k) \xrightarrow{\tilde{j}_{k+l+1}} \tilde{\mathcal{R}}_{k+l+1} \xrightarrow{\bar{D}} J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_{k+l} \xrightarrow{\bar{D}} \wedge^2 J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_{k+l-1}.$$

The vanishing of the cohomology $H^1(P_k)_{k+l,a}$, for all $a \in X$, implies the exactness of the above non-linear complexes.

PROPOSITION 7.4. *Suppose that $R_k \subset J_k(T)$ is a formally integrable Lie equation and that g_{k_0} is 2-acyclic where $k_0 \geq \sup(k, 2)$. Then for all $m \geq k_0$ the mappings*

$$\pi_m: Z^1(R_{m+1}) \to Z^1(R_m), \tag{7.9}$$

$$\pi_m: \bar{Z}^1(R_{m+1}) \to \bar{Z}^1(R_m) \tag{7.10}$$

are surjective.

Proof. Since the mapping (2.44) is compatible with the projections $\pi_m: J_{m+1}(T) \to J_m(T)$, $\pi_m: \tilde{J}_{m+1}(T) \to \tilde{J}_m(T)$, and since the mappings (7.8) are bijections, it suffices to show that (7.10) is surjective. Let $u \in \bar{Z}^1(R_m)$, with $m \geq k_0$, and choose $u_1 \in J_0(\mathcal{T})^* \otimes \tilde{R}_{m+1}$ such that $\pi_m u_1 = u$. Then $\bar{\mathcal{D}}_1 u \in \wedge^2 J_0(\mathcal{T})^* \otimes g_m$ and

$$\bar{\delta}\bar{\mathcal{D}}_1 u_1 = -\bar{D}(\bar{D}u_1 - \tfrac{1}{2}[u, u]) = [\bar{D}u, \pi_{m-1}u] = \tfrac{1}{2}[[\pi_{m-1}u, \pi_{m-1}u], \pi_{m-1}u] = 0$$

by the Jacobi identity. Since g_m is assumed to be 2-acyclic, there is an element $v \in J_0(\mathcal{T})^* \otimes g_{m+1}$ satisfying $\bar{\delta}v = \bar{\mathcal{D}}_1 u_1$. Then

$$\bar{\mathcal{D}}_1(u_1 + v) = \bar{D}u_1 - \bar{\delta}v - \tfrac{1}{2}[u, u] = \bar{\mathcal{D}}_1 u_1 - \bar{\delta}v = 0;$$

hence $u_1 + v$ belongs to $\bar{Z}^1(R_{m+1})$ and satisfies $\pi_m(u_1 + v) = u$, that is (7.10) is surjective.

Remark. It can be shown directly that the mapping (7.9) is surjective without using the isomorphisms (7.8) and, if this is carried out, one is led automatically to consider a twisted δ-operator, namely

$$\delta_v: \wedge^j T^* \otimes g_m \to \wedge^{j+1} T^* \otimes g_{m-1}, \quad \text{for } m \geq k,$$

where

$$\delta_v w = [v, w] = [v_1, w], \quad w \in \wedge^j T^* \otimes g_m,$$

and v is a section of $T^* \otimes J_0(T)$ such that $v: T \to J_0(T)$ is invertible, and v_1 is any section of $T^* \otimes J_m(T)$ such that $\pi_0 v_1 = v$. It is easy to see that δ_v coincides with δ when $v = \nu$. The cohomology of the complex (1.8) is not changed, up to an isomorphism, by replacing δ with δ_v.

We deduce immediately from Proposition 7.4:

PROPOSITION 7.5. *Suppose that $R_k \subset J_k(T)$ is a formally integrable Lie equation and $P_k \subset Q_k$ is a formally integrable finite form of R_k, and that g_{k_0} is 2-acyclic with $k_0 \geq \sup(k, 2)$. Then for all $m \geq k_0$, $a \in X$, the mappings of cohomology*

$$\pi_m\colon H^1(P_k)_{m+1,a} \to H^1(P_k)_{m,a}, \tag{7.11}$$

$$\pi_m\colon \bar{H}^1(P_k)_{m+1,a} \to \bar{H}^1(P_k)_{m,a} \tag{7.12}$$

are surjective.

PROPOSITION 7.6. *Let $R_k \subset J_k(T)$ be a formally integrable Lie equation, and assume that R_k possesses a formally integrable and integrable finite form $P_k \subset Q_k$. If the image of $\alpha \in H^1(P_k)_{m+1,a}$ (resp. $\alpha \in \bar{H}^1(P_k)_{m+1,a}$), with $m \geq k$, $a \in X$, in $H^1(P_k)_{m,a}$ (resp. $\bar{H}^1(P_k)_{m,a}$) vanishes, then $\alpha = 0$.*

Proof. According to Proposition 7.3, it suffices to prove the assertion for $\alpha \in \bar{H}^1(P_k)_{m+1,a}$. Let P_{k+l} be the l-th prolongation of P_k. Let $u \in \bar{Z}^1(R_{m+1})_a$ and assume that the cohomology class of $\pi_m u$ in $\bar{H}^1(P_k)_{m,a}$ vanishes. Then there exists $F_1 \in \tilde{\mathcal{D}}^{\cdot}_{m+1,a}$ such that $(\pi_m u)^{F_1} = 0$, and we choose $F_2 \in \tilde{\mathcal{D}}^{\cdot}_{m+2,a}$ with $\pi_{m+1} F_2 = F_1$. Then $u^{F_2} \in J_0(\mathcal{T})^* \otimes g_{m+1}$ and

$$\delta u^{F_2} = -\bar{D} u^{F_2} = -\tfrac{1}{2}\pi_m[u^{F_2}, u^{F_2}] = 0,$$

since $\pi_m(u^{F_2}) = (\pi_m u)^{F_1} = 0$. Since g_{m+1} is 1-acyclic, there exists $v \in g_{m+2}$ such that $\bar{\delta} v = u^{F_2}$. Since P_k is formally integrable, the mapping (7.4) (with $k+l = m+1$) is an isomorphism and $G = \partial^{-1} v$ belongs to $Q^{m+1}_{m+2} \cap \mathcal{D}_{m+2}$ and, by (2.38),

$$\tilde{\mathcal{D}} G = -\bar{\delta} v = -u^{F_2}.$$

Then $F_2 \cdot G \in \tilde{\mathcal{D}}_{m+2,a}$ and

$$u^{F_2 \cdot G} = u^{F_2} + \tilde{\mathcal{D}} G = 0.$$

Since P_k is integrable, there exists $f \in \mathrm{Sol}(P_k)_a$ such that $j_{m+2}(f)(a) = G(a)$; then $F = F_2 \cdot G \cdot j_{m+2}(f^{-1})$ belongs to $\tilde{\mathcal{D}}^{\cdot}_{m+2,a}$ and

$$u^F = (u^{F_2 \cdot G})^{j_{m+2}(f^{-1})} = 0^{j_{m+2}(f^{-1})} = 0,$$

showing that u is cohomologous to zero in $\bar{H}^1(P_k)_{m+1,a}$.

We suppose henceforth that $k \geq 1$ and continue to suppose that $R_k \subset J_k(T)$ is a formally integrable Lie equation and that P_k is a formally integrable finite form of R_k. Let C^1_{k+l}, $\hat{C}^1_{k+l}$ be the images of $J_0(T)^* \otimes \tilde{R}_{k+l}$, $(J_0(T)^* \otimes \tilde{R}_{k+l})^\wedge$ respectively in B^1_{k+l}. Then

$$C^1_{k+l} = (J_0(T)^* \otimes \tilde{R}_{k+l})/\bar{\delta}(g_{k+l+1})$$

and C^1_{k+l} is a vector bundle since g_{k+l+1} is, and $\hat{C}^1_{k+l} = C^1_{k+l} \cap \hat{B}^1_{k+l}$. We set

$$\hat{Z}^1(R_{k+l}) = \{u \in \hat{\mathcal{C}}^1_{k+l} \mid \hat{\mathcal{D}}_1 u = 0\}.$$

By Proposition 7.2 we obtain, for $l \geq 0$, the non-linear Spencer complex,

$$\mathrm{Sol}\,(P_k) \xrightarrow{j_{k+l}} \tilde{\mathcal{D}}_{k+l} \xrightarrow{\hat{\mathcal{D}}} \hat{\mathcal{C}}^1_{k+l} \xrightarrow{\hat{\mathcal{D}}_1} \mathcal{B}^2_{k+l},$$

which is a sub-complex of (2.48) (with k replaced by $k+l$) and which is a finite form of the complex

$$\mathrm{Sol}\,(R_k) \xrightarrow{\tilde{j}_{k+l}} C^0_{k+l} \xrightarrow{\hat{\mathcal{D}}} C^1_{k+l} \xrightarrow{\hat{\mathcal{D}}} \mathcal{B}^2_{k+l}$$

where $C^0_{k+l} = \tilde{R}_{k+l}$. According to (7.5), Proposition 7.1, (iv), Proposition 7.2 and (2.43), for $a \in X$ the group $\tilde{\mathfrak{D}}_{k+l,\,a}$ operates on the right on $\hat{Z}^1(R_{k+l})_a$ in the manner of (2.49). Set

$$\hat{H}^1(P_k)_{k+l,\,a} = \hat{Z}^1(R_{k+l})_a / \tilde{\mathfrak{D}}_{k+l,\,a};$$

this non-linear Spencer cohomology of P_k is the set of orbits under the right operations of $\tilde{\mathfrak{D}}_{k+l,\,a}$ and depends only on R_k and not on the choice of the finite form P_k. For an alternative description of this cohomology of P_k, we refer the reader to [19], § 8. We denote by $0 \in \hat{H}^1(P_k)_{k+l,\,a}$ the orbit of $0 \in \hat{Z}^1(R_{k+l})_a$, and we remark that the vanishing of the cohomology $\hat{H}^1(P_k)_{k+l,\,a}$ for all $a \in X$ implies the exactness of the above non-linear complex. For $l,\ m \geqslant 0$, $a \in X$, the mappings $\pi_{k+l}\colon R_{k+l+m} \to R_{k+l}$ induce mappings of cohomology

$$\pi_{k+l}\colon \hat{H}^1(P_k)_{k+l+m,\,a} \to \hat{H}^1(P_k)_{k+l,\,a}$$

and, for $a \in X$, we define the cohomology

$$\hat{H}^1(P_k)_a = \varprojlim \hat{H}^1(P_k)_{k+l,\,a}$$

which is a set with distinguished element 0.

Let us show that the projection $J_0(T)^* \otimes \tilde{R}_m \to C^1_m$ induces a mapping

$$\bar{Z}^1(R_m) \to \hat{Z}^1(R_m) \tag{7.13}$$

for $m \geqslant k$. Let $u \in \bar{Z}^1(R_m)$ and $\hat{u}$ be its image in C^1_m. By the exactness of (2.32), there exists $F \in \tilde{Q}_{m+1}$ such that $\bar{\mathcal{D}} F = u$. Choose $F_1 \in \tilde{Q}_{m+2}$ with $\pi_{m+1} F_1 = F$; then $u_1 = \bar{\mathcal{D}} F_1 \in \bar{Z}^1(J_{m+1}(T))$ and $\pi_m u_1 = u$. Now $\hat{\mathcal{D}}_1 \hat{u}$ is the class of $\bar{\mathcal{D}}_1 u_1$ in $\mathcal{B}^2_m$, and hence vanishes. We obtain therefore mappings of cohomology for $m \geqslant k$, $a \in X$,

$$\bar{H}^1(P_k)_{m,\,a} \to \hat{H}^1(P_k)_{m,\,a}, \tag{7.14}$$

$$\bar{H}^1(P_k)_a \to \hat{H}^1(P_k)_a. \tag{7.15}$$

The commutative diagram

$$\begin{tikzcd}
J_0(T)^* \otimes \tilde{R}_{m+1} \arrow[r] \arrow[d, "\pi_m"] & C^1_{m+1} \arrow[d, "\pi_m"] \arrow[dl] \\
J_0(T)^* \otimes \tilde{R}_m \arrow[r] & C^1_m
\end{tikzcd}$$

induces for $m \geqslant k$ a commutative diagram

$$\begin{tikzcd}
\bar{Z}^1(R_{m+1}) \arrow[r] \arrow[d, "\pi_m"] & \hat{Z}^1(R_{m+1}) \arrow[d, "\pi_m"] \arrow[dl] \\
\bar{Z}^1(R_m) \arrow[r] & \hat{Z}^1(R_m)
\end{tikzcd} \tag{7.16}$$

and therefore also a commutative diagram

$$\begin{tikzcd}
\bar{H}^1(P_k)_{m+1,a} \arrow[r] \arrow[d, "\pi_m"] & \hat{H}^1(P_k)_{m+1,a} \arrow[d, "\pi_m"] \arrow[dl] \\
\bar{H}^1(P_k)_{m,a} \arrow[r] & \hat{H}^1(P_k)_{m,a}
\end{tikzcd} \tag{7.17}$$

of cohomology, for $a \in X$.

Proposition 7.7. *Let $R_k \subset J_k(T)$ be a formally integrable Lie equation, with $k \geqslant 1$, and $P_k \subset Q_k$ a formally integrable finite form of R_k. Then:*

(i) *For $m \geqslant k$, $a \in X$, the mappings* (7.13) *and* (7.14) *are surjective and* (7.15) *is an isomorphism of cohomology.*

(ii) *If g_{k_0} is 2-acyclic, with $k_0 \geqslant \sup(k, 2)$, then all the mappings of diagram* (7.17) *are surjective for $m \geqslant k_0$, $a \in X$.*

(iii) *If P_k is integrable, then for $m \geqslant k$, $a \in X$:*

(a) *if the image of $\alpha \in \bar{H}^1(P_k)_{m,a}$ vanishes in $\hat{H}^1(P_k)_{m,a}$, then $\alpha = 0$;*

(b) *if the image of $\alpha \in \hat{H}^1(P_k)_{m+1,a}$ vanishes in $\bar{H}^1(P_k)_{m,a}$, then $\alpha = 0$;*

(c) *if the image of $\alpha \in \hat{H}^1(P_k)_{m+1,a}$ vanishes in $\hat{H}^1(P_k)_{m,a}$, then $\alpha = 0$.*

(iv) *If P_k is integrable, then for $m \geqslant k$, $a \in X$, the following assertions are equivalent:*

(a) $H^1(P_k)_{m,a} = 0$;

(b) $\bar{H}^1(P_k)_{m,a} = 0$;

(c) $\hat{H}^1(P_k)_{m,a} = 0$.

Proof. (i) We first prove that (7.13) is surjective for $m \geqslant k$. Let $\hat{u} \in \hat{Z}^1(R_m)$ be the image of $u \in (J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_m)^\wedge$. Choose $u_1 \in J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_{m+1}$ with $\pi_m u_1 = u$. Then $\hat{\mathcal{D}}_1 \hat{u}$ is the image in $\mathcal{B}^2_m$ of $\bar{\mathcal{D}}_1 u_1 \in \wedge^2 J_0(\mathcal{T})^* \otimes \tilde{\mathcal{R}}_m$. Therefore $\bar{\mathcal{D}}_1 u_1 \in \wedge^2 J_0(\mathcal{T})^* \otimes g_m$ and

$$0 = \pi_{m-1} \bar{\mathcal{D}}_1 u_1 = \bar{\mathcal{D}}_1 u;$$

so $\hat{u}$ is the image of $u \in \bar{Z}^1(R_m)$. Thus the mappings (7.14) are also surjective for $m \geqslant k$. That (7.15) is an isomorphism of cohomology follows from the commutativity of (7.17).

(ii) is a direct consequence of (i) and Proposition 7.5.

(iii) We first verify (a). Let $u \in \bar{Z}^1(R_m)_a$, with $m \geqslant k$, $a \in X$; assume that the cohomology class of the image $\hat{u}$ of u in $\hat{Z}^1(R_m)_a$ vanishes. Then there exists $F \in \tilde{\mathcal{D}}^{\boldsymbol{\cdot}}_{m,a}$ such that $\hat{u}^F = 0$. Choose $F_1 \in \tilde{\mathcal{D}}^{\boldsymbol{\cdot}}_{m+1,a}$ with $\pi_m F_1 = F$; thus u^{F_1} belongs to $\delta(g_{m+1})$ and we can write $u^{F_1} = \delta v$, with $v \in g_{m+1}$. Since P_k is formally integrable, the mapping (7.4) is an isomorphism (with $k + l = m$) and $G = \partial^{-1} v$ belongs to $Q^m_{m+1} \cap \mathcal{D}_{m+1}$ and, by (2.38),

$$\bar{\mathcal{D}} G = -\delta v = -u^{F_1}.$$

Then $F_1 \cdot G \in \tilde{\mathcal{D}}_{m+1,a}$ and

$$u^{F_1 \cdot G} = u^{F_1} + \bar{\mathcal{D}} G = 0.$$

Since P_k is integrable, there exists $f \in \operatorname{Sol}(P_k)_a$ such that $j_{m+1}(f)(a) = G(a)$; then $F = F_1 \cdot G \cdot j_{m+1}(f^{-1})$ belongs to $\tilde{\mathcal{D}}^{\boldsymbol{\cdot}}_{m+1,a}$ and $u^F = 0$, showing that u is cohomologous to 0 in $\bar{H}^1(P_k)_{m,a}$. By the commutativity of diagram (7.17), we deduce that (b) follows from (i) and Proposition 7.6, while (a) and (b) together imply (c).

(iv) The equivalence of the three assertions follows from (i), (iii), (a) and Proposition 7.3.

According to Proposition 7.7, (i), we may identify $\tilde{H}^1(R_k)_a$ and $\hat{H}^1(P_k)_a$, for $a \in X$.

PROPOSITION 7.8. *Let $R_k \subset J_k(T)$ be a formally integrable Lie equation and $P_k \subset Q_k$ a formally integrable finite form of R_k. Suppose that g_{k_0} is 2-acyclic, with $k_0 \geqslant k$. For $a \in X$, the following assertions are equivalent:*

(i) $\tilde{H}^1(R_k)_a = 0$;

(ii) *for all* $m \geqslant \sup(k_0, 2)$, $H^1(P_k)_{m,a} = 0$;

(iii) *for all* $m \geqslant \sup(k_0, 2)$, $\bar{H}^1(P_k)_{m,a} = 0$;

(iv) *for all* $m \geqslant \sup(k_0, 2)$, $\hat{H}^1(P_k)_{m,a} = 0$.

If moreover P_k is integrable, these assertions are equivalent to each of the following:

(v) *for some* $m \geqslant \sup(k_0, 2)$, $H^1(P_k)_{m,a} = 0$;

(vi) *for some* $m \geqslant \sup(k_0, 2)$, $\bar{H}^1(P_k)_{m,a} = 0$;

(vii) *for some* $m \geqslant \sup(k_0, 2)$, $\hat{H}^1(P_k)_{m,a} = 0$.

If P_k is integrable, then each of the above assertions is implied by the equivalent conditions:

(viii) *for some* $m \geq k$, $H^1(P_k)_{m,a} = 0$;

(ix) *for some* $m \geq k$, $\bar{H}^1(P_k)_{m,a} = 0$;

(x) *for some* $m \geq k$, $\hat{H}^1(P_k)_{m,a} = 0$.

Proof. The equivalence of (i)–(iv) follows from Proposition 7.5, Proposition 7.7, (ii) and [1], § 3, No. 5, Corollary 1. When P_k is integrable, we deduce from Proposition 7.6 and Proposition 7.7, (iii), (c) or (iv) that (v)–(vii) are equivalent to (i)–(iv) and that (viii)–(x) imply (i)–(iv).

The following three propositions are closely related to results in § 5 and § 6 of [9]; in particular, the following proposition and its proof are related to Theorem 6.2 of [9].

PROPOSITION 7.9. *Let R_k, $R_k^{\#} \subset J_k(T)$ be formally integrable Lie equations and let P_k, $P_k^{\#} \subset Q_k$ be formally integrable finite forms of R_k, $R_k^{\#}$ respectively. Let $m \geq k$ and F be a section of $\tilde{Q}_{m+1}$ over an open set $U \subset X$ such that $f = \pi_0 F$ is a local diffeomorphism of X and*

$$F(R_{m\,|\,U}) = R^{\#}_{m\,|\,f(U)}, \tag{7.18}$$

$$F(\tilde{R}_{m+1\,|\,U}) = \tilde{R}^{\#}_{m+1\,|\,f(U)}. \tag{7.19}$$

Let $a \in U$ and $b = f(a)$.

(i) *If R_k and $R_k^{\#}$ are formally transitive or if $F = j_{m+1}(f)$, then the germ of F in $\tilde{Q}_{m+1,a}$ induces a commutative diagram*

$$\begin{array}{ccc} H^1(P_k)_{m,a} & \longrightarrow & H^1(P_k^{\#})_{m,b} \\ \downarrow & & \downarrow \\ \bar{H}^1(P_k)_{m,a} & \longrightarrow & \bar{H}^1(P_k^{\#})_{m,b} \\ \downarrow & & \downarrow \\ \hat{H}^1(P_k)_{m,a} & \longrightarrow & \hat{H}^1(P_k^{\#})_{m,b} \end{array} \tag{7.20}$$

whose horizontal arrows are bijective. Moreover if $F = j_{m+1}(f)$, then f induces an isomorphism of cohomology

$$\tilde{H}^1(R_k)_a \longrightarrow \tilde{H}^1(R_k^{\#})_b.$$

(ii) *If either the first or the second horizontal arrow in diagram* (7.20) *is an isomorphism of cohomology, or a fortiori if $H^1(P_k^{\#})_{m,b} = 0$, there exists a local diffeomorphism g of X defined on a neighborhood U_1 of a such that $j_{m+1}(g)(a) = F(a)$ and*

$$j_{k+1}(g)(R_{k\,|\,U_1}) = R^{\#}_{k\,|\,g(U_1)}.$$

Proof. (i) Using (2.25) we infer from (7.18) and (7.19) that the restriction of $\mathcal{D}F^{-1}$ to $\tilde{R}_m^{\#}$ is a section of $\tilde{R}_m^{\#*}\otimes R_m^{\#}$ and hence, if $R_m^{\#}$ is formally transitive, that $\mathcal{D}F^{-1}$ is a section of $T^*\otimes R_m^{\#}$ over $f(U)$. From (7.18), if $u\in(\mathcal{J}^*\otimes\mathcal{R}_m)_a$, we see that $F(u)$ belongs to $(\mathcal{J}^*\otimes\mathcal{R}_m^{\#})_b$. Therefore, under one or the other of our hypotheses of (i), $u^{F^{-1}}$ belongs to $(\mathcal{J}^*\otimes\mathcal{R}_m^{\#})_b$. Thus by Lemma 2.2, (i) and (ii), $\tilde{u}^{F^{-1}}$ belongs to $(J_0(\mathcal{J})^*\otimes\tilde{\mathcal{R}}_m^{\#})_b$. By Lemma 2.2, (ii) and (iv), we therefore have a commutative diagram

$$\begin{array}{ccc} Z^1(R_m)_a & \longrightarrow & Z^1(R_m^{\#})_b \\ \downarrow & & \downarrow \\ \bar{Z}^1(R_m)_a & \longrightarrow & \bar{Z}^1(R_m^{\#})_b \\ \downarrow & & \downarrow \\ \hat{Z}^1(R_m)_a & \longrightarrow & \hat{Z}^1(R_m^{\#})_b \end{array} \tag{7.21}$$

whose vertical arrows are given by (7.8) and (7.13) and whose horizontal arrows are bijective and send $u\in Z^1(R_m)_a$ (resp. $\bar{Z}^1(R_m)_a$) into $u^{F^{-1}}\in Z^1(R_m^{\#})_b$ (resp. $\bar{Z}^1(R_m^{\#})_b$) and $\hat{u}\in\hat{Z}^1(R_m)_a$ into $\hat{u}^{\pi_m F^{-1}}\in\hat{Z}^1(R_m^{\#})_b$. We denote by Ad F: $Q_{m+1\,|\,U\times U}\to Q_{m+1\,|\,f(U)\times f(U)}$ the mapping sending G, with source $G=x\in U$, target $G=y\in U$, into $F(y)\cdot G\cdot F(x)^{-1}$. According to Lemma 6.1 of [9], we have by (7.19)

$$\operatorname{Ad} F(P_{m+1\,|\,U\times U})=P^{\#}_{m+1\,|\,f(U)\times f(U)}$$

in a neighborhood of $I_{m+1\,|\,f(U)}$, and thus Ad F induces a bijective mapping Ad F: $\tilde{\mathcal{P}}^{\cdot}_{m+1,\,a}\to\tilde{\mathcal{P}}^{\#\cdot}_{m+1,\,f(a)}$. From (2.42), we have

$$(u^G)^{F^{-1}}=(u^{F^{-1}})^{\operatorname{Ad} F\cdot G}$$

for $u\in(\mathcal{J}^*\otimes\mathcal{R}_m)_a$ or $u\in(J_0(\mathcal{J})^*\otimes\tilde{\mathcal{R}}_m)_a$, $a\in U$ and $G\in\tilde{\mathcal{P}}^{\cdot}_{m+1,\,a}$, and

$$(\hat{u}^{G_1})^{\pi_m F^{-1}}=(\hat{u}^{\pi_m F^{-1}})^{\operatorname{Ad}\pi_m F\cdot G_1}$$

for $\hat{u}\in\mathcal{C}^1_{m,\,a}$ and $G_1\in\tilde{\mathcal{P}}^{\cdot}_{m,\,a}$, where $\operatorname{Ad}\pi_m F\cdot G_1=\pi_m(\operatorname{Ad} F\cdot G)$ if $G_1=\pi_m G$. Diagram (7.21) induces the commutative diagram (7.20) whose horizontal arrows are bijective. If $F=j_{m+1}(f)$, these arrows are isomorphisms of cohomology; furthermore for all $p\geqslant k$, we have

$$j_{p+1}(f)\,(R_{p\,|\,U})=R^{\#}_{p\,|\,f(U)},$$

and the diagram

$$\begin{array}{ccc} H^1(P_k)_{p+l,\,a} & \longrightarrow & H^1(P_k^{\#})_{p+l,\,b} \\ \Big\downarrow{\scriptstyle \pi_p} & & \Big\downarrow{\scriptstyle \pi_p} \\ H^1(P_k)_{p,\,a} & \longrightarrow & H^1(P_k^{\#})_{p,\,b} \end{array},$$

whose horizontal arrows are the isomorphisms of cohomology induced by $j_{p+l+1}(f)$ and $j_{p+1}(f)$ respectively, is commutative for all $p \geq k$, $l \geq 0$. Thus we obtain an isomorphism of cohomology $H^1(P_\kappa)_a \to H^1(P_k^\#)_b$.

(ii) If either the first or the second horizontal arrow in diagram (7.20) is an isomorphism of cohomology, there exists a section G of $\tilde{\mathcal{D}}_{m+1}^\#$ over a neighborhood of b such that $\mathcal{D}G = \mathcal{D}F^{-1}$ and $G(a) = I_{m+1}(a)$. Since $\mathcal{D}(G \cdot F) = F^{-1}(\mathcal{D}G) + \mathcal{D}F = F^{-1}(\mathcal{D}G - \mathcal{D}F^{-1}) = 0$, by (2.23) we can write $G \cdot F = j_{m+1}(g)$ where g is a local diffeomorphism of X defined on a neighborhood U_1 of a; it is clear that g has the required properties.

Let $R_k \subset J_k(T)$ be a formally transitive Lie equation. An R_k-connection is a mapping of vector bundles $\omega: J_0(T) \to R_k$ satisfying $\pi_0 \circ \omega = \mathrm{id}$; we set $\tilde{\omega} = \nu^{-1} \circ \omega \circ \nu$: $T \to \tilde{R}_k$. The curvature Ω of ω is the section of $\wedge^2 T^* \otimes R_k^0$ over X defined by

$$\Omega(\xi \wedge \eta) = [\tilde{\omega}(\xi), \tilde{\omega}(\eta)] - \tilde{\omega}[\xi, \eta]$$

for $\xi, \eta \in \mathcal{T}$. An R_k-connection ω determines covariant derivatives ∇ in $J_{k-1}(T)$ and $J_k^0(T)$ by setting

$$\nabla_\xi \eta = \mathcal{L}(\tilde{\omega}(\xi))\eta, \quad \text{for } \xi \in \mathcal{T},\ \eta \in J_{k-1}(\mathcal{T}),$$

$$\nabla_\xi \zeta = [\tilde{\omega}(\xi), \zeta], \quad \text{for } \xi \in \mathcal{T},\ \zeta \in J_k^0(\mathcal{T}).$$

If the curvature of ω vanishes, then so do the curvatures of the covariant derivatives ∇ (see [9], Proposition 3.3). We say that a sub-bundle F of $J_{k-1}(T)$ (resp. $J_k^0(T)$) is stable by ∇ if $\nabla(\mathcal{F}) \subset \mathcal{T}^* \otimes \mathcal{F}$.

The following proposition generalizes one aspect of Proposition 5.5 of [9].

PROPOSITION 7.10. *Let R_k, $R_k^\# \subset J_\kappa(T)$ be formally transitive and formally integrable Lie equations. Let P_k, $P_k^\# \subset Q_k$ be formally integrable finite forms of R_k, $R_k^\#$ respectively. Let $a, b \in X$ and let $\phi \in Q_\infty(a, b)$ satisfy $\phi(R_{\infty,a}) = R_{\infty,b}^\#$. Given a local diffeomorphism $f: X \to X$ defined on a neighborhood U of a with $f(a) = b$, for all $m \geq k$ there exists a section F_{m+1} of $\tilde{Q}_{m+1}$ over a neighborhood $U_{m+1} \subset U$ of a such that $F_{m+1}(a) = \pi_{m+1}\phi$, $\pi_0 F_{m+1} = f$ and*

$$F_{m+1}(R_{m \mid U_{m+1}}) = R^\#_{m \mid f(U_{m+1})}, \tag{7.22}$$

$$F_{m+1}(\tilde{R}_{m+1 \mid U_{m+1}}) = \tilde{R}^\#_{m+1 \mid f(U_{m+1})}. \tag{7.23}$$

Furthermore we have a bijective mapping

$$\tilde{H}^1(R_k)_a \to \tilde{H}^1(R_k^\#)_b.$$

Proof. For $m \geq k$, consider $P_m(a)$, $P_m^\#(b)$ as bundles over the connected components of a and b respectively via the projection "target". For all $m \geq k$, we can find sections s_m of $P_m(a)$ over a simply connected neighborhood $U_m \subset U$ of a and $s_m^\#$ of $P_m^\#(b)$ over $U_m^\# = f(U_m)$

such that $s_m(a)=I_m(a)$, $s^{\#}_m(b)=I_m(b)$, $U_{m+1}\subset U_m$, and $\pi_m s_{m+1}=s_m$ on U_{m+1} and $\pi_m s^{\#}_{m+1}=s^{\#}_m$ on $U^{\#}_{m+1}$. Define $\tilde{\omega}_m\colon T\to\tilde{R}_m$ on U_m by $\tilde{\omega}_m(\xi)=s_{m*}(\xi)\cdot s(x)^{-1}$ for $\xi\in T_x$, $x\in U_m$, and $\tilde{\omega}^{\#}_m\colon T\to\tilde{R}^{\#}_m$ on $U^{\#}_m$ by $\tilde{\omega}^{\#}_m(\xi)=s^{\#}_{m*}(\xi)\cdot s^{\#}(y)^{-1}$ for $\xi\in T_y$, $y\in U^{\#}_m$. It is clear that $\omega_m=\nu\circ\tilde{\omega}_m\circ\nu^{-1}$ is an R_m-connection on U_m and that $\omega^{\#}_m=\nu\circ\tilde{\omega}^{\#}_m\circ\nu^{-1}$ is an $R^{\#}_m$-connection on $U^{\#}_m$ whose curvatures vanish. Let $F_m(x)=s^{\#}_m(f(x))\cdot\pi_m\phi\cdot s_m(x)^{-1}$, for $x\in U_m$; then F_m is a section of $\tilde{Q}_m$ over U_m with $\pi_0 F_m=f$ and $\pi_m F_{m+1}=F_m$ on U_{m+1}, and $F_m(a)=\pi_m\phi$. By (2.5), for $\xi\in T_y$, $y\in U^{\#}_m$,

$$F_m(\tilde{\omega}_m(f^{-1}(\xi)))=F_m\cdot s_{m*}(f^{-1}(\xi))\cdot s(f^{-1}(y))^{-1}\cdot F_m(f^{-1}(y))^{-1}$$
$$=s^{\#}_{m*}(\xi)\cdot\pi_m\phi\cdot\pi_m\phi^{-1}\cdot s^{\#}_m(y)^{-1}=\tilde{\omega}^{\#}_m(\xi)$$

and thus $F_m(\tilde{\omega}_m)=\tilde{\omega}^{\#}_m$. Then the sub-bundles $F_{m+1}(R_{m\,|\,U_{m+1}})$, $R^{\#}_{m\,|\,U^{\#}_{m+1}}$ of $J_m(T)_{|\,U^{\#}_{m+1}}$ and $F_{m+1}(R^0_{m+1\,|\,U_{m+1}})$, $R^{\#0}_{m+1\,|\,U^{\#}_{m+1}}$ of $J^0_{m+1}(T)_{|\,U^{\#}_{m+1}}$ are stable by the covariant derivatives induced by $\omega^{\#}_{m+1}$ in $J_m(T)$ and $J^0_{m+1}(T)$ respectively. Moreover, $F_{m+1}(R_{m,a})=\pi_{m+1}\phi(R_{m,a})=R^{\#}_{m,b}$ and $F_{m+1}(R^0_{m+1,a})=\pi_{m+1}\phi(R^0_{m+1,a})=R^{\#0}_{m+1,b}$. Since $U^{\#}_{m+1}$ is simply connected, from Proposition 3.2 of [9] we deduce (7.22) and

$$F_{m+1}(R^0_{m+1\,|\,U_{m+1}})=R^{\#0}_{m+1\,|\,U^{\#}_{m+1}}. \tag{7.24}$$

Since $F_{m+1}(\tilde{\omega}_{m+1}(T_{|\,U_{m+1}}))=\tilde{\omega}^{\#}_{m+1}(T_{|\,U^{\#}_{m+1}})$, clearly (7.23) follows from (7.24). According to Proposition 7.9, (i), for $m\geq k$ the germ of F_{m+1} in $\tilde{Q}_{m+1,a}$ induces a bijective mapping

$$F_{m+1}\colon H^1(P_k)_{m,a}\to H^1(P^{\#}_k)_{m,b}$$

Since $\pi_{m+1}F_{m+l+1}=F_{m+1}$ on U_{m+l+1}, the diagram

$$\begin{tikzcd} H^1(P_k)_{m+l,a} \arrow[r, "F_{m+l+1}"] \arrow[d, "\pi_m"] & H^1(P^{\#}_k)_{m+l,b} \arrow[d, "\pi_m"] \\ H^1(P_k)_{m,a} \arrow[r, "F_{m+1}"] & H^1(P^{\#}_k)_{m,b} \end{tikzcd}$$

is commutative for $m\geq k$, $l\geq 0$. Therefore we obtain a bijective mapping $H^1(P_k)_a\to H^1(P^{\#}_k)_b$.

PROPOSITION 7.11. *Assume that X is connected. Let $R_k\subset J_k(T)$ be a formally transitive and formally integrable Lie equation. Then for all a, $b\in X$, we have a bijective mapping*

$$\tilde{H}^1(R_k)_a\to\tilde{H}^1(R_k)_b.$$

Proof. By Proposition 5.4 of [9], for all a, $b\in X$, there exists $\phi\in Q_\infty(a,b)$ such that $\phi(R_{\infty,a})=R_{\infty,b}$ and so the conclusion follows from Proposition 7.10.

The following proposition is an immediate consequence of Proposition 11.2 of [10] and Proposition 7.9, (i).

PROPOSITION 7.12. *Let R_k, $R_k^{\#} \subset J_k(T)$ be two formally transitive and formally integrable Lie equations and f a local diffeomorphism of X defined on a connected neighborhood U of $x \in X$ such that*

$$j_{k+1}(f)\,(R_{k\,|\,U}) = R^{\#}_{k\,|\,f(U)}.$$

If N_{k_1}, $N^{\#}_{k_1} \subset J_{k_1}(T)$ are formally integrable Lie equations, with $k_1 \geq k$, such that

$$[\tilde{\mathcal{R}}_{k_1+1}, \mathcal{N}_{k_1}] \subset \mathcal{N}_{k_1}, \quad [\tilde{\mathcal{R}}^{\#}_{k_1+1}, \mathcal{N}^{\#}_{k_1}] \subset \mathcal{N}^{\#}_{k_1}$$

and if

$$j_{k_1+1}(f)\,(x)\,(N_{k_1,x}) = N^{\#}_{k_1, f(x)},$$

then

$$j_{k_1+1}(f)\,(N_{k_1\,|\,U}) = N^{\#}_{k_1\,|\,f(U)}$$

and f defines an isomorphism of cohomology

$$\tilde{H}^1(N_{k_1})_a \to \tilde{H}^1(N^{\#}_{k_1})_{f(a)}$$

for all $a \in U$.

COROLLARY 7.1. *Let R_k, $R_k^{\#} \subset J_k(T)$ be formally transitive and formally integrable Lie equations and let $N_{k_1} \subset R_{k_1}$, $N^{\#}_{k_1} \subset R^{\#}_{k_1}$ be formally integrable Lie equations, with $k_1 \geq k$, such that*

$$[\tilde{\mathcal{R}}_{k_1+1}, \mathcal{N}_{k_1}] \subset \mathcal{N}_{k_1}, \quad [\tilde{\mathcal{R}}^{\#}_{k_1+1}, \mathcal{N}^{\#}_{k_1}] \subset \mathcal{N}^{\#}_{k_1}.$$

Let a, $b \in X$ and let $\phi \in Q_\infty(a, b)$ satisfy $\phi(R_{\infty,a}) = R^{\#}_{\infty,b}$ and $\phi(N_{\infty,a}) = N^{\#}_{\infty,b}$. Then we have a bijective mapping

$$\tilde{H}^1(R_k)_a \to \tilde{H}^1(R_k^{\#})_b. \tag{7.25}$$

If this mapping is an isomorphism of cohomology, or a fortiori if $\tilde{H}^1(R_k^{\#})_b = 0$, we have an isomorphism of cohomology

$$\tilde{H}^1(N_{k_1})_a \to \tilde{H}^1(N^{\#}_{k_1})_b. \tag{7.26}$$

Proof. Let P_k and $P_k^{\#}$ be formally integrable finite forms of R_k and $R_k^{\#}$ respectively and let $m \geq k_1$. By Proposition 7.10, we have a section F of $\tilde{Q}_{m+1}$ over a neighborhood U of a, with $\pi_0 F = f$, satisfying (7.18) and (7.19) and $F(a) = \pi_{m+1}\phi$, and a bijective mapping (7.25) such that the diagram

$$\begin{tikzcd} \tilde{H}^1(R_k)_a \arrow[r] \arrow[d, "\pi_m"] & \tilde{H}^1(R_k^{\#})_b \arrow[d, "\pi_m"] \\ H^1(P_k)_{m,a} & H^1(P_k^{\#})_{m,b} \end{tikzcd}$$

is commutative, where the lower horizontal arrow is induced by F according to Proposition 7.9, (i). If the upper horizontal arrow of this diagram is an isomorphism of cohomology, then so is the lower horizontal arrow. Therefore, by Proposition 7.9, (ii) and Proposition 7.12, we deduce the existence of a local diffeomorphism g of X defined on a neighborhood U_1 of a such that $j_{m+1}(g)(a) = \pi_{m+1}\phi$ and

$$j_{k+1}(g)(R_{k|U_1}) = R^{\#}_{k|g(U_1)},$$

$$j_{k_1+1}(g)(N_{k_1|U_1}) = N^{\#}_{k_1|g(U_1)}.$$

The isomorphism (7.26) of cohomology is given by Propositon 7.12.

Remark. Even the assertion that (7.26) is bijective requires an additional hypothesis because N_{k_1} and $N^{\#}_{k_1}$ are in general intransitive Lie equations (cf. Proposition 7.9, (i)).

Assume that X is endowed with the structure of an analytic manifold compatible with its structure of differentiable manifold. The following theorem is an immediate consequence of Corollary 6.1 of [9] and of Theorem 10.1 of [10].

THEOREM 7.1. *Let $R_k \subset J_k(T)$ be a formally transitive and formally integrable Lie equation and $N_{k_1} \subset R_{k_1}$ a formally integrable Lie equation, with $k_1 \geq k$, satisfying $[\tilde{\mathcal{R}}_{k_1+1}, \mathcal{N}_{k_1}] \subset \mathcal{N}_{k_1}$. Let $a \in X$. There exist on a neighborhood of a an analytic formally transitive and formally integrable Lie equation $R^{\#}_k \subset J_k(T)$ and a formally integrable Lie equation $N^{\#}_{k_1} \subset R^{\#}_{k_1}$ satisfying $[\tilde{\mathcal{R}}^{\#}_{k_1+1}, \mathcal{N}^{\#}_{k_1}] \subset \mathcal{N}^{\#}_{k_1}$ and $\phi \in Q_\infty(a, a)$ such that $\phi(R_{\infty,a}) = R^{\#}_{\infty,a}$, $\phi(N_{\infty,a}) = N^{\#}_{\infty,a}$.*

The hypotheses of Corollary 7.1 are satisfied by the equations R_k, $R^{\#}_k$, N_{k_1}, $N^{\#}_{k_1}$ of Theorem 7.1. Therefore Theorem 7.1 implies that the computation of the Spencer cohomology of formally transitive and formally integrable Lie equations is always reducible to the case of analytic Lie equations. If the second fundamental theorem holds for $R^{\#}_k$, there exists a local diffeomorphism f of X, defined on a neighborhood U of $a \in X$, such that

$$j_{k+1}(f)(R_{k|U}) = R^{\#}_{k|f(U)} \tag{7.27}$$

and

$$j_{k_1+1}(f)(N_{k_1|U}) = N^{\#}_{k_1|f(U)}.$$

The same conclusions hold under the weaker assumption that (7.25) is an isomorphism of cohomology.

8. Vanishing of the non-linear cohomology of a multifoliate Lie equation

Let W be an integrable sub-bundle of T and suppose that $V \cap W$ is a vector bundle. Let $\mathcal{W}_\varrho$ be the sheaf of ϱ-projectable sections of W and $J_k(W; \varrho)$ the set of k-jets of sections of $\mathcal{W}_\varrho$. Then $J_k(W; \varrho)$ is a vector bundle and

$$J_k(W; \varrho) = J_k(W) \cap J_k(T; \varrho). \tag{8.1}$$

Since W is integrable, we have

$$[\tilde{J}_k(\mathcal{W}), \tilde{J}_k(\mathcal{W})] \subset \tilde{J}_k(\mathcal{W}), \tag{8.2}$$

where $\tilde{J}_k(W) = \nu^{-1} J_k(W)$. Since $J_1(T; \varrho)$ is a formally integrable Lie equation whose k-th prolongation is $J_{k+1}(T; \varrho)$, it follows from (8.1) and (8.2) that $J_1(W; \varrho)$ is also a formally integrable Lie equation whose k-th prolongation is $J_{k+1}(W; \varrho)$ (see [6], p. 20). The kernel $g_k(W; \varrho) \subset S^k J_0(T)^* \otimes J_0(W)$ of $\pi_{k-1}: J_k(W; \varrho) \to J_{k-1}(W; \varrho)$ is therefore 1-acyclic for $k \geq 1$.

Let $Q_1(W; \varrho)$ be a formally integrable finite form of $J_1(W; \varrho)$ whose k-th prolongation we denote by $Q_{k+1}(W; \varrho)$.

THEOREM 8.1. *For all $m \geq 1$, $a \in X$, we have*

$$\bar{H}^1(Q_1(W; \varrho))_{m,a} = 0.$$

Proof. Set $\tilde{J}_k(W; \varrho) = \nu^{-1} J_k(W; \varrho)$. Let u be a section of $(J_0(T)^* \otimes \tilde{J}_1(W; \varrho))^\wedge$ over a neighborhood of a point $a \in X$, which we shall suppose is equal to X without any loss of generality; assume that $\bar{\mathcal{D}}_1 u = 0$. Now $\tilde{u}_0 = (\pi_0 u) \circ \nu$ is a section of $T^* \otimes W$ and, since $\mathrm{id} - \tilde{u}_0$: $T \to T$ is invertible,

$$\mathrm{id} - \tilde{u}_0 \colon W \to W, \quad \mathrm{id} - \tilde{u}_0 \colon V + W \to V + W$$

are isomorphisms and

$$\mathrm{id} - \tilde{u}_0 \colon V \to V + W$$

is injective. We set $u_0 = \pi_0 u$ and

$$V^{u_0} = (\mathrm{id} - \tilde{u}_0)(V);$$

then

$$V^{u_0} \cap W = (\mathrm{id} - \tilde{u}_0)(V \cap W)$$

and

$$V^{u_0} + W = V + W.$$

Since V is integrable, the sub-bundle V^{u_0} is integrable by (6.3) and Lemma 1.3; therefore so is $V^{u_0} \cap W$. By Frobenius' theorem, replacing X by a neighborhood of a and Y by a neighborhood of $b = \varrho(a)$, if necessary, there exist manifolds Z, S, surjective submersions $\tau: X \to Z$, $\lambda: Y \to S$, $\sigma: Z \to S$, $\varrho': X \to Y$, $\lambda': Y \to S$ such that $\varrho'(a) = \varrho(a) = b$ and the diagrams

$$\begin{tikzcd} X \arrow[r, "\varrho"] \arrow[d, "\tau"'] & Y \arrow[d, "\lambda"] \\ Z \arrow[r, "\sigma"] & S \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varrho'"] \arrow[d, "\tau"'] & Y \arrow[d, "\lambda'"] \\ Z \arrow[r, "\sigma"] & S \end{tikzcd}$$

commute, and such that W, $V+W$, V^{u_0} are the bundles of vectors tangent to the fibers of the submersions $\tau\colon X\to Z$, $\sigma\circ\tau\colon X\to S$, $\varrho'\colon X\to Y$ respectively. Set

$$Y\times_S Z=\{(y,z)\in Y\times Z\,|\,\lambda(y)=\sigma(z)\},$$

$$(Y\times_S Z)'=\{(y,z)\in Y\times Z\,|\,\lambda'(y)=\sigma(z)\}.$$

Then $V\cap W$ and $V^{u_0}\cap W$ are the bundles of vectors tangent to the fibers of the submersions $(\varrho,\tau)\colon X\to Y\times_S Z$ and $(\varrho',\tau)\colon X\to(Y\times_S Z)'$ respectively. By the implicit-function theorem, there exists a local diffeomorphism $g\colon Y\to Y$ defined on a neighborhood of b such that $g(b)=b$ and the diagram

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "\lambda"] & Y \arrow[d, "\lambda'"] \\
S \arrow[r, "\mathrm{id}"] & S
\end{tikzcd}$$

commutes. Then $(g,\mathrm{id})\colon Y\times_S Z\to(Y\times_S Z)'$ is a local diffeomorphism defined on a neighborhood of $(b,\tau(a))$ and, by the implicit-function theorem, there exists a local diffeomorphism $f\colon X\to X$ defined on a neighborhood of a such that $f(a)=a$ and the diagram

$$\begin{tikzcd}
X \arrow[r, "f^{-1}"] \arrow[d, "(\varrho,\tau)"] & X \arrow[d, "(\varrho',\tau)"] \\
Y\times_S Z \arrow[r, "(g,\mathrm{id})"] & (Y\times_S Z)'
\end{tikzcd}$$

of local mappings commutes. Therefore the diagram

$$\begin{tikzcd}
& X \arrow[rr, "f^{-1}"] \arrow[dl, "\varrho"'] \arrow[dd, "\tau"] & & X \arrow[dl, "\varrho'"'] \arrow[dd, "\tau"] \\
Y \arrow[rr, "g"] \arrow[dd, "\lambda"] & & Y \arrow[dd, "\lambda'"] & \\
& Z \arrow[rr, "\mathrm{id}"] \arrow[dl, "\sigma"'] & & Z \arrow[dl, "\sigma"] \\
S \arrow[rr, "\mathrm{id}"] & & S &
\end{tikzcd}$$

of local mappings is commutative. Thus we have a diffeomorphism $f: X \to X$ defined on a neighborhood U of a which is τ-projectable onto the identity $Z \to Z$ and satisfies $f(a)=a$ and

$$f^{-1}(V_{|f(U)}) = V^{u_0}_{|U}, \tag{8.3}$$

$$f^{-1}(W_{|f(U)}) = W_{|U}. \tag{8.4}$$

For $k \geq 1$, let $Q_k(W) \subset Q_k$ be the finite form of $J_k(W)$ consisting of all k-jets of local diffeomorphisms $X \to X$ which are τ-projectable onto the identity mapping $Z \to Z$; it is easily seen that $Q_k(W) \cap Q_k(\varrho)$ is a formally integrable and integrable finite form of $J_k(W; \varrho)$ whose l-th prolongation is $Q_{k+l}(W) \cap Q_{k+l}(\varrho)$. We shall henceforth assume that

$$Q_k(W; \varrho) = Q_k(W) \cap Q_k(\varrho)$$

for $k \geq 1$, and set

$$\tilde{Q}_k(W; \varrho) = Q_k(W) \cap \tilde{Q}_k(\varrho).$$

By Lemma 2.3, (ii),

$$F = j_1(f) - f \circ \tilde{u}_0$$

is a section of $\tilde{Q}_1$ over U; from Lemma 2.3, (iii), we deduce that $\tilde{D}F = u_0$ on U. Clearly $j_1(f)$ is a section of $Q_1(W)$ over U. By (8.4), $(f \circ \tilde{u}_0)(x)$ belongs to $T^*_x \otimes W_{f(x)}$, for all $x \in U$; thus by Proposition 6.1, (ii) and (2.20), F is also a section of $Q_1(W)$ over U. By Lemma 2.3, (i), (8.3) is equivalent to the fact that $F\xi$ belongs to $J_0(V)_{f(x)}$ for all $\xi \in J_0(V)_x$, $x \in U$; from Proposition 6.1, (i), we deduce that F is a section of $\tilde{Q}_1(\varrho)$ and hence of $\tilde{Q}_1(W; \varrho)$ over U satisfying $\tilde{D}F = u$ and $(\pi_0 F)(a) = a$.

Finally, we also denote by F and u the germs of the sections F and u in $\tilde{Q}_1(W; \varrho)_a$ and $(J_0(\mathcal{T})^* \otimes \tilde{J}_1(\mathcal{W}; \varrho))_a$ respectively. The following argument then resembles that used to prove Proposition 7.6. Choose $F_1 \in \tilde{Q}_2(W; \varrho)_a$ such that $\pi_1 F_1 = F$. Then $\pi_0(u^{F_1^{-1}}) = 0$; hence $u^{F_1^{-1}} \in (J_0(\mathcal{T})^* \otimes g_1(W; \varrho))_a$ and (see § 1)

$$\tilde{\delta} u^{F_1^{-1}} = -\tilde{D} u^{F_1^{-1}} = -\tfrac{1}{2}\pi_0[u^{F_1^{-1}}, u^{F_1^{-1}}] = 0.$$

Since $g_1(W; \varrho)$ is 1-acyclic, there exists $v \in g_2(W; \varrho)$ such that $\tilde{\delta} v = u^{F_1^{-1}}$. Since $Q_1(W; \varrho)$ is formally integrable the mapping (7.4), namely

$$\partial: Q_2^1 \cap Q_2(W; \varrho) \to g_2(W; \varrho),$$

is an isomorphism of Lie groups over X and thus $G = \partial^{-1} v$ belongs to $Q_2^1 \cap Q_2(W; \varrho)$. By (2.38)

$$\tilde{D}G = -\tilde{\delta} v = -u^{F_1^{-1}},$$

and $F_1^{-1} \cdot G \in \tilde{Q}_2(W; \varrho)_a$ with $\pi_0(F_1^{-1} \cdot G)(a) = a$ and

$$u^{F_1^{-1}\cdot G} = u^{F_1^{-1}} + \bar{\mathcal{D}}G = 0.$$

Since $Q_1(W;\varrho)$ is integrable, there exists $h \in \operatorname{Sol}(Q_1(W;\varrho))_a$ such that $j_2(h)(a) = F_1^{-1}(a)\cdot G(a)$; therefore $F_2 = F_1^{-1}\cdot G \cdot j_2(h^{-1})$ belongs to $\tilde{Q}_2(W;\varrho)^{\cdot}_a$ and

$$u^{F_2} = (u^{F_1^{-1}\cdot G})^{j_2(h^{-1})} = 0^{j_2(h^{-1})} = 0,$$

showing that u is cohomologous to zero in $\bar{H}^1(Q_1(W;\varrho))_{1,a}$. Therefore $\bar{H}^1(Q_1(W;\varrho))_{1,a} = 0$, and the desired result holds by Proposition 7.8.

9. Non-linear cohomology sequences for projectable Lie equations

In this section we prove our main theorems concerning non-linear cohomology sequences. Before taking these theorems up, however, we accumulate various facts about ϱ-projectable Lie equations which are needed in the proofs, and we begin with the following lemma which is an easy consequence of the implicit-function theorem.

LEMMA 9.1. *Let $\pi: E \to X$ be a fibered manifold over X and F a fibered manifold over Y. Let $\varphi: E \to F$ be a morphism of fibered manifolds over ϱ such that the rank of*

$$\varphi_*: V_e(E) \to T_{\varphi(e)}(F/Y)$$

is independent of $e \in E$. Then, if $e_0 \in E$, there exist an open neighborhood U of $x_0 = \pi(e_0)$ in X, an open fibered submanifold E' of $E_{|U}$ containing e_0, a fibered submanifold F' of $F_{|\varrho(U)}$ such that $\varphi(E') = F'$ and $\varphi: E' \to F'$ is an epimorphism of fibered manifolds over $\varrho: U \to \varrho(U)$. If s' is a section of F' over a neighborhood of $y_0 = \varrho(x_0)$ and if $u \in J_1(E;\varphi)$ satisfies $\pi_0 u = e_0$ and $\varphi u = j_1(s')(y_0)$, there exists a section s of E' over a neighborhood of x_0 such that $j_1(s)(x_0) = u$ and $\varphi \circ s = s' \circ \varrho$.

Let $R_k \subset J_k(T;\varrho)$ be a Lie equation, and assume that there exists a differential equation $R''_k \subset J_k(T_Y; Y)$ such that $\varrho(R_{k,a}) = R''_{k,\varrho(a)}$ for all $a \in X$. Then $\varrho: (\bar{R}_k)_{\varrho,a} \to \bar{R}''_{k,\varrho(a)}$ is surjective for all $a \in X$ and, by (6.7), R''_k is a Lie equation.

Let $P_k \subset Q_k(\varrho)$, $P''_k \subset Q_k(Y)$ be finite forms of R_k and R''_k respectively, and consider the mapping $\varrho: P_k \to Q_k(Y)$. If $a \in X$ then, by Lemma 9.1, there exist an open neighborhood U of a, an open fibered submanifold E of $P_{k|U}$ containing $I_k(a)$, a fibered submanifold E'' of $Q_k(Y)_{|\varrho(U)}$ such that $\varrho(E) = E''$. For $p \in E$, the image of

$$\varrho_*: V_p(P_k) \to T_{\varrho(p)}(Q(Y)/Y)$$

is equal to $\tilde{R}''_{k,\varrho(x)}\cdot\varrho(p)$, where $x =$ source p, and so

$$T_{\varrho(p)}(E''/\varrho(U)) = \tilde{R}''_{k,\varrho(x)}\cdot\varrho(p).$$

Since E'' and P_k'' are integral submanifolds of the same distribution, we obtain the equality $E''=P_k''$ on a neighborhood of $I_{Y,k}(\varrho(a))$.

From these remarks and Lemma 9.1, we deduce:

LEMMA 9.2. *Let $a\in X$ and $b=\varrho(a)$. The following assertions hold:*

(i) *If $F\in \mathcal{D}_{k,a}$, $F(a)=I_k(a)$, then $\varrho F\in \mathcal{D}''_{k,X}\subset Q_k(Y)_X$ with $\varrho F(a)=I_{Y,k}(b)$.*

(ii) *If $\phi\in \mathcal{D}''_{k,X,a}$, with $\phi(a)=I_{Y,k}(b)$, and if there is an element $G\in J_1(P_k)$ with $J_1(\varrho)G=j_1(\phi)(a)$, $\pi_0 G=I_k(a)$, then there exists $F\in\mathcal{D}_{k,a}$ satisfying $\varrho F=\phi$ and $j_1(F)(a)=G$.*

(iii) *If $\phi\in \mathcal{D}''_{k,b}$, with $\phi(b)=I_{Y,k}(b)$, and if there is an element $G\in J_1(P_k;\varrho)$ with $\varrho G=j_1(\phi)(b)$, $\pi_0 G=I_k(a)$, then there exists $F\in\mathcal{D}_{k,\varrho,a}$ satisfying $\varrho F=\phi$ and $j_1(F)(a)=G$.*

Definition 9.1. A differential equation $R_k\subset J_k(T;\varrho)$ is ϱ-projectable if, for each $l\geqslant 0$, R_{k+l} is a vector bundle and if there exists a differential equation $R''_{k+l}\subset J_{k+l}(T_Y;\,Y)$ such that $\varrho(R_{k+l,a})=R''_{k+l,\varrho(a)}$ for all $a\in X$.

If $Y=X$ and $\varrho\colon X\to X$ is a diffeomorphism, then $J_k(T;\varrho)=J_k(T)$ and $\varrho\colon J_k(T)_a\to J_k(T)_{\varrho(a)}$, for $a\in X$, is the isomorphism $j_{k+1}(\varrho)(a)$ and every differential equation $R_k\subset J_k(T)$ all of whose prolongations are vector bundles is ϱ-projectable.

We shall consider a formally integrable Lie equation $R_k\subset J_k(T;\varrho)$ satisfying the following conditions:

(I) R_k is ϱ-projectable;

(II) $\pi_0\tilde{R}_k=W$ and $V\cap W$ are sub-bundles of T and $R_k\subset J_k(W)$.

Let $R''_{k+l}\subset J_{k+l}(T_Y;\,Y)$ be the Lie equation such that $\varrho(R_{k+l,a})=R''_{k+l,\varrho(a)}$ for all $a\in X$. Since $\bar{R}_{k+l}=R_{k+l}\cap J_{k+l}(V)$ is the kernel of the epimorphism $\varrho\colon R_{k+l}\to\varrho^{-1}R''_{k+l}$, it is a vector bundle. The third condition assumed satisfied is:

(III) for all l, $m\geqslant 0$, the projections $\pi_{k+l}\colon \bar{R}_{k+l+m}\to\bar{R}_{k+l}$ are of constant rank.

For the most part, we assume only conditions (I) and (II) as, for example, in Theorem 9.1 and Proposition 9.1; condition (III) is used only at the end of this section.

If X and the fibers of ϱ are connected and if R_k is formally transitive, or more generally if there exists a formally transitive and formally integrable Lie equation $N_{k+1}\subset J_{k+1}(T;\varrho)$ such that

$$[\tilde{\mathcal{N}}_{k+1},\mathcal{R}_k]\subset\mathcal{R}_k,$$

then condition (I) above holds by Theorem 11.1 of [10] and (II), (III) hold by Lemma 10.3, (ii) and Proposition 10.3, (i) of [10].

Let $R_k\subset J_k(T;\varrho)$ be a formally integrable Lie equation satisfying conditions (I) and (II). Let $P_k\subset Q_k(\varrho)$ be a formally integrable finite form of R_k and let $P_{k+l}\subset Q_{k+l}(\varrho)$ be the

l-th prolongation of P_k; for $m \geq k$, let $P''_m \subset Q_m(Y)$ be a finite form of R''_m. Since R_k satisfies (I) and (II), W and $V \cap W$ are integrable sub-bundles of T; moreover the image W_Y of $\tilde{R}''_k$ in T_Y is an integrable sub-bundle of T_Y such that $\varrho W_a = W_{Y,\varrho(a)}$ for all $a \in X$. Since π_0: $\tilde{R}''_m \to W_Y$ is surjective for $m \geq k$, its kernel R''^0_m is a vector bundle. Therefore $P''^0_m = P''_m \cap Q^0_m(Y)$ is a sub-bundle of Lie groups of $Q^0_m(Y)$ whose Lie algebra we identify with R''^0_m under the mappings (5.23). Thus (4.6) gives us a sub-complex of (6.38), namely

$$\mathcal{P}''^0_{m,X} \xrightarrow{\mathcal{D}_{X/Y}} \mathcal{V}^* \otimes (\mathcal{R}''^0_m)_X \xrightarrow{\mathcal{D}_{1,X/Y}} \wedge^2 \mathcal{V}^* \otimes (\mathcal{R}''^0_m)_X. \tag{9.1}$$

For $m \geq k$, let

$$(\mathcal{T}^* \otimes \mathcal{R}_m)_\varrho = (\mathcal{T}^* \otimes \mathcal{R}_m) \cap \mathcal{T}^* \otimes J_m(\mathcal{T};\varrho))_\varrho,$$

$$Z^1_\varrho(R_m) = Z^1(R_m) \cap (\mathcal{T}^* \otimes \mathcal{R}_m)_\varrho$$

and let

$$\tilde{\mathcal{P}}_{m,\varrho} = \mathcal{P}_m \cap \tilde{\mathcal{Q}}_m(\varrho)_\varrho,$$

$$\tilde{\mathcal{P}}^{\cdot}_{m,\varrho,a} = \{F \in \tilde{\mathcal{P}}_{m,\varrho,a} \mid F(a) = I_m(a)\}$$

for $a \in X$. According to Proposition 6.4, (iv), the group $\tilde{\mathcal{P}}^{\cdot}_{m+1,\varrho,a}$ operates on $Z^1_\varrho(R_m)_a$ and so we define the cohomology

$$H^1_\varrho(P_k)_{m,a} = Z^1_\varrho(R_m)_a / \tilde{\mathcal{P}}^{\cdot}_{m+1,\varrho,a},$$

for $m \geq k$, $a \in X$, to be the set of orbits under the right operations of the group $\tilde{\mathcal{P}}^{\cdot}_{m+1,\varrho,a}$ on $Z^1_\varrho(R_m)_a$. We denote by 0 the orbit of $0 \in Z^1_\varrho(R_m)_a$. This cohomology is therefore a set with distinguished element 0 and clearly does not depend on the choice of the finite form P_k. We have the mapping of cohomology

$$H^1_\varrho(P_k)_{m,a} \to H^1(P_k)_{m,a}$$

which sends the orbit of $H^1_\varrho(P_k)_{m,a}$ passing through $u \in Z^1_\varrho(R_m)_a$ into the orbit $\{u^F \mid F \in \tilde{\mathcal{P}}^{\cdot}_{m+1,a}\}$.

Let $k_0 \geq \sup(k, 2)$ be an integer such that g_{k_0} is 2-acyclic.

THEOREM 9.1. *Assume that $R_k \subset J_k(T;\varrho)$ is a formally integrable Lie equation satisfying the conditions* (I) *and* (II). *Then, for all $m \geq k_0$, $a \in X$, the mapping*

$$H^1_\varrho(P_k)_{m,a} \to H^1(P_k)_{m,a} \tag{9.2}$$

is an isomorphism of cohomology. Moreover, if $u \in Z^1(R_m)_a$, then there exists $F \in \tilde{\mathcal{P}}^{\cdot}_{m+1,a}$ such that $u^F(a) = 0$ and $u^F \in Z^1_\varrho(R_m)_a$.

Proof. If u_1, $u_2 \in Z^1_\varrho(R_m)_a$ and if there is an $F \in \tilde{\mathcal{P}}^{\cdot}_{m+1,a}$ with $u_1^F = u_2$, then by Proposition 6.4, (ii), $F \in \tilde{\mathcal{P}}^{\cdot}_{m+1,\varrho,a}$ and so (9.2) is injective.

Let $u \in Z^1(R_m)_a$; then since g_m is 2-acyclic, there exists, by Proposition 7.4, $u_1 \in Z^1(R_{m+3})_a$ with $\pi_m u_1 = u$. By Lemma 7.1, there exists $F_1 \in \tilde{\mathcal{P}}^{\cdot}_{m+4,a}$ such that $u_1^{F_1}(a) = 0$. We set $u_2 = u_1^{F_1}$.

Let $Q_2(W;\varrho)$ be a finite form of the Lie equation $J_2(W;\varrho)$. Since $R_1\subset J_1(W;\varrho)$ there exists, by Theorem 8.1, $F_2\in\tilde{Q}_2(W;\varrho)^{\cdot}_a$ satisfying $(\pi_1 u_2)^{F_2}=0$. Since $(\pi_1 u_2)(a)=0$, it follows that $(\mathcal{D}F_2)(a)=(\pi_1 u_2)^{F_2}(a)=0$; hence by (2.27) we have $j_1(\pi_1 F_2)(a)=j_1(I_1)(a)$. Therefore, if $f=\pi_0 F_2$, we have $j_1(f)(a)=j_1(I_0)(a)$. Let $Q_0(W)\subset X\times X$ be a finite form of the Lie equation $J_0(W)\subset J_0(T)$; since $\pi_0\colon J_2(W;\varrho)\to J_0(W)$ is surjective, f belongs to $\tilde{Q}_0(W)^{\cdot}_a$ by Proposition 7.1, (iii). Because $\pi_0\colon R_{m+4}\to J_0(W)$ is surjective there exists, by Proposition 7.1, (iii), $F_3\in\tilde{\mathcal{P}}^{\cdot}_{m+4,a}$ such that $\pi_0 F_3=f$ and $j_1(F_3)(a)=j_1(I_{m+4})(a)$. Since $(\mathcal{D}F_3)(a)=u_2(a)=0$, we see that $u_2^{F_3}(a)=0$. As $\pi_0\varrho((\pi_1 u_2)^{F_2})=0$ and $\pi_0 F_2=\pi_0 F_3$, we have by Lemma 6.4

$$\pi_0\varrho(u_2^{F_3})=0.$$

Therefore $w=\varrho(u_2^{F_3})$ belongs to $(\mathcal{V}^*\otimes(\mathcal{R}''^0_{m+3})_X)_a$ and $w(a)=0$; by Proposition 6.3, we have

$$\mathcal{D}_{1,X/Y}w=\pi_{m+2}\cdot d_{X/Y}w-\tfrac{1}{2}[w,w]=0.$$

Set $w_1=\pi_{m+2}w\in\mathcal{V}^*\otimes(\mathcal{R}''^0_{m+2})_X$; then $w_1(a)=0$ and

$$\mathcal{D}_{1,X/Y}w_1=d_{X/Y}w_1-\tfrac{1}{2}[w_1,w_1]=0,$$

where $\mathcal{D}_{1,X/Y}$ is the operator of the complex (9.1) with $m+2$ replacing m. By Proposition 4.1 applied to this complex, there exists $\phi\in\mathcal{P}''^0_{m+2,X,a}$ satisfying $\mathcal{D}_{X/Y}\phi=w_1$ and $j_1(\phi)(a)=j_1(I_{Y,m+2}\circ\varrho)(a)$. By Lemma 9.2, (ii) (with $G=j_1(I_{m+2})(a)$), there exists $F_4\in\mathcal{P}_{m+2,a}$ satisfying $j_1(F_4)(a)=j_1(I_{m+2})(a)$ and $\varrho F_4=\phi$; clearly $F_4\in\tilde{\mathcal{P}}^{\cdot}_{m+2,a}$.

Set $w_2=\pi_{m+1}w_1$ and write

$$u_3=(\pi_{m+1}(u_2^{F_3}))^{F_4^{-1}};$$

since $(\mathcal{D}F_4^{-1})(a)=0$, we have $u_3(a)=0$ and

$$\varrho(u_3^{F_4})=w_2=\mathcal{D}_{X/Y}\phi,$$

where $\mathcal{D}_{X/Y}$ is the operator $\tilde{Q}_{m+2}(Y)_X\to\mathcal{V}^*\otimes J_{m+1}(\mathcal{T}_Y;\ Y)_X$. Since $\pi_0 F_4$ is ϱ-projectable onto the germ of the identity $Y\to Y$, it follows from Lemma 6.5 that $\varrho(u_3)=0$ or equivalently $u_3\in F_1^1(J_{m+1}(\mathcal{T});\varrho)$. We have

$$\mathcal{D}_1 u_3=Du_3-\tfrac{1}{2}[u_3,u_3]=0,$$

where $[u_3,u_3]\in F_2^2(J_m(\mathcal{T});\varrho)$ by (6.9). Hence $Du_3\in F_2^2(J_m(\mathcal{T});\varrho)$. Set $u_4=\pi_m u_3$; by Proposition 4, (i) of [6], we see that $u_4\in(\mathcal{T}^*\otimes J_m(\mathcal{T};\varrho))_\varrho$. Finally, we note that $u_4=u^F$ and $u_4(a)=0$, where $F=\pi_{m+1}F_1\cdot\pi_{m+1}F_3\cdot\pi_{m+1}F_4^{-1}\in\tilde{\mathcal{P}}^{\cdot}_{m+1,a}$. Hence $u_4\in Z^1_\varrho(R_m)_a$ belongs to the same cohomology class in $H^1(P_k)_{m,a}$ as u, showing that (9.2) is surjective and completing the proof of the theorem.

We now recall some facts which may be found in the papers [6], [10]. For $l\geq 0$, we have $\bar{R}_{k+l}=(\bar{R}_k)_{+l}$; since $\pi_m\colon R''_{m+1}\to R''_m$ is surjective for $m\geq k$ and $R''_{m+1}\subset(R''_m)_{+1}$, there exists by the Cartan-Kuranishi prolongation theorem an integer $k_1 \geqslant \sup(k, 1)$ such that $(R''_{k_1})_{+l} = R''_{k_1+l}$ for all $l \geqslant 0$ and R''_{k_1} is a formally integrable Lie equation in $J_{k_1}(T_Y; Y)$.

For $m \geqslant k$ and $a \in X$, we define the group

$$H^0(P''_{k_1})_{m,a} = \left\{ f'' \in (\operatorname{Aut} Y)_{\varrho(a)} \,\middle|\, \begin{array}{l} j_m(f'') \in \tilde{\mathcal{D}}''_{m,\varrho(a)} \text{ and there exists } G \in (Q_{(1,m)}(\varrho) \cap J_1(P_m))_a \\ \text{such that } \pi_0 G = I_m(a) \text{ and } \varrho G = j_1(j_m(f''))(\varrho(a)) \end{array} \right\},$$

or equivalently, by Lemma 9.2, (iii),

$$H^0(P''_{k_1})_{m,a} = \left\{ f'' \in (\operatorname{Aut} Y)_{\varrho(a)} \,\middle|\, \begin{array}{l} j_m(f'') \in \tilde{\mathcal{D}}''_{m,\varrho(a)} \text{ and there exists } F \in \tilde{\mathcal{D}}^{\cdot}_{m,\varrho,a} \\ \text{such that } \varrho F = j_m(f'') \end{array} \right\}.$$

We note that this group is independent of the choice of the finite forms P_k and P''_m and depends therefore only on the Lie equations R_k and R''_{k_1}. Since $P_m \subset Q_m(\varrho)$, the elements of $H^0(P_k)_{m,a}$ are ϱ-projectable; hence by Lemma 9.2, (i), we have the homomorphism of groups

$$\varrho: H^0(P_k)_{m,a} \to H^0(P''_{k_1})_{m,a}.$$

For $m \geqslant k$, let $\bar{P}_m$ be a finite form of $\bar{R}_m$. It is easily seen that $\bar{P}_m = P_m \cap Q_m(V)$ in a neighborhood of I_m and hence that

$$\tilde{\bar{\mathcal{D}}}^{\cdot}_{m,a} = \tilde{\mathcal{D}}^{\cdot}_{m,a} \cap \tilde{Q}(V)_a$$

for $a \in X$.

We now define the operation of the group $H^0(P''_{k_1})_{m+1,a}$ on $H^1(\bar{P}_k)_{m,a}$. Let $u \in Z^1(\bar{R}_m)_a$ and $f'' \in H^0(P''_{k_1})_{m+1,a}$. If $F \in \tilde{\mathcal{D}}^{\cdot}_{m+1,\varrho,a}$ satisfies $\varrho F = j_{m+1}(f'')$, then by Proposition 6.4, (iii)

$$\varrho \mathcal{D} F = \mathcal{D} \varrho F = \mathcal{D} j_{m+1}(f'') = 0,$$

and so $\mathcal{D} F \in \mathcal{J}^* \otimes \bar{R}_m$. By (7.6) and the commutativity of (6.25), P_{m+1} preserves $\bar{R}_m$ and so $F^{-1}(u)$ belongs to $\mathcal{J}^* \otimes \bar{R}_m$. Therefore $u^F \in Z^1(\bar{R}_m)_a$. If $[u]$ is the cohomology class in $H^1(\bar{P}_k)_{m,a}$ of the cocycle u, we define $[u]^{f''}$ to be the cohomology class $[u^F]$ of u^F in $H^1(\bar{P}_k)_{m,a}$. We now verify that $[u]^{f''}$ is well-defined, i.e., that it does not depend on the choice of F or of u. To show that $[u]^{f''}$ is independent of the choice of F, let $F_1 \in \tilde{\mathcal{D}}^{\cdot}_{m+1,\varrho,a}$ with $\varrho F_1 = j_{m+1}(f'')$. Then $G' = F^{-1} \cdot F_1$ belongs to $\tilde{\mathcal{D}}^{\cdot}_{m+1,\varrho,a} \cap \tilde{Q}_{m+1}(V)_a$, that is to $\tilde{\bar{\mathcal{D}}}^{\cdot}_{m+1,a}$. Since $F_1 = F \cdot G'$, we have $u^{F_1} = (u^F)^{G'}$, and it follows that u^{F_1} belongs to the same cohomology class as u^F in $H^1(\bar{P}_k)_{m,a}$. To show that $[u^F]$ does not depend on the choice of u, we replace u by u^G, another point on the same orbit, where $G \in \tilde{\bar{\mathcal{D}}}^{\cdot}_{m+1,a}$. Then $G' = F^{-1} \cdot G \cdot F$ belongs to $\tilde{\bar{\mathcal{D}}}_{m+1,a}$ and $(u^G)^F = (u^F)^{G'}$; therefore $(u^G)^F$ is cohomologous to u^F in $H^1(\bar{P}_k)_{m,a}$. Finally, let $f''_1 \in H^0(P''_{k_1})_{m+1,a}$ and $\alpha \in H^1(\bar{P}_k)_{m,a}$; then

$$(\alpha^{f''})^{f''_1} = \alpha^{f'' \circ f''_1}, \quad \alpha^{I_{Y,0}} = \alpha, \tag{9.3}$$

and hence we have an action of the group $H^0(P''_{k_1})_{m+1,a}$ on $H^1(\bar{P}_k)_{m,a}$. In fact, let $F_1 \in \tilde{\mathcal{D}}_{m+1,\varrho,a}$ with $\varrho F_1 = j_{m+1}(f''_1)$; then

$$([u]^{f''})^{f_1''} = [u^F]^{f_1''} = [(u^F)^{F_1}] = [u^{F\cdot F_1}] = [u]^{f''\circ f_1''}.$$

We define

$$\partial^{\#}: H^0(P''_{k_1})_{m+1,a} \to H^1(\bar{P}_k)_{m,a}$$

to be the mapping sending the element $f'' \in H^0(P''_{k_1})_{m+1,a}$ into $\partial^{\#} f = [0]^{f''} = 0^{f''}$.

By Proposition 6.4, (iii) and Lemma 9.2, (i), for $m \geqslant k$ and $a \in X$, we have the commutative diagram

$$\begin{tikzcd}
\tilde{\bar{\mathcal{D}}}^{\cdot}_{m+1,a} \arrow[r, "\mathcal{D}"] \arrow[d] & (\mathcal{J}^* \otimes \bar{\mathcal{R}}_m)^{\wedge}_a \arrow[r, "\mathcal{D}_1"] \arrow[d] & (\wedge^2 \mathcal{J}^* \otimes \bar{\mathcal{R}}_{m-1})_a \arrow[d] \\
\tilde{\mathcal{D}}^{\cdot}_{m+1,\varrho,a} \arrow[r, "\mathcal{D}"] \arrow[d, "\varrho"] & (\mathcal{J}^* \otimes \mathcal{R}_m)^{\wedge}_{\varrho,a} \arrow[r, "\mathcal{D}_1"] \arrow[d, "\varrho"] & (\wedge^2 \mathcal{J}^* \otimes \mathcal{R}_{m-1})_{\varrho,a} \arrow[d, "\varrho"] \\
\mathcal{D}''^{\cdot}_{m+1,\varrho(a)} \arrow[r, "\mathcal{D}"] & (\mathcal{J}^*_Y \otimes \mathcal{R}''_m)^{\wedge}_{\varrho(a)} \arrow[r, "\mathcal{D}_1"] & (\wedge^2 \mathcal{J}^*_Y \otimes \mathcal{R}''_{m-1})_{\varrho(a)}
\end{tikzcd} \tag{9.4}$$

where $\bar{R}_{k-1} = J_{k-1}(V)$, $R_{k-1} = J_{k-1}(T;\varrho)$ and $R''_{k-1} = J_{k-1}(T_Y; Y)$. The inclusion $\bar{R}_k \subset R_k$ gives us therefore a commutative diagram

$$\begin{tikzcd}
H^1(\bar{P}_k)_{m,a} \arrow[r] \arrow[dr] & H^1_{\varrho}(P_k)_{m,a} \arrow[d] \\
& H^1(P_k)_{m,a}
\end{tikzcd}$$

for $m \geqslant k$, $a \in X$. For $m \geqslant k_1$, $a \in X$, the mappings ϱ of diagram (9.4) induce, according to Proposition 6.4, (iv), a mapping of cohomology

$$\varrho: H^1_{\varrho}(P_k)_{m,a} \to H^1(P''_{k_1})_{m,\varrho(a)}$$

sending the cohomology class of $u \in Z^1_{\varrho}(R_m)_a$ in $H^1_{\varrho}(P_k)_{m,a}$ into the cohomology class of $\varrho u \in Z^1(R''_m)_{\varrho(a)}$ in $H^1(P''_{k_1})_{m,\varrho(a)}$. If $m \geqslant \sup(k_0, k_1)$, combining this map with the isomorphism (9.2) of Theorem 9.1, we obtain a mapping of cohomology

$$\varrho: H^1(P_k)_{m,a} \to H^1(P''_{k_1})_{m,\varrho(a)}$$

for $a \in X$. One verifies easily that for $m \geqslant \sup(k_0, k_1)$, $l \geqslant 1$ and $a \in X$, the diagram of cohomology

$$\begin{tikzcd}
H^1(\bar{P}_k)_{m+l,a} \arrow[r] \arrow[d, "\pi_m"] & H^1(P_k)_{m+l,a} \arrow[r, "\varrho"] \arrow[d, "\pi_m"] & H^1(P''_{k_1})_{m+l,\varrho(a)} \arrow[d, "\pi_m"] \\
H^1(\bar{P}_k)_{m,a} \arrow[r] & H^1(P_k)_{m,a} \arrow[r, "\varrho"] & H^1(P''_{k_1})_{m,\varrho(a)}
\end{tikzcd}$$

is commutative; therefore we obtain a mapping of cohomology

$$\varrho: H^1(P_k)_a \to H^1(P''_{k_1})_{\varrho(a)}$$

for all $a \in X$.

If A, B, C are sets with distinguished elements 0, we say that the sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is exact (or exact at B) if $\beta^{-1}(0) = \alpha(A)$ (and, of course, $\alpha(0)=0$, $\beta(0)=0$).

PROPOSITION 9.1. *Assume that $R_k \subset J_k(T;\varrho)$ is a formally integrable Lie equation satisfying the conditions* (I) *and* (II) *and $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ possesses a finite form which is formally integrable and integrable. Then for $m \geq \sup(k_0, k_1)$, $a \in X$, the cohomology sequence*

$$H^0(\bar{P}_k)_{m+1,a} \longrightarrow H^0(P_k)_{m+1,a} \xrightarrow{\varrho} H^0(P''_{k_1})_{m+1,a} \xrightarrow{\partial^{\#}} H^1(\bar{P}_k)_{m,a} \longrightarrow H^1(P_k)_{m,a} \xrightarrow{\varrho} H^1(P''_{k_1})_{m,\varrho(a)} \tag{9.5}$$

is exact. Moreover, if f''_1, $f''_2 \in H^0(P''_{k_1})_{m+1,a}$ have the same image in $H^1(\bar{P}_k)_{m,a}$, i.e., $\partial^{\#} f''_1 = \partial^{\#} f''_2$, then $f''_1 = f'' \circ f''_2$ where, for some $f \in H^0(P_k)_{m+1,a}$, $f'' = \varrho f$; if α_1, $\alpha_2 \in H^1(\bar{P}_k)_{m,a}$ have the same image in $H^1(P_k)_{m,a}$ then, for some $f'' \in H^0(P''_{k_1})_{m+1,a}$, we have $\alpha_1^{f''} = \alpha_2$.

Proof. The sequence is clearly exact at $H^0(P_k)_{m+1,a}$. If $f \in H^0(P_k)_{m+1,a}$, $f'' = \varrho f$, then $0^{f''} = [\mathcal{D} j_{m+1}(f)] = 0$, and so $\partial^{\#} \cdot \varrho = 0$. Let f''_1, $f''_2 \in H^0(P''_{k_1})_{m+1,a}$, and suppose that $0^{f''_1} = 0^{f''_2}$. Then, if F_1, $F_2 \in \tilde{\mathfrak{D}}^{\cdot}_{m+1,\varrho,a}$ with $\varrho F_1 = j_{m+1}(f''_1)$, $\varrho F_2 = j_{m+1}(f''_2)$, there exists $G \in \tilde{\mathfrak{D}}^{\cdot}_{m+1,a}$ such that $\mathcal{D}F_2 = (\mathcal{D}F_1)^G = \mathcal{D}(F_1 \cdot G)$. Hence $F_1 \cdot G = j_{m+1}(f) \cdot F_2$ for some $f \in H^0(P_k)_{m+1,a}$; taking the projections of both sides of this equation by ϱ, we obtain

$$j_{m+1}(f''_1) = j_{m+1}(f'') \cdot j_{m+1}(f''_2) = j_{m+1}(f'' \circ f''_2),$$

where $\varrho f = f''$ and hence $f''_1 = f'' \circ f''_2$. In particular, if f''_2 is the identity $Y \to Y$, in which case $0^{f''_2} = 0$, we obtain $f''_1 = f'' = \varrho f$. This proves exactness at $H^0(P''_{k_1})_{m+1,a}$.

Next, if $f'' \in H^0(P''_{k_1})_{m+1,a}$ and $F \in \tilde{\mathfrak{D}}^{\cdot}_{m+1,\varrho,a}$ with $\varrho F = j_{m+1}(f'')$, then the image of $0^{f''}$ in $H^1(P_k)_{m,a}$ is the cohomology class of $\mathcal{D}F = 0^F$, and so therefore vanishes. Let u_1, $u_2 \in Z^1(\bar{R}_m)_a$ and suppose that the cohomology classes of u_1 and u_2 in $H^1(P_k)_{m,a}$ are equal, i.e., that there exists $F \in \tilde{\mathfrak{D}}^{\cdot}_{m+1,a}$ such that

$$u_1^F = F^{-1}(u_1) + \mathcal{D}F = u_2. \tag{9.6}$$

By Proposition 6.4, (ii), we see that $F \in \tilde{\mathfrak{D}}^{\cdot}_{m+1,\varrho,a}$. By (7.6) and the commutativity of (6.25), P_{m+1} preserves $\bar{R}_m$ and so $F^{-1}(u_1) \in \mathcal{T}^* \otimes \bar{\mathcal{R}}_m$. Hence (9.6) implies, by Proposition 6.4, (iv), that $0 = \varrho(\mathcal{D}F) = \mathcal{D}F''$, where $F'' = \varrho F \in \tilde{\mathfrak{D}}^{\cdot\prime\prime}_{m+1,\varrho(a)}$. Therefore $F'' = j_{m+1}(f'')$ for some $f'' \in H^0(P''_{k_1})_{m+1,a}$, and we have, by (9.6), $[u_1]^{f''} = [u_2]$ in $H^1(\bar{P}_k)_{m,a}$. If $u_1 = 0$, then $[u_2] = 0^{f''}$. Thus the sequence (9.5) is exact at $H^1(\bar{P}_k)_{m,a}$.

Finally we prove exactness at $H^1(P_k)_{m,a}$. Let $\alpha \in H^1(P_k)_{m,a}$ with $\varrho\alpha = 0$. By Proposition 7.5 and Theorem 9.1, there exists $u \in Z^1_\varrho(R_{m+1})_a$ such that $u(a)=0$ and $\pi_m[u]=\alpha$, if $[u]$ is the cohomology class of u in $H^1(P_k)_{m+1,a}$. Then $\varrho[u]$ is equal to the cohomology class of $\varrho u \in Z^1(R''_{m+1})_{\varrho(a)}$. Since $\pi_m \varrho[u] = \varrho\pi_m[u] = \varrho\alpha = 0$, our hypothesis concerning R''_{k_1} and Proposition 7.6 imply that $\varrho[u]=0$. Therefore if $b=\varrho(a)$, there exists $F'' \in \tilde{\mathfrak{D}}''_{m+2,b}$ satisfying $(\varrho u)^{F''}=0$. Since $(\varrho u)(b)=0$, we have $(\mathfrak{D}F'')(b) = (\varrho u)^{F''}(b) = 0$; hence by (2.27), we have $j_1(\pi_{m+1}F'')(b) = j_1(I_{Y,m+1})(b)$. By Lemma 9.2, (iii) (with $G = j_1(I_{m+1})(a)$), there exists $F \in \tilde{\mathfrak{D}}_{m+1,\varrho,a}$ satisfying $j_1(F)(a) = j_1(I_{m+1})(a)$ and $\varrho F = \pi_{m+1}F''$. By Proposition 6.4, (iv), we have $\varrho((\pi_m u)^F) = (\varrho\pi_m u)^{\pi_{m+1}F''} = 0$, and $(\pi_m u)^F \in Z^1(\bar{R}_m)_a$. Thus $\alpha = [\pi_m u] = [(\pi_m u)^F]$ belongs to the image of $H^1(\bar{P}_k)_{m,a}$.

Up to this point we have used only the hypothesis that the formally integrable Lie equation $R_k \subset J_k(T;\varrho)$ satisfies conditions (I) and (II); now, however, we require condition (III) since we shall construct from the $\bar{R}_{k+l}$, in the manner of the papers [5] and [6], a formally integrable Lie equation $R'_{m_0} \subset J_{m_0}(V)$ whose non-linear cohomology will replace that of $\bar{R}_k$ in a sequence which is a modification of (9.5).

Let us then assume that the formally integrable Lie equation R_k satisfies condition (III) as well as (I) and (II). For $l \geqslant 0$ and $m \geqslant k$, let $\bar{R}^{(l)}_m$ be the sub-bundle $\pi_m \bar{R}_{m+l}$ of $J_m(V)$. According to Theorem 1 of [6] (see also [5] and [10]), there exist integers $m_0 \geqslant \sup(k_0, k_1)$, $l_0 \geqslant 0$ such that $R'_{m_0} = \bar{R}^{(l_0)}_{m_0}$ is a formally integrable Lie equation in $J_{m_0}(V)$, whose r-th prolongation is equal to

$$R'_{m_0+r} = \bar{R}^{(l_0)}_{m_0+r} = \bar{R}^{(l)}_{m_0+r}$$

for all $l \geqslant l_0$, and g'_{m_0} is 2-acyclic.

For $m \geqslant m_0$, let P'_m be a finite form of R'_m. For $m \geqslant m_0$, $a \in X$, the inclusions $R'_{m_0} \subset \bar{R}_{m_0}$, $R'_{m_0} \subset R_{m_0}$ give us a commutative diagram of cohomology

$$\begin{tikzcd}
H^1(P'_{m_0})_{m,a} \arrow[rr] \arrow[dr] & & H^1(\bar{P}_k)_{m,a} \arrow[dl] \\
& H^1(P_k)_{m,a} &
\end{tikzcd} \tag{9.7}$$

For all $m \geqslant m_0$ and $l \geqslant l_0$, we have projections $\pi_m\colon \bar{R}_{m+l} \to R'_m$ which induce, by Proposition 7.1, (iii), surjective mappings $\pi_m\colon \tilde{\tilde{\mathfrak{D}}}_{m+l,a} \to \tilde{\mathfrak{D}}'_{m,a}$ and therefore mappings of cohomology

$$\pi_m\colon H^1(\bar{P}_k)_{m+l,a} \to H^1(P'_{m_0})_{m,a}, \tag{9.8}$$

for $a \in X$, such that the diagram of cohomology

$$\begin{tikzcd}[column sep=large, row sep=huge]
H^1(P'_{m_0})_{m+l,a} \arrow[r] \arrow[d, "\pi_m"] & H^1(\bar{P}_k)_{m+l,a} \arrow[r] \arrow[d, "\pi_m"] \arrow[dl, "\pi_m"'] & H^1(P_k)_{m+l,a} \arrow[d, "\pi_m"] \\
H^1(P'_{m_0})_{m,a} \arrow[r] & H^1(\bar{P}_k)_{m,a} \arrow[r] & H^1(P_k)_{m,a}
\end{tikzcd} \tag{9.9}$$

commutes. Since the mapping $\pi_m: H^1(P'_{m_0})_{m+l,a} \to H^1(P'_{m_0})_{m,a}$ is surjective by Proposition 7.5, it follows that the mapping (9.8) is also surjective. Moreover, the mappings $H^1(P'_{m_0})_{m,a} \to H^1(\bar{P}_k)_{m,a}$ induce an isomorphism of cohomology $H^1(P'_{m_0})_a \to H^1(\bar{P}_k)_a$, for $a \in X$.

For $m \geq m_0$, $l \geq l_0$, we now define the operation of the group $H^0(P''_{k_1})_{m+l+1,a}$ on $H^1(P'_{m_0})_{m,a}$ in such a way that

$$(\pi_m \alpha_1)^{f''} = \pi_m(\alpha_1^{f''}) \tag{9.10}$$

for $\alpha_1 \in H^1(\bar{P}_k)_{m+l,a}$, $f'' \in H^0(P''_{k_1})_{m+l+1,a}$, where π_m is the mapping (9.8). Let $\alpha \in H^1(P'_{m_0})_{m,a}$ and $f'' \in H^0(P''_{k_1})_{m+l+1,a}$; if $\alpha_1 \in H^1(\bar{P}_k)_{m+l,a}$ satisfies $\pi_m \alpha_1 = \alpha$, we define $\alpha^{f''}$ to be the image $\pi_m(\alpha_1^{f''})$ of $\alpha_1^{f''}$ under the mapping (9.8). We now verify that $\alpha^{f''}$ does not depend on the choice of α_1. Let $\alpha_2 \in H^1(\bar{P}_k)_{m+l,a}$ satisfy $\pi_m \alpha_2 = \alpha$ and let u_1, $u_2 \in Z^1(\bar{R}_{m+l})_a$ be elements whose cohomology classes in $H^1(\bar{P}_k)_{m+l,a}$ are equal to α_1, α_2 respectively. Then $\pi_m u_1$ and $\pi_m u_2$ are cohomologous in $H^1(P'_{m_0})_{m,a}$ and so there exists $G \in \tilde{\mathfrak{D}}'^{\cdot}_{m+1,a}$ such that $(\pi_m u_1)^G = \pi_m u_2$. Then $G = \pi_{m+1} G_1$ with $G_1 \in \tilde{\tilde{\mathfrak{D}}}^{\cdot}_{m+l+1,a}$. Let $F_1 \in \tilde{\mathfrak{D}}^{\cdot}_{m+l+1,\varrho,a}$ with $\varrho F_1 = j_{m+l+1}(f'')$; then $G'_1 = F_1^{-1} \cdot G_1 \cdot F_1$ belongs to $\tilde{\tilde{\mathfrak{D}}}^{\cdot}_{m+l+1,a}$ and $G' = \pi_{m+1} G'_1$ belongs to $\tilde{\mathfrak{D}}'^{\cdot}_{m+1,a}$. Since $G \cdot F = F \cdot G'$ where $F = \pi_{m+1} F_1$, we have

$$\pi_m(u_2^{F_1}) = (\pi_m u_2)^F = (\pi_m u_1)^{G \cdot F} = (\pi_m u_1)^{F \cdot G'} = (\pi_m(u_1^{F_1}))^{G'},$$

and so $\pi_m(\alpha_1^{f''}) = \pi_m(\alpha_2^{f''})$. Finally, (9.3) holds for $\alpha \in H^1(P'_{m_0})_{m,a}$ and f'', $f''_1 \in H^0(P''_{k_1})_{m+l+1,a}$. We define

$$\partial^{\#}: H^0(P''_{k_1})_{m+l+1,a} \to H^1(P'_{m_0})_{m,a}$$

to be the mapping sending $f'' \in H^0(P''_{k_1})_{m+l+1,a}$ into $\partial^{\#} f'' = 0^{f''}$.

For $m \geq m_0$, $l \geq l_0$, $a \in X$, consider the cohomology sequence

$$H^0(P''_{k_1})_{m+l+1,a} \xrightarrow{\partial^{\#}} H^1(P'_{m_0})_{m,a} \longrightarrow H^1(P_k)_{m,a} \xrightarrow{\varrho} H^1(P''_{k_1})_{m,\varrho(a)}; \tag{9.11}$$

since (9.5) is a complex, diagrams (9.7) and (9.9) are commutative and (9.10) holds, it follows that (9.11) is a complex.

THEOREM 9.2. *Assume that R_k is a formally integrable Lie equation satisfying the conditions* (I), (II) *and* (III). *Then:*

(i) *If R_k possesses a finite form which is formally integrable and integrable, the sequence* (9.11) *is exact at $H^1(P'_{m_0})_{m,a}$ for all $m \geq m_0$, $l \geq l_0$, $a \in X$.*

(ii) *If R''_{k_1} possesses a finite form which is formally integrable and integrable, the sequence* (9.11) *is exact at $H^1(P_k)_{m,a}$ for all $m \geq m_0$, $a \in X$.*

Proof. (i) Let $\alpha \in H^1(P'_{m_0})_{m,a}$ and assume that the image of α in $H^1(P_k)_{m,a}$ vanishes. Choose $\alpha_1 \in H^1(\bar{P}_k)_{m+l,a}$ such that $\pi_m \alpha_1 = \alpha$. Then by the commutativity of (9.9), our hypothesis concerning R_k and Proposition 7.6, the image of α_1 in $H^1(P_k)_{m+l,a}$ vanishes. Therefore by Proposition 9.1, there exists $f'' \in H^0(P''_{k_1})_{m+l+1,a}$ such that $\alpha_1 = 0^{f''}$. By (9.10), we have $\alpha = \pi_m(0^{f''}) = \partial^\# f''$, proving the exactness of the complex (9.11) at $H^1(P'_{m_0})_{m,a}$.

(ii) Let $\alpha \in H^1(P_k)_{m,a}$ with $\varrho\alpha = 0$. By Proposition 7.5 there exists $\alpha_1 \in H^1(P_k)_{m+l,a}$ such that $\pi_m \alpha_1 = \alpha$. Since $\pi_m \varrho \alpha_1 = \varrho\alpha = 0$, we have $\varrho\alpha_1 = 0$ by our hypothesis concerning R''_{k_1} and Proposition 7.6. By Proposition 9.1, there exists $\beta_1 \in H^1(\bar{P}_k)_{m+l,a}$ whose image in $H^1(P_k)_{m+l,a}$ is equal to α. Then the image of $\pi_m \beta_1 \in H^1(P'_{m_0})_{m,a}$ in $H^1(P_k)_{m,a}$ is equal to α. Thus the complex (9.11) is exact at $H^1(P_k)_{m,a}$.

Let $m_1 \geq m_0$ be an integer such that g''_{m_1} is 2-acyclic.

THEOREM 9.3. *Assume that R_k is a formally integrable Lie equation satisfying the conditions* (I), (II) *and* (III) *and suppose that $R'_{m_0} = 0$. Then:*

(i) *The mapping*

$$\varrho: H^1(P_k)_{m,a} \to H^1(P''_{k_1})_{m,\varrho(a)}$$

is surjective for all $m \geq m_1$, $a \in X$.

(ii) *If $\alpha_1, \alpha_2 \in H^1(P_k)_{m+l+1,a}$, where $m \geq m_0$, $l \geq l_0$, $a \in X$, have the same image in $H^1(P''_k)_{m+l+1,\varrho(a)}$, then $\pi_m \alpha_1 = \pi_m \alpha_2$ as elements of $H^1(P_k)_{m,a}$.*

(iii) *The mapping*

$$\varrho: H^1(P_k)_a \to H^1(P''_{k_1})_{\varrho(a)}$$

is an isomorphism of cohomology for all $a \in X$.

Proof. (i) Let $\alpha \in H^1(P''_{k_1})_{m,b}$ where $b = \varrho(a)$; by Proposition 7.5 and Lemma 7.1, there exists $u \in Z^1(R''_{m+l_0+1})_b$ with $u(b) = 0$ and $\pi_m[u] = \alpha$. Choose $v \in (\mathcal{J}^* \otimes \mathcal{R}_{m+l_0+1})_{\varrho,a}$ with $v(a) = 0$ and $\varrho v = u$. Then $v \in (\mathcal{J}^* \otimes \mathcal{R}_{m+l_0+1})^\wedge_{\varrho,a}$ and $\varrho \mathcal{D}_1 v = \mathcal{D}_1 u = 0$ by Proposition 6.4, (iii). It follows that $\mathcal{D}_1 v \in \wedge^2 \mathcal{J}^* \otimes \bar{\mathcal{R}}_{m+l_0}$ and hence $\pi_m \mathcal{D}_1 v \in \wedge^2 \mathcal{J}^* \otimes \mathcal{R}'_m$. Therefore, writing $v' = \pi_m v$, we have $\mathcal{D}_1 v' = \pi_{m-1} \mathcal{D}_1 v = 0$ since $R'_m = 0$, and $v' \in Z^1(R_k)_{m,a}$ satisfies $\varrho[v'] = [\pi_m u] = \alpha$.

(ii) By Theorem 9.1, choose representatives $u_1, u_2 \in Z^1_\varrho(R_k)_{m+l+1,a}$ of α_1, α_2 respectively with $u_1(a) = u_2(a) = 0$. Our hypothesis implies that there exists $F'' \in \tilde{\mathcal{D}}''_{m+l+2,b}$, where $b = \varrho(a)$, such that $(\varrho u_1)^{F''} = \varrho u_2$. Since $(\varrho u_1)(b) = (\varrho u_2)(b) = 0$, we have $(\mathcal{D}F'')(b) = 0$, which implies by (2.27) that $j_1(\pi_{m+l+1}F'')(b)=j_1(I_{Y,m+l+1})(b)$. Hence by Lemma 9.2, (iii), there exists $F\in\tilde{\mathcal{D}}_{m+l+1,\varrho,a}$ satisfying $j_1(F)(a)=j_1(I_{m+l+1})(a)$ and $\varrho F=\pi_{m+l+1}F''$. By Proposition 6.4, (iv), we have $\varrho((\pi_{m+l}u_1)^F)=\varrho(\pi_{m+l}u_2)$ and it follows that $(\pi_{m+l}u_1)^F-\pi_{m+l}u_2$ belongs to $\mathcal{T}^*\otimes\bar{R}_{m+l}$. Since $R'_m=0$, we obtain the equality $(\pi_m u_1)^{\pi_{m+1}F}=\pi_m u_2$, i.e., $\pi_m u_1$ and $\pi_m u_2$ represent the same class in $H^1(P_k)_{m,a}$.

(iii) The injectivity of ϱ: $H^1(P_k)_a\to H^1(P''_{k_1})_{\varrho(a)}$ follows immediately from (ii). To prove that ϱ is surjective, it suffices by the Mittag-Leffler theorem (see [1], § 3, No. 5, Corollary 2) to show that if $(\alpha''_m)\in H^1(P''_{k_1})_{\varrho(a)}$, with $\alpha''_m\in H^1(P''_{k_1})_{m,\varrho(a)}$, $m\geqslant k_1$, then, for all $m\geqslant m_1$ and all $r\geqslant m+l_0+1$ and all $\alpha\in H^1(P_k)_{m+l_0+1,a}$ such that $\varrho(\alpha)=\alpha''_{m+l_0+1}$, there exists $\alpha'\in H^1(P_k)_r$ such that $\pi_m\alpha'=\pi_m\alpha$, $\varrho\alpha'=\alpha''_r$. To verify that this condition is satisfied, by (i) choose $\alpha'\in H^1(P_k)_{r,a}$ with $\varrho(\alpha')=\alpha''_r$. Then $\pi_{m+l_0+1}\alpha'$ and α have the same image α''_{m+l_0+1} in $H^1(P''_{k_1})_{m+l_0+1,\varrho(a)}$. Hence by (ii), $\pi_m\alpha'=\pi_m\alpha$.

10. Non-linear cohomology of transitive Lie algebras

Consider the real line $\mathbf{R}$ endowed with the discrete topology and linearly compact topological vector spaces over $\mathbf{R}$, i.e., those which are topological duals of real vector spaces endowed with the discrete topology. A transitive Lie algebra L is a topological Lie algebra over $\mathbf{R}$ whose underlying topological vector space is linearly compact and which possesses a neighborhood of 0 containing no ideals other than 0. A monomorphism (resp. epimorphism) of transitive Lie algebras is a continuous monomorphism (resp. epimorphism) of Lie algebras and an isomorphism of transitive Lie algebras is an isomorphism of Lie algebras which is also an isomorphism of the underlying topological vector spaces.

A transitive Lie algebra L possesses an open subalgebra L^0 containing no ideals of L other than 0, which we call fundamental. We define subalgebras $D^k_L L^0$ of L by induction on k by setting:

$$D^0_L L^0=L^0,\quad D^k_L L^0=\{\xi\in D^{k-1}_L L^0\,|\,[L,\xi]\subset D^{k-1}_L L^0\},\quad\text{for}\quad k\geqslant 1;$$

then $D^k_L L^0$ is a fundamental subalgebra of L and $\{D^k_L L^0\}_{k\geqslant 0}$ is a fundamental system of neighborhoods of 0 and $\bigcap_{k=0}^{\infty} D^k_L L^0=0$.

If $a\in X$, let $J^k(T)_a$ denote the subalgebra of $J_\infty(T)_a$ which is the kernel of the projection π_k: $J_\infty(T)_a\to J_k(T)_a$. Then $J_\infty(T)_a$ is a transitive Lie algebra whose subalgebras $J^k(T)_a$ are fundamental and $D^k_{J_\infty(T)_a}J^0(T)_a=J^k(T)_a$. If $\phi\in Q_\infty(a,a)$, then ϕ: $J_\infty(T)_a\to J_\infty(T)_a$ is an isomorphism of transitive Lie algebras such that $\phi(J^0_\infty(T)_a)=J^0_\infty(T)_a$. A closed subalgebra L of $J_\infty(T)_a$ such that $\pi_0 L=J_0(T)_a$ is a transitive Lie algebra whose subalgebras $L^k=L\cap J^k(T)_a$ are fundamental, and is said to be a transitive subalgebra of $J_\infty(T)_a$; in fact $L^k=D^k_L L^0$. By Theorem III of [13], if L is a transitive Lie algebra and $L^0\subset L$ is a fundamental subalgebra and if the dimension of L/L^0 is equal to the dimension of X, then, for $a \in X$, there exists a monomorphism of transitive Lie algebras $i: L \to J_\infty(T)_a$ such that $i(L) \cap J^0_\infty(T)_a = i(L^0)$ and $i(L)$ is a transitive subalgebra of $J_\infty(T)_a$; then i induces an isomorphism $L/L^0 \to J_0(T)_a$. Thus every transitive Lie algebra is isomorphic to a transitive subalgebra of $J_\infty(T)_a$ for some manifold X and $a \in X$.

Let $R_k \subset J_k(T)$ be a formally transitive and formally integrable Lie equation; for $a \in X$, the subalgebra $R_{\infty,a}$ of $J_\infty(T)_a$ is a transitive subalgebra of $J_\infty(T)_a$. If $N_{k_1} \subset R_{k_1}$ is a formally integrable Lie equation, with $k_1 \geqslant k$, such that $[\tilde{\mathcal{R}}_{k_1+1}, \mathcal{N}_{k_1}] \subset \mathcal{N}_{k_1}$, then, for $a \in X$, by Lemma 10.3, (iii) of [10], $N_{\infty,a}$ is a closed ideal of $R_{\infty,a}$. We shall always consider such Lie algebras $R_{\infty,a}$ and $N_{\infty,a}$ endowed with the topologies induced by $J_\infty(T)_a$.

Definition 10.1. We say that a formally integrable and ϱ-projectable Lie equation $R_k \subset J_k(T;\varrho)$ is a *prolongation* of the formally integrable Lie equation $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ if $\varrho(R_{m,a}) = R''_{m,\varrho(a)}$ for all $a \in X$ and $m \geqslant \sup(k, k_1)$ and if $\varrho: R_{\infty,a} \to R''_{\infty,\varrho(a)}$ is an isomorphism for all $a \in X$.

If a formally integrable and ϱ-projectable Lie equation $R_k \subset J_k(T;\varrho)$ is a prolongation of a formally integrable Lie equation $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ and satisfies conditions (II) and (III) of § 9, then the equation R'_{m_0} constructed from the equations $\bar{R}_{k+l} = R_{k+l} \cap J_{k+l}(V)$ vanishes and the hypotheses of Theorem 9.3 hold for R_k; hence for all $a \in X$, we have an isomorphism of cohomology $\varrho: \tilde{H}^1(R_k)_a \to \tilde{H}^1(R''_{k_1})_{\varrho(a)}$.

Taking $Y = X$ and ϱ to be the identity map of X, we see that the l-th prolongation R_{k+l} of a formally integrable Lie equation $R_k \subset J_k(T)$ is a prolongation in the above sense.

THEOREM 10.1. *Let L, L'' be transitive Lie algebras and $\phi: L \to L''$ an epimorphism of transitive Lie algebras. Let $I \subset L$, $I'' \subset L''$ be closed ideals of L and L'' such that $\phi(I) = I''$. Let I' be the closed ideal of L which is the kernel of $\phi: I \to I''$. There exist connected analytic manifolds X, Y, points $x \in X$, $y \in Y$, an analytic submersion $\varrho: X \to Y$ with $\varrho(x) = y$, formally integrable and formally transitive analytic Lie equations $R_k \subset J_k(T;\varrho)$, $R''_{k_1} \subset J_{k_1}(T_Y; Y)$, with $k_1 \geqslant k$, formally integrable analytic Lie equations $N_k \subset R_k$, $N'_k \subset R_k$, $N''_{k_1} \subset R''_{k_1}$ and isomorphisms of transitive Lie algebras $\psi: L \to R_{\infty,x}$, $\psi'': L'' \to R''_{\infty,y}$ such that*

$$[\tilde{\mathcal{R}}_{k+1}, \mathcal{N}_k] \subset \mathcal{N}_k, \quad [\tilde{\mathcal{R}}_{k+1}, \mathcal{N}'_k] \subset \mathcal{N}'_k, \quad [\tilde{\mathcal{R}}''_{k_1+1}, \mathcal{N}''_{k_1}] \subset \mathcal{N}''_{k_1}, \tag{10.1}$$

and R_k, N_k are ϱ-projectable and

$$\varrho(R_{k_1+l,a}) = R''_{k_1+l,\varrho(a)}, \tag{10.2}$$

$$\varrho(N_{k_1+l,a}) = N''_{k_1+l,\varrho(a)},$$

for all $l \geqslant 0$, $a \in X$, and the diagram

$$\begin{array}{ccc} L & \xrightarrow{\psi} & R_{\infty,x} \\ \downarrow \phi & & \downarrow \varrho \\ L'' & \xrightarrow{\psi''} & R''_{\infty,y} \end{array} \tag{10.3}$$

is commutative and

$$\psi(I) = N_{\infty,x}, \quad \psi(I') = N'_{\infty,x}, \quad \psi''(I'') = N''_{\infty,y}. \tag{10.4}$$

Furthermore, if V is the bundle of vectors tangent to the fibers of $\varrho: X \to Y$, there exists an integer $l_0 \geq 0$ such that

$$N'_m = \pi_m(N_{m+l} \cap J_{m+l}(V)) \tag{10.5}$$

for all $m \geq k$, $l \geq l_0$. If $\phi: I \to I''$ is an isomorphism, then $N'_k = 0$ and N_k is a prolongation of N''_{k_1}.

Proof. Let L''^0 be a fundamental subalgebra of L''. By Corollary 6.1 of [9], there exist a formally transitive and formally integrable analytic Lie equation $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ on an analytic simply connected manifold Y, a point $y \in Y$ and an isomorphism of transitive Lie algebras $\psi''_1: L'' \to R''_{\infty,y}$ such that $\psi''_1(L''^0) = R''^0_{\infty,y}$. Let L^0 be a fundamental subalgebra of L such that $\phi(L^0) \subset L''^0$. By Theorem 12.2 of [10], there exist an analytic simply connected manifold X, an analytic submersion $\varrho: X \to Y$, a point $x \in X$ with $\varrho(x) = y$, a formally transitive and formally integrable analytic Lie equation $R_k \subset J_k(T; \varrho)$ and isomorphisms of transitive Lie algebras $\psi: L \to R_{\infty,x}$, $\psi'': L'' \to R''_{\infty,y}$ such that $\psi(L^0) = R^0_{\infty,x}$ and $\psi''(L''^0) = R''^0_{\infty,y}$ and such that diagram (10.3) commutes and (10.2) holds. Replacing R_k by one of its prolongations R_{k+l} and R''_{k_1} by one of its prolongations R''_{k_1+m} if necessary, we may assume that $k_1 \geq k$ and according to Theorem 10.1 of [10], there exist formally integrable analytic Lie equations $N_k \subset R_k$, $N'_k \subset R_k$, $N''_{k_1} \subset R''_{k_1}$ such that (10.1) and (10.4) hold. From Theorem 11.2 of [10], we deduce the remaining properties of N_k, N'_k and N''_{k_1}.

Let Z be a differentiable manifold whose tangent bundle we denote by T_Z. Let $R''_p \subset J_p(T_Y; Y)$, $R''^{\#}_q \subset J_q(T_Z; Z)$ be two formally transitive and formally integrable Lie equations. Let $N''_{p_1} \subset R''_{p_1}$, $N''^{\#}_{q_1} \subset R''^{\#}_{q_1}$ be two formally integrable Lie equations, with $p_1 \geq p$, $q_1 \geq q$, such that

$$[\tilde{R}''_{p_1+1}, \mathcal{N}''_{p_1}] \subset \mathcal{N}''_{p_1}, \quad [\tilde{R}''^{\#}_{q_1+1}, \mathcal{N}''^{\#}_{q_1}] \subset \mathcal{N}''^{\#}_{q_1}.$$

Let $y \in Y$, $z \in Z$. Assume that Y and Z are endowed with structures of analytic manifolds compatible with their structures of differentiable manifolds.

THEOREM 10.2. *Suppose that R''_p and $R''^{\#}_q$ are analytic Lie equations. If the pairs of topological Lie algebras $(R''_{\infty,y}, N''_{\infty,y})$ and $(R''^{\#}_{\infty,z}, N''^{\#}_{\infty,z})$ are isomorphic, we have a commutative diagram of cohomology*

$$\begin{CD}
\tilde{H}^1(N''_{p_1})_y @>>> \tilde{H}^1(N''^{\#}_{q_1})_z \\
@VVV @VVV \\
\tilde{H}^1(R''_p)_y @>>> \tilde{H}^1(R''^{\#}_q)_z
\end{CD} \tag{10.6}$$

whose horizontal arrows are isomorphisms of cohomology.

Proof. By Theorem 12.4, (i) of [10], our hypotheses imply the existence of a differentiable manifold X, submersions $\varrho: X \to Y$, $\varrho^{\#}: X \to Z$, a point $x \in X$ satisfying $\varrho(x) = y$, $\varrho^{\#}(x) = z$, a formally transitive and formally integrable Lie equation $R_k \subset J_k(T; \varrho) \cap J_k(T; \varrho^{\#})$ and a formally integrable Lie equation $N_k \subset R_k$ such that

$$[\tilde{\mathcal{R}}_{k+1}, \mathcal{N}_k] \subset \mathcal{N}_k$$

and such that R_k is a prolongation of R''_p and of $R''^{\#}_q$ and N_k a prolongation of N''_{p_1} and $N''^{\#}_{q_1}$. Replacing X, if necessary, by a neighborhood of x, according to the remarks at the beginning of § 9 we may suppose that R_k and N_k satisfy conditions (I), (II) and (III) of § 9 with respect to both submersions ϱ and $\varrho^{\#}$. The equations R_k and N_k therefore satisfy the hypotheses of Theorem 9.3 with respect to both submersions ϱ and $\varrho^{\#}$. So Theorem 9.3 yields a commutative diagram

$$\begin{CD}
\tilde{H}^1(N''_{p_1})_y @<{\varrho}<< \tilde{H}^1(N_k)_x @>{\varrho^{\#}}>> \tilde{H}^1(N''^{\#}_{q_1})_z \\
@VVV @VVV @VVV \\
\tilde{H}^1(R''_p)_y @<{\varrho}<< \tilde{H}^1(R_k)_x @>{\varrho^{\#}}>> \tilde{H}^1(R''^{\#}_q)_z
\end{CD},$$

whose vertical arrows are induced by inclusions of Lie equations and whose horizontal arrows are isomorphisms of cohomology, from which we deduce diagram (10.6).

The following result is a consequence of Theorem 7.1, Corollary 7.1 and Theorem 10.2:

THEOREM 10.3. *If the transitive Lie algebras $R''_{\infty, y}$ and $R''^{\#}_{\infty, z}$ are isomorphic as topological Lie algebras, we have a bijective mapping*

$$\tilde{H}^1(R''_p)_y \to \tilde{H}^1(R''^{\#}_q)_z. \tag{10.7}$$

If the pairs of topological Lie algebras $(R''_{\infty, y}, N''_{\infty, y})$ and $(R''^{\#}_{\infty, z}, N''^{\#}_{\infty, z})$ are isomorphic, and if the mapping (10.7) is an isomorphism of cohomology (or a fortiori if $\tilde{H}^1(R''_p)_y = 0$), then we have an isomorphism of cohomology

$$\tilde{H}^1(N''_{p_1})_y \to \tilde{H}^1(N''^{\#}_{q_1})_z.$$

According to [10], the linear Spencer cohomology $H^*(R_k)_x = \oplus_{j \geq 0} H^j(R_k)_x$ of a formally integrable Lie equation $R_k \subset J_k(T)$ at $x \in X$ is a graded Lie algebra whose bracket on $H^0(R_k)_x$ is the Lie bracket of germs of vector fields.

Henceforth, we shall identify two graded Lie algebras of linear cohomology which are isomorphic, and two non-linear cohomologies if there is an isomorphism of cohomology between them.

Let L be a transitive Lie algebra and I be a closed ideal of L. According to Corollary 6.1 of [9] and Theorem 10.1 of [10] (see also Theorem 10.1), there exist a formally transitive and formally integrable analytic Lie equation $R_k \subset J_k(T)$ on an analytic manifold X, a point $x \in X$, a formally integrable Lie equation $N_{k_1} \subset R_{k_1}$, with $k_1 \geq k$, such that

$$[\tilde{\mathcal{R}}_{k_1+1}, \mathcal{N}_{k_1}] \subset \mathcal{N}_{k_1}$$

and $(R_{\infty,x}, N_{\infty,x})$ and (L, I) are isomorphic as pairs of topological Lie algebras. We set

$$H^*(L) = H^*(R_k)_x, \quad H^*(L, I) = H^*(N_{k_1})_x,$$

$$\tilde{H}^1(L) = \tilde{H}^1(R_k)_x, \quad \tilde{H}^1(L, I) = \tilde{H}^1(N_{k_1})_x,$$

and call $H^*(L)$ and $\tilde{H}^1(L)$ respectively the linear and non-linear Spencer cohomology of L, and $H^*(L, I)$ and $\tilde{H}^1(L, I)$ respectively the linear and non-linear Spencer cohomology of the closed ideal I of L. We have $H^*(L, L) = H^*(L)$ and $\tilde{H}^1(L, L) = \tilde{H}^1(L)$. These linear cohomologies are graded Lie algebras and these non-linear cohomologies are sets with distinguished elements 0. The linear cohomology was introduced in [10] and was shown to be well-defined; we now extend certain properties of the linear cohomology to the non-linear cohomology.

THEOREM 10.4. (i) *The non-linear Spencer cohomology $\tilde{H}^1(L, I)$ of a closed ideal I of a transitive Lie algebra L is well-defined and depends only on the isomorphism class of (L, I) as a pair of topological Lie algebras.*

(ii) *Let $z \in Z$ and let $R_q^\# \subset J_q(T_Z; Z)$ be a formally transitive and formally integrable Lie equation and $N_{q_1}^\# \subset R_{q_1}^\#$ be a formally integrable Lie equation, with $q_1 \geq q$, such that*

$$[\tilde{\mathcal{R}}^\#_{q_1+1}, \mathcal{N}^\#_{q_1}] \subset \mathcal{N}^\#_{q_1}$$

and such that the pairs of topological Lie algebras (L, I) and $(R^\#_{\infty,z}, N^\#_{\infty,z})$ are isomorphic. Then we have a bijective mapping

$$\tilde{H}^1(L) \to \tilde{H}^1(R_q^\#)_z.$$

If this mapping is an isomorphism of cohomology, or a fortiori if $\tilde{H}^1(L) = 0$, then

$$\tilde{H}^1(L, I) = \tilde{H}^1(N^\#_{q_1})_z.$$

If $\tilde{H}^1(L) = 0$, then

$$H^*(L) = H^*(R^{\#}_q)_z, \quad H^*(L, I) = H^*(N^{\#}_{q_1})_z;$$

moreover, if Z is connected, the equations $R^{\#}_q$, $N^{\#}_{q_1}$ are integrable.

(iii) *Let $\phi: L \to L''$ be an epimorphism of transitive Lie algebras and $I \subset L$, $I'' \subset L''$ be closed ideals of L and L'' such that $\phi(I) = I''$. Let I' be the closed ideal of L which is the kernel of $\phi: I \to I''$. If $\tilde{H}^1(L, I') = 0$ and $\tilde{H}^1(L'', I'') = 0$, then $\tilde{H}^1(L, I) = 0$.*

(iv) *If $\phi: I \to I''$ is an isomorphism, we have an isomorphism of cohomology*

$$\tilde{H}^1(L, I) \to \tilde{H}^1(L'', I'').$$

Proof. (i) follows directly from Theorem 10.2. The statements of (ii) concerning non-linear cohomology follow from Theorem 10.3. As for the remainder of (ii), if $\tilde{H}^1(L) = 0$, then by Theorem 7.1 and the results of the end of § 7, there exist on a neighborhood of z an analytic formally transitive and formally integrable Lie equation $R^b_q \subset J_q(T_Z; Z)$, an analytic formally integrable Lie equation $N^b_{q_1} \subset R^b_{q_1}$ and a local diffeomorphism f of Z defined on a neighborhood U of $z \in Z$ such that $f(z) = z$ and

$$[\tilde{\mathcal{R}}^b_{q_1+1}, \mathcal{N}^b_{q_1}] \subset \mathcal{N}^b_{q_1},$$

$$j_{q+1}(f)\,(R^{\#}_{q\,|\,U}) = R^b_{q\,|\,f(U)}, \quad j_{q_1+1}(f)\,(N^{\#}_{q_1\,|\,U}) = N^b_{q_1\,|\,f(U)}$$

Since R^b_q, $N^b_{q_1}$ are integrable differential equations, so are $R^{\#}_{q\,|\,U}$, $N^{\#}_{q_1\,|\,U}$. Thus if Z is connected, it follows by Proposition 5.4 of [9] that $R^{\#}_q$ and $N^{\#}_{q_1}$ are integrable. By Proposition 11.2 of [10], f induces isomorphisms

$$f: H^*(R^{\#}_q)_z \to H^*(R^b_q)_z, \quad f: H^*(N^{\#}_{q_1})_z \to H^*(N^b_{q_1})_z,$$

implying the remaining assertions of (ii).

(iii)–(iv) We apply Theorem 10.1 to $\phi: L \to L''$ and to the ideals I, I' of L and I'' of L'', and consider the various objects and relations connecting them whose existence is asserted by that theorem. We may assume that $k \geqslant 2$ and that the kernels of $\pi_{k-1}: N_k \to J_{k-1}(T)$, $\pi_{k-1}: N'_k \to J_{k-1}(T)$ and $\pi_{k_1-1}: N''_{k_1} \to J_{k_1-1}(T_Y; Y)$ are 2-acyclic. Let $P_k \subset Q_k(\varrho)$, $P'_k \subset Q_k(V)$ and $P''_{k_1} \subset Q_{k_1}(Y)$ be formally integrable analytic finite forms of $N_k \subset J_k(T; \varrho)$, $N'_k \subset J_k(V)$ and $N''_{k_1} \subset J_{k_1}(T_Y; Y)$ respectively. Since P''_{k_1} is integrable and N_k satisfies conditions (I), (II) and (III) of § 9 (see the remarks at the beginning of § 9) and N'_k satisfies (10.5) for all $m \geqslant k$, $l \geqslant l_0$, Theorem 9.2, (ii) gives the exact sequence of cohomology

$$H^1(P'_k)_{m,x} \to H^1(P_k)_{m,x} \to H^1(P''_{k_1})_{m,\varrho(x)} \tag{10.8}$$

for all $m \geqslant k_1$. If $\tilde{H}^1(L, I') = 0$ and $\tilde{H}^1(L'', I'') = 0$, then by (i) we have $H^1(P'_k)_x = 0$ and $H^1(P''_{k_1})_{\varrho(x)} = 0$. According to Proposition 7.8, it follows that $H^1(P'_k)_{m,x} = 0$ for all $m \geqslant k$ and $H^1(P''_{k_1})_{m,\varrho(x)} = 0$ for all $m \geqslant k_1$. The exactness of (10.8) now implies that $H^1(P_k)_{m,x} = 0$ for all $m \geqslant k_1$, and hence that $H^1(P_k)_x = 0$. By (i) and the properties of N_k we have $\tilde{H}^1(L, I) =$

$\tilde{H}^1(N_k)_x = 0$, proving (iii). If $\phi: I \to I''$ is an isomorphism, by Theorem 10.1 we know that $N'_k = 0$; we may therefore apply Theorem 9.3, (iii) to N_k and deduce that $\varrho: H^1(P_k)_x \to H^1(P''_{k_1})_{\varrho(x)}$ is an isomorphism of cohomology, giving us the desired isomorphism by (i) and concluding the proof of the theorem.

COROLLARY 10.1. *Let $\phi: L \to L''$ be an epimorphism of transitive Lie algebras and let J be the kernel of ϕ. If $\tilde{H}^1(L, J) = 0$ and $\tilde{H}^1(L'') = 0$, then $\tilde{H}^1(L) = 0$.*

THEOREM 10.5. *Let L be a transitive Lie algebra, L^0 a fundamental subalgebra of L. Let M be a closed subalgebra of L such that $L = M + L^0$. Then M is a transitive Lie algebra. If J is a closed ideal of M contained in a closed ideal I of L, then we have a mapping of cohomology*

$$\tilde{H}^1(M, J) \to \tilde{H}^1(L, I). \tag{10.9}$$

If I is a closed ideal of L contained in M, we have an isomorphism of cohomology

$$\tilde{H}^1(M, I) \to \tilde{H}^1(L, I).$$

Proof. By Theorem 13.2 of [10], there exist formally transitive and formally integrable analytic Lie equations R'_k, R_k in $J_k(T)$ on an analytic manifold X and formally integrable analytic Lie equations $N'_k \subset R'_k$, $N_k \subset R_k$ and a point $x \in X$ such that

$$N'_k \subset N_k,$$

$$[\tilde{\mathcal{R}}'_{k+1}, \mathcal{N}'_k] \subset \mathcal{N}'_k, \quad [\tilde{\mathcal{R}}_{k+1}, \mathcal{N}_k] \subset \mathcal{N}_k$$

and (M, J) and $(R'_{\infty, x}, N'_{\infty, x})$ (resp. (L, I) and $(R_{\infty, x}, N_{\infty, x})$) are isomorphic as pairs of topological Lie algebras; moreover, if $I = J$, then $N'_k = N_k$. The mapping (10.9) is determined by the map

$$\tilde{H}^1(N'_k)_x \to \tilde{H}^1(N_k)_x$$

given by the inclusion $N'_k \subset N_k$.

11. Abelian Lie equations and their cohomology

Definition 11.1. A formally integrable Lie equation $R_k \subset J_k(T)$ is said to be *abelian* if $[R_{k+1}, R_{k+1}] = 0$.

From Lemma 1.4, we deduce that if $R_k \subset J_k(T)$ is an abelian Lie equation, then, for all $l \geq 0$,

$$[R_{k+l+1}, R_{k+l+1}] = 0,$$

and if ξ, η are solutions of R_k, then $[\xi, \eta] = 0$.

We now construct examples of abelian Lie equations. Theorem 11.1 implies that under mild assumptions integrable abelian Lie equations are locally of the type of these examples.

Let Z be a manifold, $\tau: X \to Z$, $\sigma: Z \to Y$ be surjective submersions such that the diagram

$$\begin{tikzcd} X \arrow[d, "\tau"'] \arrow[dr, "\varrho"] & \\ Z \arrow[r, "\sigma"] & Y \end{tikzcd}$$

is commutative. Let A be an affine bundle over Y whose associated vector bundle we denote by F. Assume that $\tau: X \to Z$ is equal to the induced affine bundle $\sigma^{-1}A$ over Z, whose associated vector bundle is $\sigma^{-1}F$. If W is the integrable sub-bundle of T of vectors tangent to the fibers of τ, we have a canonical morphism of vector bundles $\lambda: W \to F$ over ϱ such that the corresponding mapping

$$\lambda: W \to \varrho^{-1}F \tag{11.1}$$

is an isomorphism of vector bundles over X. A section f of F over Y determines a diffeomorphism $\gamma_f: X \to X$ sending x into $x + f(\varrho(x))$ and a vector field μ_f on X given by

$$\mu_f(x) = \frac{d}{dt}(x + tf(\varrho(x)))\,|_{t=0}, \quad x \in X,$$

which is a section of $\mathcal{W}_\lambda$. If f_1, f_2 are sections of F over Y, then

$$\gamma_{f_1} \circ \gamma_{f_2} = \gamma_{f_2} \circ \gamma_{f_1} = \gamma_{f_1+f_2}, \tag{11.2}$$

$$[\mu_{f_1}, \mu_{f_2}] = 0. \tag{11.3}$$

We obtain the injective mapping

$$\gamma: \varrho^{-1}J_k(F;\, Y) \to Q_k(W)$$

sending $(x, j_k(f)(y))$ into $j_k(\gamma_f)(x)$, where $x \in X$ and $y = \varrho(x)$. The mapping

$$\lambda: J_k(W; \lambda) \to J_k(F;\, Y)$$

given by (3.1) is a morphism of vector bundles over ϱ sending $j_k(\mu_f)(x)$ into $j_k(f)(y)$ such that the corresponding mapping

$$\lambda: J_k(W; \lambda) \to \varrho^{-1}J_k(F;\, Y)$$

is an isomorphism of vector bundles over X. From (11.3), we deduce that

$$[J_k(W; \lambda), J_k(W; \lambda)] = 0 \tag{11.4}$$

and that $J_1(W;\lambda)$ is a formally integrable Lie equation. The image $Q_k(W;\lambda)$ of γ is a sub-bundle of $Q_k(W)$ and a finite form of $J_k(W;\lambda)$. Let

$$\alpha\colon Q_k(W;\lambda)\to J_k(W;\lambda),$$
$$\beta\colon Q_k(W;\lambda)\to \varrho^{-1}J_k(F;\,Y)$$

be the bijective mappings sending $j_k(\gamma_f)(x)$ into $j_k(\mu_f)(x)$ and $(x, j_k(f)(y))$ respectively. Then $\beta=\lambda\circ\alpha$ and $\alpha(I_k)=0$ and $\beta(I_k)=0$.

We shall identify $J_0(F;\,Y)$ with F. Let $\tilde{\mathfrak{F}}_X$ be the sub-sheaf of $\mathfrak{F}_X$ of sections v of $\mathfrak{F}_X$ satisfying the following condition: the section $\lambda+d_{X/Z}v$ of $W^*\otimes_X F$ is invertible, where λ is the isomorphism (11.1). If $v\in\mathfrak{F}_X$, one verifies easily that $v\in\tilde{\mathfrak{F}}_X$ if and only if $\beta^{-1}(v)$ belongs to $\tilde{Q}_0$. Moreover, if $u\in T^*\otimes J_k(W;\,\lambda)$, then $u\in(T^*\otimes J_k(W;\,\lambda))^\wedge$ if and only if the element $\lambda+\lambda(\pi_0u)$ of $W^*\otimes_X F$ is invertible, where $\lambda(\pi_0 u)$ is defined by

$$\lambda(\pi_0 u)(\xi)=\lambda\pi_0 u(\xi),\quad \text{for } \xi\in W.$$

We set $\tilde{Q}_k(W;\lambda)=\tilde{Q}_k\cap Q_k(W;\lambda)$.

PROPOSITION 11.1. (i) *The diagram*

$$\begin{array}{ccccc}
\tilde{Q}_{k+1}(W;\lambda) & \xrightarrow{\mathcal{D}} & \mathcal{T}^*\otimes J_k(\mathcal{W};\lambda) & \xrightarrow{\mathcal{D}_1} & \wedge^2\,\mathcal{T}^*\otimes J_{k-1}(\mathcal{W};\lambda)\\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\mathrm{id}}\\
J_{k+1}(\mathcal{W};\lambda) & \xrightarrow{D} & \mathcal{T}^*\otimes J_k(\mathcal{W};\lambda) & \xrightarrow{D} & \wedge^2\,\mathcal{T}^*\otimes J_{k-1}(\mathcal{W};\lambda)
\end{array}\tag{11.5}$$

is commutative.

(ii) *If $\phi\in Q_{k+1}(W;\lambda)$, then ϕ belongs to $\tilde{Q}_{k+l}(W;\lambda)$ if and only if $D\alpha(\phi)$ belongs to $(\mathcal{T}^*\otimes J_k(\mathcal{W};\lambda))^\wedge$.*

Proof. (i) The commutativity of the left-hand square of (11.5) follows from formula (5.3) of [19] and the definition of D given in [19], § 1. As for the commutativity of the right-hand square of (11.5), it is a consequence of (11.4).

(ii) Let $\phi\in Q_{k+1}(W;\lambda)$; then by the commutativity of (3.2)

$$\lambda(\pi_0 D\alpha(\phi))=\pi_0\cdot\lambda(D\alpha(\phi))=\pi_0\cdot d_{X/Z}\lambda(\alpha(\phi))=\pi_0\cdot d_{X/Z}\beta(\phi)=d_{X/Z}\beta(\pi_0\phi).$$

Thus $D\alpha(\phi)\in(\mathcal{T}^*\otimes J_k(\mathcal{W};\lambda))^\wedge$ if and only if $\beta(\pi_0\phi)\in\tilde{\mathfrak{F}}_X$, or equivalently if $\phi\in\tilde{Q}_{k+1}(W;\lambda)$.

Let $R''_k\subset J_k(F;\,Y)$ be a formally integrable differential equation. Let $R_{k+l}\subset J_{k+l}(W;\lambda)$ be the inverse image of $\varrho^{-1}R''_{k+l}$ under the isomorphism $\lambda\colon J_{k+l}(W;\lambda)\to\varrho^{-1}J_{k+l}(F;\,Y)$. According to Proposition 5, (ii) of [6], $R_{k+l}=(R_k)_{+l}$, for $l\geqslant 0$, and R_k is formally integrable. Theorem 3 of [6] gives an isomorphism

$$H^j(R_k)_a \to H^j(R''_k)_{\varrho(a)}$$

for all $j \geqslant 0$ and $a \in X$. By (11.4), we have

$$[R_{k+l}, R_{k+l}] = 0, \quad \text{for all } l \geqslant 0;$$

therefore by Proposition 4.4 of [19], R_k is an abelian Lie equation. Let $P_{k+l} = \alpha^{-1}(R_{k+l})$; by (11.2), P_{k+l} is a groupoid. If $a \in X$ and f is a section of F over a neighborhood of $b = \varrho(a)$ such that $j_{k+l}(f)(b) \in R''_{k+l}$, then the element of $\tilde{R}_{k+l,a}$

$$\tilde{j}_{k+l}(\mu_f)(a) = \frac{d}{dt} j_{k+l}(\gamma_{tf})(a)\big|_{t=0}$$

belongs to $V_{I_{k+l}(a)}(P_{k+l})$, since $j_{k+l}(\gamma_{tf})(a) \in P_{k+l}$. Thus $\tilde{R}_{k+l,a} \subset V_{I_{k+l}(a)}(P_{k+l})$; as the dimension of these vector spaces are equal, we see that P_{k+l} is a finite form of R_{k+l}.

PROPOSITION 11.2. *Let $a \in X$ and $b = \varrho(a)$. If $H^1(R''_k)_b = 0$, or equivalently if $H^1(R_k)_a = 0$, and if R''_k is integrable, then $\tilde{H}^1(R_k)_a = 0$.*

Proof. Let $m_1 \geqslant k$ be an integer such that $H^1(R_k)_{m,a} = 0$ for all $m \geqslant m_1$. Let $m \geqslant m_1$ and $u \in (\mathcal{T}^* \otimes \mathcal{R}_m)^\wedge_a$ satisfy $\mathcal{D}_1 u = Du = 0$; by our hypothesis, $u = Dv$ for some $v \in \mathcal{R}_{m+1,a}$. Then $\lambda v(a) \in R''_{m+1,b}$ and we can write $\lambda v(a) = j_{m+1}(f)(b)$, for some solution f of R''_k over a neighborhood of b. We see that $\xi = \mu_f$ is a λ-projectable section of W over a neighborhood of a which is a solution of R_k and satisfies $j_{m+1}(\xi)(a) = v(a)$. If we also denote by ξ the germ of ξ in $\mathcal{W}_a$, clearly $v_1 = v - j_{m+1}(\xi)$ belongs to $\mathcal{R}_{m+1,a}$ and satisfies $v_1(a) = 0$ and $Dv_1 = u$. We set $\phi = \alpha^{-1}(v_1)$. Then $\phi(a) = I_{m+1}(a)$ and ϕ belongs to $\tilde{\mathcal{Q}}_{m+1}(W; \lambda)$ according to Proposition 11.1, (ii); furthermore $\mathcal{D}\phi = u$, by Proposition 11.1, (i). Since $P_{k+l} = \alpha^{-1}(R_{k+l})$ is a finite form of R_{k+l}, for $l \geqslant 0$, we see that $\phi \in \tilde{\mathcal{P}}_{m+1,a}$ satisfies $\mathcal{D}\phi = u$, showing that $H^1(P_k)_{m,a} = 0$.

LEMMA 11.1. *Let W, V be integrable sub-bundles of T, with $W \subset V$, and let $\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s$ be vector fields such that $\{\xi_1, \ldots, \xi_r\}$ is a frame for W and $\{\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s\}$ is a frame for V and*

$$[\xi_i, \xi_j] = 0, \quad [\xi_i, \eta_l] = 0,$$

for $i, j = 1, \ldots, r$, $l = 1, \ldots, s$. For all $x \in X$, there exist coordinates $x^1, \ldots, x^r, z^1, \ldots, z^s, y^1, \ldots, y^m$ on a neighborhood U of x such that $\xi_i = \partial/\partial x^i$, $i = 1, \ldots, r$, and $\{\partial/\partial x^1, \ldots, \partial/\partial x^r, \partial/\partial z^1, \ldots, \partial/\partial z^s\}$ is a frame for V over U.

Proof. We proceed by induction on s. For $s = 0$ or 1, the lemma is a standard consequence of Frobenius' theorem. Assume now that $s > 1$ and that the lemma holds for $s - 1$. Since $\xi_1, \ldots, \xi_r, \eta_1$ are commuting vector fields, the lemma with $s = 0$ gives us a function g defined on a neighborhood of x such that $\eta_1 \cdot g = 1$ and $\xi_i \cdot g = 0$, for $i = 1, \ldots, r$. We set, for $l = 2, \ldots, s$,

$$\eta'_l = \eta_l - (\eta_l \cdot g)\eta_1;$$

then $\{\xi\ , ..., \xi_r, \eta_1, \eta'_2, ..., \eta'_s\}$ is a frame for V over a neighborhood of x. For $l=2, ..., s$ and $i=1, ..., r$, we have $\eta'_l \cdot g = 0$ and

$$[\xi_i, \eta'_l] = [\xi_i, \eta_l] - (\eta_l \cdot g)[\xi_i, \eta_1] - (\xi_i \cdot \eta_l \cdot g)\eta_1 = (\eta_l \cdot \xi_i \cdot g)\eta_1 = 0.$$

Since $[\eta'_l, \eta'_p] \cdot g = 0$, for $l, p = 2, ..., s$, we have

$$[\eta'_l, \eta'_p] = \sum_{q=2}^{s} c^q_{lp} \eta'_q + \sum_{i=1}^{r} d^i_{lp} \xi_i;$$

similarly $[\eta_1, \eta'_l] \cdot g = 0$, which implies the relation

$$[\eta_1, \eta'_l] = \sum_{q=2}^{s} c^q_l \eta'_q + \sum_{i=1}^{r} d^i_l \xi_i. \tag{11.6}$$

By our induction hypothesis applied to W and the integrable sub-bundle V' of T generated by the vector fields $\xi_1, ..., \xi_r, \eta'_2, ..., \eta'_s$ over a neighborhood of x, there are vector fields $\eta''_2, ..., \eta''_s$ and functions $f^1, ..., f^r, g^2, ..., g^s$ on a neighborhood of x such that $\{\xi_1, ..., \xi_r, \eta''_2, ..., \eta''_s\}$ is a frame for V' and

$$[\xi_i, \eta''_l] = 0, \quad [\eta''_l, \eta''_p] = 0, \quad \xi_i \cdot f^j = \delta^j_i, \quad \xi_i \cdot g^p = 0, \quad \eta''_l \cdot f^j = 0, \quad \eta''_l \cdot g^p = \delta^p_l,$$

for $i, j = 1, ..., r, \ \ l, p = 2, ..., s$, on a neighborhood of x. Then by (11.6),

$$[\eta_1, \eta''_l] = \sum_{q=2}^{s} a^q_l \eta''_q + \sum_{i=1}^{r} b^i_l \xi_i \tag{11.7}$$

for $l = 2, ..., s$. We set

$$\eta''_1 = \eta_1 - \sum_{l=2}^{s} (\eta_1 \cdot g^l)\eta''_l - \sum_{i=1}^{r} (\eta_1 \cdot f^i)\xi_i.$$

For $i = 1, ..., r$, we have

$$[\xi_i, \eta''_1] = [\xi_i, \eta_1] - \sum_{l=2}^{s} \{(\eta_1 \cdot g^l)[\xi_i, \eta''_l] + (\xi_i \cdot \eta_1 \cdot g^l)\eta''_l\} = -\sum_{l=2}^{s} (\eta_1 \cdot \xi_i \cdot g^l)\eta''_l = 0.$$

Since $\eta''_1 \cdot f^i = 0$ and $\eta''_1 \cdot g^l = 0$, we have

$$[\eta''_1, \eta''_l] \cdot f^i = 0, \quad [\eta''_1, \eta''_l] \cdot g^p = 0, \tag{11.8}$$

for $i = 1, ..., r, \ \ l, p = 2, ..., s$; on the other hand by (11.7),

$$[\eta''_1, \eta''_l] = \sum_{q=2}^{s} \tilde{a}^q_l \eta''_q + \sum_{i=1}^{r} \tilde{b}^i_l \xi_i.$$

From (11.8), we deduce that $\tilde{a}^q_l = 0$ and $\tilde{b}^i_l = 0$ and so $[\eta''_1, \eta''_l] = 0$. Therefore, we obtain a frame $\{\xi_1, \ldots, \xi_r, \eta''_1, \ldots, \eta''_s\}$ of commuting vector fields for V over a neighborhood of x; the lemma with $s=0$ gives us coordinates $x^1, \ldots, x^r, z^1, \ldots, z^s, y^1, \ldots, y^m$ on a neighborhood U of x such that $\xi_i = \partial/\partial x^i$, $\eta''_l = \partial/\partial z^l$, for $i=1, \ldots, r$, $l=1, \ldots, s$.

THEOREM 11.1. *Let $R_k \subset J_k(T)$, with $k \geq 1$, be an integrable and formally integrable abelian Lie equation such that $\pi_0 \tilde{R}_k$ is a sub-bundle W of T. Assume that there exists an integrable and formally integrable Lie equation $N_k \subset J_k(T)$ such that $R_k + N_k$ is a sub-bundle of $J_k(T)$ and $\pi_0(\tilde{R}_k + \tilde{N}_k)$ is a sub-bundle V of T and*

$$[N_{k+1}, R_{k+1}] = 0. \tag{11.9}$$

Then, for all $x \in X$, with X replaced if necessary by a neighborhood of x, there exist manifolds Y, Z, surjective submersions $\varrho: X \to Y$, $\tau: X \to Z$, $\sigma: Z \to Y$, an affine bundle A over Y whose associated vector bundle we denote by F, a diffeomorphism $\varphi: X \to \sigma^{-1}A$ of X onto an open subset of the induced affine bundle $\sigma^{-1}A$ over Z, whose associated vector bundle is $\sigma^{-1}F$, and an integrable and formally integrable differential equation $R''_k \subset J_k(F; Y)$ such that:

(i) *the diagrams*

$$\begin{tikzcd} X \arrow[d, "\tau"'] \arrow[dr, "\varrho"] & \\ Z \arrow[r, "\sigma"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\tau"'] & \sigma^{-1}A \arrow[dl] \\ Z & \end{tikzcd}$$

are commutative;

(ii) *W, V are the bundles of vectors tangent to the fibers of $\tau: X \to Z$, $\varrho: X \to Y$ respectively;*

(iii) *identifying X with an open subset of $\sigma^{-1}A$ via φ, if $\lambda: W \to F$ is the canonical morphism over ϱ given by the structure of affine bundle of $\sigma^{-1}A$ over Z, we have $R_{k+l} \subset J_{k+l}(W; \lambda)$, for all $l \geq 0$;*

(iv) *if $\lambda: J_{k+l}(W; \lambda) \to J_{k+l}(F; Y)$ is the morphism given by* (3.1), *then*

$$\lambda(R_{k+l,\, a}) = R''_{k+l,\, \varrho(a)},$$

for all $l \geq 0$ and $a \in X$;

(v) *if $a \in X$ and $b = \varrho(a)$ and if $H^1(R''_k)_b = 0$, or equivalently if $H^1(R_k)_a = 0$, then $\tilde{H}^1(R_k)_a = 0$.*

Proof. Since R_k is a Lie equation, W is an integrable sub-bundle of T. If $a \in X$ and $u \in R_{k,a}$, since R_k is integrable, we can write $u = j_k(\xi)(a)$, for some solution ξ of R_k over a neighborhood of a; as ξ is a section of W, we see that $u \in J_k(W)$ and $R_k \subset J_k(W)$. Since R_k is integrable, there exist sections $\xi_1, \ldots, \xi_r$ of W which are solutions of R_k over a neighborhood of x such that $\{\xi_1, \ldots, \xi_r\}$ is a frame for W over that neighborhood. As

$$N_{k+1} + R_{k+1} \subset (N_k + R_k)_{+1}, \tag{11.10}$$

it follows from Proposition 4.4 of [19] and (11.9) that $N_k + R_k$ is a Lie equation. Thus V is an integrable sub-bundle of T containing W. Since N_k is integrable, there exist sections $\eta_1, \ldots, \eta_s$ of V which are solutions of N_k over a neighborhood of x such that $\{\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s\}$ is a frame for V over this neighborhood. Since R_k is abelian and (11.9) holds, we deduce from Lemma 1.4 that

$$[\xi_i, \xi_j] = 0, \quad [\xi_i, \eta_l] = 0,$$

for $1 \leqslant i, j \leqslant r$, $1 \leqslant l \leqslant s$. By Lemma 11.1, there exist a neighborhood U of x and coordinates $x^1, \ldots, x^r, z^1, \ldots, z^s, y^1, \ldots, y^m$ on U such that the mapping

$$\varphi: U \to \mathbf{R}^r \times \mathbf{R}^s \times \mathbf{R}^m$$

given by these coordinates is a diffeomorphism of U onto an open subset $U_1 \times U_2 \times U_3$ of $\mathbf{R}^r \times \mathbf{R}^s \times \mathbf{R}^m$, where $U_1 \subset \mathbf{R}^r$, $U_2 \subset \mathbf{R}^s$, $U_3 \subset \mathbf{R}^m$ are connected open subsets, and $\xi_i = \partial/\partial x^i$, for $i = 1, \ldots, r$, and $\{\partial/\partial x^1, \ldots, \partial/\partial x^r, \partial/\partial z^1, \ldots, \partial/\partial z^s\}$ is a frame for V over U. Replacing X by U, setting $Y = U_3$, $Z = U_2 \times U_3$ and letting A be the trivial vector bundle F of rank r over Y, and $\sigma: Z \to Y$ be the projection onto the second factor, $\varrho: X \to Y$ be the composition of φ and the projection of $U_1 \times U_2 \times U_3$ onto the last factor and $\tau: X \to Z$ the composition of φ and the natural projection of $U_1 \times U_2 \times U_3$ onto Z, we thus obtain the mappings satisfying (i) and (ii). During the remainder of the proof, we shall identify X with its image by $\varphi: X \to \sigma^{-1}A$; then $\tau: X \to Z$ is a fibered submanifold of the affine bundle $\sigma^{-1}A \to Z$. As $\sigma^{-1}F$ is the associated vector bundle of $\sigma^{-1}A$, we have a canonical morphism of vector bundles $\lambda: W \to F$ over ϱ; if $a \in X$ and $f \in F_{\varrho(a)}$, then

$$\lambda(d(a + tf)/dt\,|_{t=0}) = f$$

and the corresponding mapping $\lambda: W \to \varrho^{-1}F$ is an isomorphism of vector bundles. Denoting by $\varepsilon_1, \ldots, \varepsilon_r$ the sections of the canonical frame of F over Y, we see from the construction of φ and ϱ that

$$\lambda(\xi_i(a)) = \varepsilon_i(\varrho(a)), \quad i = 1, \ldots, r, \quad a \in X. \tag{11.11}$$

Now let ξ be a solution of R_k over an open set $U' \subset X$; then we may write $\xi = \sum_{j=1}^r c^j \xi_j$ and by Lemma 1.4 and (11.9),

$$0 = [\xi_i, \xi] = \sum_{j=1}^{r} (\xi_i \cdot c^j)\, \xi_j, \quad 0 = [\eta_l, \xi] = \sum_{j=1}^{r} (\eta_l \cdot c^j)\, \xi_j,$$

for $i = 1, \ldots, r$, $l = 1, \ldots, s$. Therefore $\xi_i \cdot c^j = 0$, $\eta_l \cdot c^j = 0$; since $\{\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s\}$ is a frame for V over X, we have $d_{X/Y} c^j = 0$ for $j = 1, \ldots, r$. Any point $a_0 \in U'$ possesses a neighborhood

$U'' \subset U'$ such that the fibers of $\varrho: U'' \to \varrho U''$ are connected. Thus there exist functions b^j on $\varrho U''$ such that $c^j = b^j \circ \varrho$ on U'', for $j = 1, \ldots, r$, and

$$\xi = \sum_{j=1}^{r} (b^j \circ \varrho)\, \xi_j \tag{11.12}$$

on U''; by (11.11), we have

$$\lambda \xi(a) = \sum_{j=1}^{r} b^j(\varrho(a))\, \varepsilon_j(\varrho(a))$$

for all $a \in U''$. Therefore ξ is a λ-projectable section of W over U''. As R_k is integrable, we have $R_{k+l} \subset J_{k+l}(W; \lambda)$ for all $l \geqslant 0$.

Let us show that, for $a, b \in X$, we have

$$\lambda(R_{k+l,a}) = \lambda(R_{k+l,b}) \tag{11.13}$$

whenever $\varrho(a) = \varrho(b)$. Let $a_0 \in X$ and $l \geqslant 0$; since R_k is integrable, we choose sections $\xi'_1, \ldots, \xi'_p$ of W over a neighborhood of a_0 which are solutions of R_k and which can be written in the form (11.12) such that $\{j_{k+l}(\xi'_1), \ldots, j_{k+l}(\xi'_p)\}$ is a frame for R_{k+l} over this neighborhood. For $t = (t^1, \ldots, t^{r+s}) \in \mathbf{R}^{r+s}$, with $|t| = |t^1| + \ldots + |t^{r+s}| < \varepsilon$, let ϕ_t be the local diffeomorphism of X defined on a neighborhood U_0 of a_0 sending (x, z, y) into $(x^1 + t^1, \ldots, x^r + t^r, z^1 + t^{r+1}, \ldots, z^s + t^{r+s}, y)$. We may assume that the vector fields ξ'_i are defined on U_0 and $\phi_t(U_0)$ for $|t| < \varepsilon$. Then for $a \in U_0$, $i = 1, \ldots, p$, $|t| < \varepsilon$,

$$\phi_{t*}(\xi'_i)(a) = \xi'_i(\phi_t(a)).$$

Thus

$$j_{k+l+1}(\phi_t)(a) \cdot j_{k+l}(\xi'_i)(a) = j_{k+l}(\xi'_i)(\phi_t(a));$$

because of our condition on $\xi'_1, \ldots, \xi'_p$, we therefore have

$$j_{k+l+1}(\phi_t)(a)(R_{k+l,a}) = R_{k+l,\phi_t(a)}.$$

Furthermore, the diagram

$$\begin{array}{ccc} R_{k+l,a} & \xrightarrow{j_{k+l+1}(\phi_t)(a)} & R_{k+l,\phi_t(a)} \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\ J_{k+l}(F;Y)_{\varrho(a)} & \xrightarrow{\text{id}} & J_{k+l}(F;Y)_{\varrho(a)} \end{array}$$

is commutative. Hence $\lambda(R_{k+l,a}) = \lambda(R_{k+l,\phi_t(a)})$, for $a \in U_0$, $|t| < \varepsilon$. Since $\{\phi_t(a_0) \mid |t| < \varepsilon\}$ is a neighborhood of a_0 in the fiber of ϱ passing through a_0 and the fibers of $\varrho: X \to Y$ are connected, we obtain (11.13). Therefore there exists a differential equation $R''_{k+l} \subset J_{k+l}(F; Y)$ whose fiber at $\varrho(a)$ is equal to $\lambda(R_{k+l,a})$. From Proposition 5, (i) of [6], we deduce that

R''_{k+l} is the l-th prolongation of R''_k and that R''_k is formally integrable. Since R_k is integrable, so is R''_k, proving (iv). Therefore, R_k is the restriction to an open set of $\sigma^{-1}A$ of the equation on $\sigma^{-1}A$ obtained from R''_k and so Proposition 11.2 implies (v).

Let $X=G$ be a Lie group and let E be a vector bundle over X. Assume that E is a G-bundle, that is possesses the structure of a G-space such that $g\colon E\to E$ is a morphism of vector bundles over the left-translation $g\colon X\to X$, for $g\in X$. Then E has a natural trivialization $E\simeq X\times E_{x_0}$, where x_0 is the identity element of G. We have a morphism of vector bundles

$$g\colon J_k(E)\to J_k(E)$$

over $g\colon X\to X$ defined by

$$g\cdot j_k(s)(x)=j_k(g\cdot s\cdot g^{-1})(g\cdot x),$$

where s is a section of E over X and $x\in X$; thus $J_k(E)$ is a G-bundle.

We say that a differential equation $R_k\subset J_k(E)$ is G-invariant if R_k is a G-invariant sub-bundle of $J_k(E)$; for such an equation, there exist a G-vector bundle F over X and a G-morphism of vector bundles $\varphi\colon J_k(E)\to F$ such that $\ker\varphi=R_k$.

If $G=\mathbf{R}^n$, we say that a G-invariant differential equation $R_k\subset J_k(E)$ is a differential equation with constant coefficients. For such an equation R_k, the theorem of Ehrenpreis-Malgrange implies that $H^j(R_k)=0$, for $j>0$.

LEMMA 11.2. *Let G be a Lie group, E a vector bundle over an open set $X\subset G$ and $R_k\subset J_k(E)$ a differential equation. Assume that there are an open set $U\subset X$, a neighborhood $H\subset G$ of the identity element of G and a mapping*

$$\psi\colon H\times E_{|U}\to E$$

sending (g, e) into $\psi_g(e)$ such that $\psi_g\colon E_{|U}\to E$ is a morphism of vector bundles over the left-translation $g\colon U\to X$ for all $g\in H$ and

$$\psi_{g_1}\circ\psi_{g_2}=\psi_{g_1\cdot g_2}$$

as mappings $E_a\to E_{g_1\cdot g_2\cdot a}$, for all $g_1, g_2\in H$, $a\in U$ with $g_1\cdot g_2\in H$ and $g_2\cdot a\in U$. If $\psi_g\colon J_k(E)_{|U}\to J_k(E)$ is the morphism of vector bundles over ψ_g sending $j_k(s)(a)$ into $j_k(\psi_g\cdot s\cdot g^{-1})(g\cdot a)$, where s is a section of E over a neighborhood of $a\in U$ and $g\in H$ and if

$$\psi_g(R_{k|U})=R_{k|gU} \tag{11.14}$$

for all $g\in H$, then for each point $x\in U$ there are a neighborhood $U'\subset U$ of x, a G-vector bundle E' over G, a G-invariant differential equation $R'_k\subset J_k(E'; G)$ and a morphism of vector bundles $\chi\colon E_{|U'}\to E'_{|U'}$ such that $J_k(\chi)(R_{k|U'})=R'_{k|U'}$. If R_k is formally integrable, so is R'_k and χ induces isomorphisms

$$\chi: H^j(R_k)_{|U'} \to H^j(R'_k)_{|U'} \tag{11.15}$$

for $j \geq 0$.

Proof. Let F be the quotient $J_k(E)/R_k$ and $\varphi: J_k(E) \to F$ the natural projection. For $g \in H$, it follows from (11.14) that $\psi_g: J_k(E)_{|U} \to J_k(E)$ induces a morphism of vector bundles $\psi_g: F_{|U} \to F$ over the left-translation $g: U \to X$ such that the diagram

$$\begin{CD} J_k(E)_{|U} @>\varphi>> F_{|U} \\ @V\psi_g VV @VV\psi_g V \\ J_k(E) @>\varphi>> F \end{CD}$$

commutes. Thus for $g \in G$ and a section s of E over U, we have

$$\varphi j_k(\psi_g \cdot s \cdot g^{-1}) = \psi_g \cdot (\varphi j_k(s)) \cdot g^{-1} \tag{11.16}$$

on gU. Let x be a fixed point of U. Let $e_1^0, \ldots, e_r^0$ be a basis for E_x and $f_1^0, \ldots, f_q^0$ a basis for F_x; consider the frames $\{e_1, \ldots, e_r\}$ for E and $\{f_1, \ldots, f_q\}$ for F over $H \cdot x$, where the sections e_i of E and f_j of F are defined by

$$e_i(g \cdot x) = \psi_g(e_i^0), \quad f_j(g \cdot x) = \psi_g(f_j^0),$$

for $1 \leq i \leq r$, $1 \leq j \leq q$. Then for $g, h \in H$ with $g \cdot h \in H$, $h \cdot x \in U$, we have

$$\psi_g e_i(h \cdot x) = e_i(g \cdot h \cdot x), \quad \psi_g f_j(h \cdot x) = f_j(g \cdot h \cdot x);$$

hence for $1 \leq i \leq r$, $1 \leq j \leq q$, we see that

$$\psi_g e_i(a) = e_i(g \cdot a), \quad \psi_g f_j(a) = f_j(g \cdot a),$$

for all a belonging to a neighborhood $U_1 \subset U$ of x and all g belonging to a neighborhood $H_1 \subset H$ of the identity element of G, with $H_1 \cdot U_1 \subset H \cdot x$. Therefore, if s is a section of E over U_1 and t is a section of F over U_1 and

$$s = \sum_{i=1}^{r} s^i e_i, \quad t = \sum_{j=1}^{q} t^j f_j,$$

then for $g \in H_1$ we have

$$(\psi_g \cdot s \cdot g^{-1})(a) = \sum_{i=1}^{r} s^i(g^{-1} \cdot a)\, e_i(a), \quad (\psi_g \cdot t \cdot g^{-1})(a) = \sum_{j=1}^{q} t^j(g^{-1} \cdot a)\, f_j(a), \tag{11.17}$$

for $a \in gU_1$. Let $\{p_\alpha\}_{\alpha \in A}$ be a basis for the space of left-invariant differential operators of order $\leq k$ on G. There exist functions $c_i^{\alpha, j}$ on $H \cdot x$ such that

$$\varphi j_k(s) = \sum_{j=1}^{q} (D^j s) f_j,$$

with

$$D^j s = \sum_{\substack{\alpha \in A \\ i=1,\dots,r}} c_i^{\alpha,j} p_\alpha s^i, \quad j = 1, \dots, q,$$

for all sections $s = \sum_{i=1}^r s^i e_i$ of E over $H \cdot x$. From (11.16) and (11.17), since the differential operators p_α are left-invariant, we deduce that for $g \in H_1$

$$c_i^{\alpha,j}(g^{-1} a) = c_i^{\alpha,j}(a)$$

for all $a \in gU_1$, $\alpha \in A$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant q$; hence

$$c_i^{\alpha,j}(gx) = c_i^{\alpha,j}(x)$$

for all $g \in H_1$ and so the functions $c_i^{\alpha,j}$ are constant on the neighborhood $U' = H_1 \cdot x$ of x. Let E', F' be the trivial G-vector bundles $G \times \mathbf{R}^r$, $G \times \mathbf{R}^q$ respectively, that is, for $g \in G$, the morphisms g: $E' \to E'$, g: $F' \to F'$ send (h, u) into $(g \cdot h, u)$, where $h \in G$, $u \in \mathbf{R}^r$ or $u \in \mathbf{R}^q$. Let χ: $E_{|U'} \to E'_{|U'}$ be the isomorphism of vector bundles sending $\sum_{i=1}^r b^i e_i(a)$ into $(a, b^1, \dots, b^r)$, with $a \in U'$, and let φ': $J_k(E'; G) \to F'$ be the morphism of vector bundles sending $j_k(s)$, where $s = (s^1, \dots, s^r)$ is a section of E' over G, into the section $((\varphi' j_k(s))^1, \dots, (\varphi' j_k(s))^q)$ of F' over G, where

$$(\varphi' j_k(s))^j = \sum_{\substack{\alpha \in A \\ i=1,\dots,r}} c_i^{\alpha,j}(x) \, p_\alpha s^i, \quad j = 1, \dots, q.$$

Clearly the kernel $R'_k \subset J_k(E'; G)$ is a G-invariant sub-bundle and $J_k(\chi)(R_{k|U'}) = R'_{k|U'}$. Since

$$J_{k+l}(\chi)(R_{k+l|U'}) = R'_{k+l|U'} \tag{11.18}$$

for all $l \geqslant 0$, if R_k is formally integrable, then so is $R'_{k|U'}$; as R'_k is G-invariant, it is therefore formally integrable over G. From Lemma 1.1 and (11.18), it follows that χ induces the isomorphisms (11.15).

THEOREM 11.2. *Assume that the hypotheses of Theorem* 11.1 *hold. Suppose that* $R_{k+1} + N_{k+1}$ *is a vector bundle and that there exists an integrable and formally integrable Lie equation* $S_k \subset J_k(T)$ *such that* $\pi_1 S_k$ *is a vector bundle and*

$$[S_{k+1}, S_{k+1}] = 0, \quad [S_{k+1}, R_{k+1}] \subset R_k, \tag{11.19}$$

$$[S_{k+1}, N_{k+1}] \subset J_k(V), \tag{11.20}$$

and

$$V + \pi_0 \tilde{S}_k = T. \tag{11.21}$$

Then, for $x \in X$, *we may assume that the manifold* Y *given by Theorem* 11.1 *is equal to an open subset of* $\mathbf{R}^m$, *that there is an* $\mathbf{R}^m$*-vector bundle* F' *on* $\mathbf{R}^m$, *a formally integrable differential* *equation* $R'_k \subset J_k(F'; \mathbf{R}^m)$ *with constant coefficients and an isomorphism of vector bundles* $\chi\colon F \to F'_{|Y}$ *such that* $J_k(\chi)(R''_k) = R'_{k|Y}$. *Furthermore* $H^j(R_k)=0$ *for* $j>0$ *and* $\tilde{H}^1(R_k)_a = 0$ *for all* $a \in X$.

Proof. We fix $x \in X$ and then consider the objects described in the course of the proof of Theorem 11.1. By (11.21), since S_k is integrable we choose vector fields $\zeta_1, \ldots, \zeta_m$ which are solutions of S_k over a neighborhood of x such that $\{\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s, \zeta_1, \ldots, \zeta_m\}$ is a frame for T over this neighborhood. By (11.19) and Lemma 1.4, we have

$$[\zeta_\alpha, \zeta_\beta] = 0, \tag{11.22}$$

for $1 \leqslant \alpha, \beta \leqslant m$. Since $\pi_1 S_k$ is a vector bundle, S_k is integrable and

$$[S_{k+1}, R_{k+1} + N_{k+1}] \subset R_k + J_k(V) \subset J_k(V),$$

and since $\pi_0(R_{k+1} + N_{k+1}) = J_0(V)$ and (11.10) holds, we deduce from Lemma 6.1 that $S_k \subset J_k(T; \varrho)$. Therefore replacing X and Y by neighborhoods of x and $\varrho(x)$ respectively, we may assume that $\zeta_1, \ldots, \zeta_m$ are ϱ-projectable vector fields on X and, by Frobenius' theorem, that there are coordinates $y'^1, \ldots, y'^m$ on Y such that

$$\varrho_* \zeta_\alpha(a) = \frac{\partial}{\partial y'^\alpha}(\varrho(a))$$

for all $a \in X$, $\alpha = 1, \ldots, m$. We shall consider Y as an open subset of $\mathbf{R}^m$ by means of these coordinates. Let $S^\#_k \subset S_k$ be the formally integrable Lie equation generated by the sections $j_k(\zeta_1), \ldots, j_k(\zeta_m)$ of S_k. Then

$$S^\#_k \cap J_k(V) = 0; \tag{11.23}$$

this implies that $S^\#_k + R_k$ is a vector bundle. As

$$S^\#_{k+1} + R_{k+1} \subset (S^\#_k + R_k)_{+1},$$

by Proposition 4.4 of [19] and (11.19), it follows that $S^\#_k + R_k$ is a Lie equation, which by (11.23) satisfies

$$(S^\#_k + R_k) \cap J_k(V) = R_k, \tag{11.24}$$

since $R_k \subset J_k(V)$. Then by Proposition 7.1, (v), there are neighborhoods $U' \subset X$ of x and $H \subset \mathbf{R}^m$ of 0 such that for $t = (t^1, \ldots, t^m) \in H$ the local diffeomorphism of X

$$\psi_t = (\exp t^1\zeta_1)\circ \ldots \circ(\exp t^m\zeta_m) \tag{11.25}$$

is defined on U' and is a solution of a finite form of $S^\#_k$ and of a finite form of $S^\#_k + R_k$ and satisfies $\varrho\psi_t = \bar{\psi}_t\varrho$, where $\bar{\psi}_t$ is the translation of $\mathbf{R}^m$ by t sending $b \in \varrho U'$ into $t + b \in Y$. Therefore

$$j_{k+1}(\psi_t)(a)(R_{k,a}) \subset (S^\#_k + R_k)_{\psi_t(a)}$$

for all $a \in U'$, $t \in H$. As $j_{k+1}(\psi_t)$ is a section of $Q_{k+1}(\varrho)$, we have

$$j_{k+1}(\psi_t)(a)(R_{k,a}) \subset J_k(V)_{\psi_t(a)}$$

and by (11.24),

$$j_{k+1}(\psi_t)(a)(R_{k,a}) = R_{k,\psi_t(a)} \tag{11.26}$$

for all $a \in U'$, $t \in H$. We may assume that the fibers of $\varrho\colon U' \to \varrho U'$ are connected by replacing U' by a smaller neighborhood of x if necessary. From (11.26), we deduce that $\psi_{t*}(\xi_i)$ is a solution of R_k over $\psi_t(U')$ for all $i=1, \ldots, r$, $t \in H$. Therefore there are functions g_j^i on $H \times \varrho U'$ such that

$$\psi_{t*}(\xi_i(a)) = \sum_{j=1}^{r} g_i^j(t, \varrho(a))\, \xi_j(\psi_t(a)) \tag{11.27}$$

for all $a \in U'$, $t \in H$, $i = 1, \ldots, r$. For $t \in H$, let $\psi_t^{\#}\colon F_{|\varrho U'} \to F_{|\bar{\psi}_t(\varrho U')}$ be the morphism of vector bundles over $\bar{\psi}_t$ defined by

$$\psi_t^{\#}(\varepsilon_i(b)) = \sum_{j=1}^{r} g_i^j(t, b)\, \varepsilon_j(t+b),$$

for $b \in \varrho U'$. By (11.27), for $a \in U'$, $t \in H$, the diagram

$$\begin{array}{ccc} W_a & \xrightarrow{\psi_{t*}} & W_{\psi_t(a)} \\ \Big\downarrow{\scriptstyle\lambda} & & \Big\downarrow{\scriptstyle\lambda} \\ F_{\varrho(a)} & \xrightarrow{\psi_t^{\#}} & F_{t+\varrho(a)} \end{array} \tag{11.28}$$

is commutative; from (11.22) and (11.25) we now deduce the equality

$$\psi_{t_1}^{\#} \circ \psi_{t_2}^{\#} = \psi_{t_1+t_2}^{\#} \tag{11.29}$$

of mappings $F_b \to F_{t_1+t_2+b}$, where $b \in \varrho U'$, $t \in H$ satisfy $t_1 + t_2 \in H$ and $t_2 + b \in \varrho U'$. For $t \in H$, let

$$\psi_t^{\#}\colon J_k(F; Y)_{|\varrho U'} \to J_k(F; Y)$$

be the morphism of vector bundles over $\bar{\psi}_t$ sending $j_k(s)(b)$ into $j_k(\psi_t^{\#} \cdot s \cdot \bar{\psi}_{-t})(t+b)$, where s is a section of F over a neighborhood of $b \in \varrho U'$. The commutativity of (11.28) implies that, for $a \in U'$, $t \in H$, the diagram

$$\begin{array}{ccc} J_k(W; \lambda)_a & \xrightarrow{j_{k+1}(\psi_t)(a)} & J_k(W; \lambda)_{\psi_t(a)} \\ \Big\downarrow{\scriptstyle\lambda} & & \Big\downarrow{\scriptstyle\lambda} \\ J_k(F; Y)_{\varrho(a)} & \xrightarrow{\psi_t^{\#}} & J_k(F; Y)_{t+\varrho(a)} \end{array}$$

commutes; from (11.13) and (11.26), it follows that

$$\psi_t^{\#}(R''_{k\,|\,\varrho U'}) = R''_{k\,|\,\overline{\psi}_t(\varrho U')} \tag{11.30}$$

for all $t \in H$. Replacing X and Y by neighborhoods of x and $\varrho(x)$ respectively if necessary, because of (11.29) and (11.30), Lemma 11.2 gives us the vector bundle F' on $\mathbf{R}^m$, the differential equation $R'_k \subset J_k(F'; \mathbf{R}^m)$ and the isomorphism $\chi: F \to F'_{|Y}$ satisfying the desired conditions. Since R'_k is a differential equation with constant coefficients, by the theorem of Ehrenpreis-Malgrange we have $H^j(R'_k) = 0$ for $j > 0$ and hence, by Lemma 11.2, $H^j(R''_k) = 0$ for $j > 0$. From Theorem 11.1, we deduce that $H^j(R_k) = 0$ for $j > 0$ and $\tilde{H}^1(R_k)_a = 0$ for all $a \in X$.

LEMMA 11.3. *Assume that X is connected and let $x \in X$. Let $R_k \subset J_k(T)$ be a formally transitive and formally integrable Lie equation. Let N_k, $N'_k \subset J_k(T)$ be formally integrable differential equations such that*

$$[\tilde{\mathcal{R}}_{k+2}, \mathcal{N}_{k+1}] \subset \mathcal{N}_{k+1}, \quad [\tilde{\mathcal{R}}_{k+2}, \mathcal{N}'_{k+1}] \subset \mathcal{N}'_{k+1}.$$

Then $N_k + N'_k$ and $N_{k+1} + N'_{k+1}$ are vector bundles. Moreover if $[N_{k+1,x}, N'_{k+1,x}] = 0$, then

$$[N_{k+1}, N'_{k+1}] = 0. \tag{11.31}$$

Proof. Let ω be an R_{k+2}-connection defined on an open subset of X. Clearly, the sub-bundles N_{k+1}, N'_{k+1} of $J_{k+1}(T)$ are stable by the covariant derivative in $J_{k+1}(T)$ determined by ω and the sub-bundles N_k, N'_k of $J_k(T)$ are stable by the covariant derivative in $J_k(T)$ determined by $\pi_{k+1}\omega$. Jacobi's identity

$$[\tilde{\xi}, [\eta_1, \eta_2]] = [[\tilde{\xi}, \eta_1], \eta_2] + [\eta_1, [\tilde{\xi}, \eta_2]],$$

for $\tilde{\xi} \in \tilde{\mathcal{R}}_{k+2}$, $\eta_1 \in \mathcal{N}_{k+1}$, $\eta_2 \in \mathcal{N}'_{k+1}$, implies that the bracket $N_{k+1} \otimes N'_{k+1} \to J_k(T)$ is compatible in the sense of § 3 of [9] with the covariant derivatives determined by ω in $N_{k+1} \otimes N'_{k+1}$ and by $\pi_{k+1}\omega$ in $J_k(T)$. Propositions 5.1, 3.3 and 3.2 of [9] imply that $N_{k+1} + N'_{k+1}$ and $N_k + N'_k$ are vector bundles and that the set of points $a \in X$ such that $[N_{k+1,a}, N'_{k+1,a}] = 0$ is both open and closed. Since X is connected, if this set is non-empty, (11.31) holds.

THEOREM 11.3. *Let L be a transitive Lie algebra and I a closed abelian ideal of L. If $H^1(L, I) = 0$, then $\tilde{H}^1(L, I) = 0$.*

Proof. By Corollary 6.1 of [9] and Theorem 10.1 of [10], there exist a formally transitive and formally integrable analytic Lie equation $R_k^{\#} \subset J_k(T)$ on a connected analytic manifold X, a point $x \in X$, a formally integrable Lie equation $R_k \subset R_k^{\#}$ such that $[\tilde{\mathcal{R}}^{\#}_{k+1}, \mathcal{R}_k] \subset \mathcal{R}_k$ and $(R^{\#}_{\infty,x}, R_{\infty,x})$ and (L, I) are isomorphic as pairs of topological Lie algebras. Then by Lemmas 1.5 and 11.3, R_k is an abelian Lie equation and, by Lemma 10.3, (ii) of [10],

$\pi_0 R_k$ is a vector bundle. Therefore the hypotheses of Theorem 11.1 hold for R_k (with $N_k=0$); thus by Theorem 11.1, (v), if $H^1(R_k)_x=0$, then $\tilde{H}^1(R_k)_x=0$.

LEMMA 11.4. *Let L be a transitive Lie algebra. Let L^0 be a fundamental subalgebra of L and let A be an abelian subalgebra of L and B a subspace of L satisfying*

$$L = L^0 + A + B,$$

$$[A, B] = 0.$$

Then $L^0 \cap A = 0$ and A is a closed finite-dimensional subalgebra. If $B=0$, then

$$L = L^0 \oplus A.$$

Proof. For $k>0$, set $L^k = D_L^k L^0$; then $[L^0, L^k] \subset L^k$ for all $k \geqslant 0$ and $\bigcap_{k=0}^{\infty} L^k = 0$. If $\xi \in L^k \cap A$, with $k \geqslant 0$, then

$$[L, \xi] = [L^0 + A + B, \xi] = [L^0, \xi] \subset L^k,$$

and so $\xi \in L^{k+1}$; therefore $\xi = 0$. Since the codimension of L^0 in L is finite, A must be finite-dimensional.

THEOREM 11.4. *Let L be a transitive Lie algebra, L^0 a fundamental subalgebra of L and A, B closed subalgebras of L. Assume that A is abelian and that*

$$L = L^0 + A + B, \qquad [A, B] = 0.$$

Let I be a closed ideal of L satisfying $[B, I] = 0$. Then there exist formally transitive and formally integrable analytic Lie equations R_k, $R'_k \subset J_k(T)$, formally integrable analytic Lie equations S_k, $B_k \subset R'_k$, $N_k \subset R_k$ on a connected analytic manifold X, a point $x \in X$, isomorphisms of transitive Lie algebras

$$\psi: L \to R_{\infty,x}, \quad \psi': A + B \to R'_{\infty,x},$$

such that, for all $l \geqslant 0$,

$$\psi(L^0) = R^0_{\infty,x}, \tag{11.32}$$

$$\psi(I) = N_{\infty,x}, \tag{11.33}$$

$$\psi'(A) = S_{\infty,x}, \quad \psi'(B) = B_{\infty,x}, \tag{11.34}$$

$$R'_k \subset R_k, \tag{11.35}$$

$$[\tilde{\mathcal{R}}_{k+l+1}, \mathcal{N}_{k+l}] \subset \mathcal{N}_{k+l}, \quad [R_{k+1}, N_{k+1}] \subset N_k, \tag{11.36}$$

$$[\tilde{\mathcal{R}}'_{k+l+1}, \mathcal{S}_{k+l}] \subset \mathcal{S}_{k+l}, \quad [\tilde{\mathcal{R}}'_{k+l+1}, \mathcal{B}_{k+l}] \subset \mathcal{B}_{k+l}, \tag{11.37}$$

$$[S_{k+1}, S_{k+1}] = 0, \quad [S_{k+1}, B_{k+1}] = 0, \tag{11.38}$$

$$[B_{k+1}, N_{k+1}] = 0, \tag{11.39}$$

and

$$\pi_0(S_k + B_k) = J_0(T), \quad R_\infty = R_\infty^0 + S_\infty + B_\infty. \tag{11.40}$$

Furthermore, $\pi_0 N_k$, $\pi_1 S_k$, $N_k + B_k$, $N_{k+1} + B_{k+1}$ *and* $\pi_0(N_k + B_k)$ *are vector bundles and, if* I *is an abelian ideal,* N_k *is an abelian Lie equation.*

Proof. We begin by following the first part of the proof of Theorem 13.2 of [10]. We see that $A+B$ is a transitive Lie algebra and that $L'^0 = L^0 \cap (A+B)$ is a fundamental subalgebra of $A+B$. Clearly A and B are closed ideals of $A+B$. Let us consider the filtrations induced by L^0 and L'^0 on L, I and $A+B$, A, B respectively in the sense of § 10 of [10] and the corresponding graded Lie algebras. By Lemma 10.1 of [10], there exists an integer $k \geqslant 1$ such that

$$H^{m,j}(\operatorname{gr} L) = H^{m,1}(\operatorname{gr} I) = 0,$$

$$H^{m,j}(\operatorname{gr}(A+B)) = H^{m,1}(\operatorname{gr} A) = H^{m,1}(\operatorname{gr} B) = 0,$$

for all $m \geqslant k$ and $j = 1, 2$. Let X be an analytic manifold whose dimension is equal to the dimension of L/L^0 and let $x \in X$. By Theorem III of [13], there exists a monomorphism $i: L \to J_\infty(T)_x$ of transitive Lie algebras such that $i(L^0) = i(L) \cap J_\infty^0(T)_x$ and $i(L)$ is a transitive subalgebra of $J_\infty(T)_x$; then $i(A+B)$ is also a transitive subalgebra of $J_\infty(T)_x$ and $i(L'^0) = i(A+B) \cap J_\infty^0(T)_x$. We apply Corollary 6.1 of [9] to the subalgebras $i(L)$ and $i(A+B)$ of $J_\infty(T)_x$ and replace X by a simply connected neighborhood of x if necessary to obtain the existence of formally transitive and formally integrable analytic Lie equations $R_k \subset J_k(T)$, $R'_k \subset J_k(T)$ and ϕ, $\phi' \in Q_\infty(x, x)$ such that $R'_k \subset R_k$ and $\pi_{k+2}\phi = \pi_{k+2}\phi' = I_{k+2}(x)$ and

$$\phi \cdot i(L) = R_{\infty, x}, \quad \phi' \cdot i(A+B) = R'_{\infty, x}.$$

Then

$$\pi_{k+1} \cdot i(L) = R_{k+1, x}, \quad \pi_{k+1} \cdot i(A+B) = R'_{k+1, x}.$$

Set $\psi = \phi \cdot i$ and $\psi' = \phi' \cdot i$. Then (11.32) holds. Since A, B are closed ideals of $A+B$, by Theorem 10.1 of [10], there exist formally integrable analytic Lie equations $N_k \subset R_k$, S_k, $B_k \subset R'_k$ satisfying (11.33), (11.34), (11.36) and (11.37). Then $S_k + B_k = R'_k$ and so (11.40) holds. From Lemma 11.3 and the relations $[A, A] = 0$ and $[A, B] = 0$, we deduce (11.38); moreover (11.39) holds since

$$[B_{k+1,x}, N_{k+1,x}] = \pi_k i[B, I] = 0$$

and $[\tilde{\mathcal{R}}'_{k+2}, \mathcal{N}_{k+1}] \subset \mathcal{N}_{k+1}$ by (11.36). Lemma 11.3 and (11.37) tell us that $N_k + B_k$ and $N_{k+1} + B_{k+1}$ are vector bundles. Lemma 10.3, (ii) of [10] says that $\pi_0 N_1$, $\pi_1 S_k$ and $\pi_0(N_k + B_k)$ are vector bundles. If I is abelian, by Lemma 11.3 it follows that N_k is an abelian Lie equation.

From Theorems 11.4, 11.1 and 11.2, we obtain:

THEOREM 11.5. *Let L be a transitive Lie algebra and I a closed abelian ideal of L. If there exist a fundamental subalgebra L^0 of L, closed subalgebras A, B of L such that A is abelian and*

$$L = L^0 + A + B, \quad [A, B] = 0, \quad [B, I] = 0,$$

then $H^j(L, I) = 0$ for $j > 0$ and $\tilde{H}^1(L, I) = 0$.

12. Prolongations of Lie equations

Let $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ be a formally transitive and formally integrable Lie equation and let $k \geqslant k_1$. Assume that Y is connected and let $P''_k \subset Q_k(Y)$ be a finite form of R''_k. Let $y_0 \in Y$ and consider $X = P''_k(y_0)$ as a bundle over Y by means of the target projection $\varrho: X \to Y$; it is a principal bundle with structure group $G = \varrho^{-1}(y_0)$. We may assume that X is connected. The Lie algebra $\mathfrak{g}$ of G is identified with V_{x_0}, where $x_0 = I_{Y,k}(y_0)$; the natural identification (5.23) gives us an anti-isomorphism of Lie algebras $R''^{0}_{k,y} \to \mathfrak{g}$.

A section ϕ of P''_k over an open set U of Y induces a mapping $\tau(\phi): X_{|U} \to X$ sending a into $\phi(\varrho(a)) \cdot a$; if ϕ is a section of $\tilde{\mathcal{P}}''_k$, this mapping is an immersion. If $a \in X_{|U}$, $g \in G$, then

$$(\tau(\phi) a) g = \tau(\phi)(ag). \tag{12.1}$$

The mapping τ induces for all $a \in X$ an isomorphism

$$\tau_a: \tilde{R}''_{k,\varrho(a)} \to T_a;$$

if ξ is a section of $\tilde{R}''_k$ over Y, then the vector field $\tau(\xi)$ on X defined by

$$\tau(\xi)(a) = \tau_a(\xi(\varrho(a))), \qquad a \in X,$$

is G-invariant and in fact every G-invariant vector field on X is of this form. The map τ induces a monomorphism of Lie algebras

$$\tau_a: \tilde{\mathcal{R}}''_{k,\varrho(a)} \to \mathcal{J}_{\varrho,\, a}. \tag{12.2}$$

Denote by

$$\varphi: T \to \tilde{R}''_k$$

the morphism of vector bundles over ϱ sending $\xi \in T_a$, with $a \in X$, into the unique element η of $\tilde{R}''_{k,\varrho(a)}$ satisfying $\tau_a(\eta) = \xi$. The induced mapping $\varphi: T \to \varrho^{-1} R''_k$ is an isomorphism of vector bundles; the morphism (3.1) of vector bundles

$$\varphi\colon J_l(T;\varphi) \to J_l(\tilde{R}''_k;\, Y) \tag{12.3}$$

over ϱ therefore induces an isomorphism

$$\varphi\colon J_l(T;\varphi) \to \varrho^{-1}J_l(\tilde{R}''_k;\, Y)$$

of vector bundles over X. The diagram

$$\begin{tikzcd} T \arrow[d, "\varphi"'] \arrow[dr, "\varrho"] & \\ \tilde{R}''_k \arrow[r, "\pi_0"] & T_Y \end{tikzcd}$$

is commutative; hence $\mathcal{J}_\varphi \subset \mathcal{J}_\varrho$ and $J_l(T;\varphi) \subset J_l(T;\varrho)$. Moreover $[\mathcal{J}_\varphi, \mathcal{J}_\varphi] \subset \mathcal{J}_\varphi$ and for all $a \in X$ the image of (12.2) belongs to $\mathcal{J}_{\varphi,a}$, and

$$\varphi\colon \mathcal{J}_{\varphi,a} \to \tilde{R}''_{k,\varrho(a)}$$

is an isomorphism of Lie algebras. Therefore

$$[J_l(T;\varphi), J_l(T;\varphi)] \subset J_{l-1}(T;\varphi).$$

Since R''_k is a Lie equation, the bracket (1.33) gives by restriction a bracket

$$J_l(\tilde{R}''_k;\, Y) \times_Y J_l(\tilde{R}''_k;\, Y) \to J_{l-1}(\tilde{R}''_k;\, Y) \tag{12.4}$$

and hence also a structure of Lie algebra on $J_\infty(\tilde{R}''_k;\, Y)_b$ for all $b \in Y$. From the above remarks, we see that

$$\varphi[\xi, \eta] = [\varphi\xi, \varphi\eta], \tag{12.5}$$

for all $\xi, \eta \in J_l(T;\varphi)$, where the right-hand side is defined in terms of the bracket (12.4). Thus for $a \in X$, the mappings (12.3) determine an isomorphism of Lie algebras

$$\varphi\colon J_\infty(T;\varphi)_a \to J_\infty(\tilde{R}''_k;\, Y)_{\varrho(a)}.$$

For $a \in X$, the mapping $G \to X$ sending g into $a \cdot g$ induces a canonical isomorphism

$$\iota_a\colon \mathfrak{g} \to V_a,$$

and a monomorphism of Lie algebras

$$\iota\colon \mathfrak{g} \to \Gamma(X, V)$$

which satisfies by (12.1)

$$[\tau(\xi), \iota(\eta)] = 0 \tag{12.6}$$

for all $\xi \in \Gamma(Y, \tilde{R}''_k)$, $\eta \in \mathfrak{g}$. Let $C_1 \subset J_1(V)$ be the formally integrable differential equation generated by the sections $\{j_1(\iota(\eta))\}_{\eta\in\mathfrak{g}}$ of $J_1(V)$. Since ι is a monomorphism, C_1 is a Lie equation; clearly, $\pi_0\colon C_1\to J_0(V)$ and $\pi_1\colon C_\infty\to C_1$ are isomorphisms and Sol $(C_1)_a\simeq\iota(\mathfrak{g})$ for all $a\in X$. From (12.6), it follows that

$$[J_{l+1}(T;\varphi), C_{l+1}]=0 \tag{12.7}$$

for all $l\geqslant 0$.

Let $N''_k\subset R''_k$ be a formally integrable Lie equation and let W be the sub-bundle of T whose fiber at $a\in X$ is equal to $\tau_a(\tilde{N}''_{k,\varrho(a)})$. Then φ induces a morphism of vector bundles

$$\varphi\colon W\to\tilde{N}''_k$$

over ϱ such that $\varphi\colon W\to\varrho^{-1}\tilde{N}''_k$ and $\tau_a\colon\tilde{\mathcal{N}}''_{k,\varrho(a)}\to\mathcal{W}_{\varphi,a}$ are isomorphisms, with $a\in X$. Thus $\mathcal{W}_{\varphi,a}$ is a Lie subalgebra of $\mathcal{T}_{\varphi,a}$ and we see that W is an integrable sub-bundle of T. Moreover, (12.3) restricts to give us a morphism of vector bundles

$$\varphi\colon J_l(W;\varphi)\to J_l(\tilde{N}''_k;\,Y) \tag{12.8}$$

over ϱ whose corresponding mapping

$$\varphi\colon J_l(W;\varphi)\to\varrho^{-1}J_l(\tilde{N}''_k;\,Y)$$

is an isomorphism.

For $l\geqslant 0$, let $N'_l\subset J_l(\tilde{N}''_k;\,Y)$ be the image under the map $\tilde{\lambda}_l\colon J_{k+l}(T_Y;\,Y)\to J_l(\tilde{J}_k(T_Y;\,Y);\,Y)$ of the l-th prolongation N''_{k+l} of N''_k. By Lemma 1.2, $N'_{l+1}=(N'_1)_{+l}$ and so N'_1 is formally integrable. According to the commutativity of (1.37), we have

$$[N'_{l+1}, N'_{l+1}]\subset N'_l, \tag{12.9}$$

for all $l\geqslant 1$, with respect to the bracket (1.33) or (12.4); moreover, the mappings $\tilde{\lambda}_l\colon N''_{k+l}\to J_l(\tilde{N}''_k;\,Y)$ induce, for all $b\in Y$, a monomorphism of Lie algebras $N''_{\infty,b}\to J_\infty(\tilde{N}''_k;\,Y)_b$ whose image is the Lie subalgebra $N'_{\infty,b}$ of $J_\infty(\tilde{N}''_k;\,Y)_b$ or of $J_\infty(\tilde{R}''_k;\,Y)_b$. Let $N_l\subset J_l(W;\varphi)$ be the inverse image of N'_l under the mappings (12.3) or (12.8), so that $N_l\subset J_l(W;\varrho)$ and

$$\varphi(N_{l,a})=N'_{l,\varrho(a)},\quad\text{for } l\geqslant 0,$$

and

$$\varrho(N_{l,a})=N''_{l,\varrho(a)},\quad\text{for } l\geqslant k,$$

for all $a\in X$. By Proposition 5, (i) of [6], N_1 is formally integrable and $N_{l+1}=(N_1)_{+l}$, for all $l\geqslant 1$. Since $\pi_k\colon N''_{k+1}\to N''_k$ is surjective, so is the mapping $\pi_0\colon N_1\to J_0(W)$. From (12.9) and (12.5), we deduce that

$$[N_{l+1}, N_{l+1}]\subset N_l$$

for all $l\geqslant 1$ and hence from Proposition 4.4 of [19] that N_1 is a Lie equation. Furthermore for all $a\in X$,

$$\varphi: N_{\infty,a} \to N'_{\infty,\varrho(a)},$$

$$\varrho: N_{\infty,a} \to N''_{\infty,\varrho(a)}$$

are isomorphisms of Lie algebras and

$$0 \longrightarrow N^0_{\infty,a} \longrightarrow N_{\infty,a} \xrightarrow{\pi_k \circ \varrho} N''_{k,\varrho(a)} \longrightarrow 0$$

is an exact sequence, where $N^0_{\infty,a}$ is the kernel of π_0: $N_{\infty,a} \to J_0(W)_a$.

In particular, if we apply these constructions to R''_k instead of N''_k, we see that the inverse image R_1 of $R'_1 = \bar{\lambda}_1(R''_{k+1})$ under (12.3) is a formally transitive and formally integrable Lie equation, that $N_1 \subset R_1$, that

$$\varphi: R_{\infty,a} \to R'_{\infty,\varrho(a)}, \tag{12.10}$$

$$\varrho: R_{\infty,a} \to R''_{\infty,\varrho(a)} \tag{12.11}$$

are isomorphisms of Lie algebras, and that $R^0_{\infty,a}$ is the kernel of $\varrho \circ \pi_k$: $R_{\infty,a} \to R''_{k,\varrho(a)}$ for all $a \in X$. If $a \in X$ and $L'' = R''_{\infty,\varrho(a)}$, $L''^0 = R''^0_{\infty,\varrho(a)}$, then $D^k_{L''} L''^0$ is the kernel of π_k: $L'' \to R''_{k,\varrho(a)}$ and so ϱ: $R^0_{\infty,a} \to D^k_{L''} L''^0$ is an isomorphism. Moreover

$$R_l \cap C_l = 0, \tag{12.12}$$

for all $l \geqslant 1$. Indeed, let $a \in X$ and $u \in (R_l \cap C_l)_a$; then there is an element η of $\mathfrak{g}$ such that $u = j_l(\iota(\eta))(a)$. Let $S_1 \subset C_1$ be the formally integrable differential equation generated by the section $j_1(\iota(\eta))$ of C_1. From (12.7) and Lemma 1.5, we deduce that

$$[\tilde{R}_{l+1}, S_l] \subset S_l.$$

Since $R_l \subset J_l(T)$ is a formally transitive Lie equation, we therefore see from (7.1) and Lemma 11.3 that $R_l \cap S_l$ is a vector bundle. As S_l is the line bundle generated by $j_l(\iota(\eta))$ and $(R_l \cap S_l)_a = S_{l,a}$, we conclude that $R_l \cap S_l = S_l$ over X and that $j_l(\iota(\eta))$ is a section of R_l over X. Hence $\iota(\eta)$ is a solution of R_1 over X and so $j_\infty(\iota(\eta))(a)$ belongs to $(R_\infty \cap C_\infty)_a$. Because $\iota(\eta)$ is a section of V, we have

$$\varrho j_\infty(\iota(\eta))(a) = 0;$$

since (12.11) is an isomorphism, we see that $j_\infty(\iota(\eta))(a) = 0$ and therefore that $\eta = 0$. Hence $u = 0$ and (12.12) holds.

From (12.12), it follows that $N_l + C_l \subset J_l(T)$ is a differential equation for all $l \geqslant 1$. Clearly, we have

$$N_{l+1} + C_{l+1} \subset (N_l + C_l)_{+1} \tag{12.13}$$

for all $l \geqslant 1$. For $l \geqslant 1$, let $h_l \subset S^l J_0(T)^* \otimes J_0(T)$ denote the kernel of π_{l-1}: $N_l \to J_{l-1}(T)$. By (12.12), h_{l+1} is equal to the kernel of the surjective map π_{l-1}: $N_{l+1} + C_{l+1} \to N_l + C_l$, for all

$l \geq 1$. Since h_l is 1-acyclic, for $l \geq 2$ the kernel of the surjective map π_l: $(N_l + C_l)_{+1} \to N_l + C_l$ is also equal to h_{l+1}. From (12.13), we conclude that

$$N_{l+1} + C_{l+1} = (N_l + C_l)_{+1}$$

for all $l \geq 2$. Using (12.7) and Proposition 4.4 of [19], we see that $N_2^{\#} = N_2 + C_2$ is a formally integrable Lie equation such that

$$N_{l+2}^{\#} = N_{l+2} + C_{l+2}, \quad \text{for } l \geq 0,$$
$$N_\infty^{\#} = N_\infty + C_\infty,$$

where

$$N_\infty \cap C_\infty = 0.$$

Let B_k'', $S_k'' \subset R_k''$ be formally integrable Lie equations and let B_1, $S_1 \subset R_1$ be the formally integrable Lie equations which are respectively the inverse images of $B_1' = \lambda_1(B_{k+1}'')$ and of $S_1' = \lambda_1(S_{k+1}'')$ under the map (12.3). If $B_{l+2}^{\#} = B_{l+2} + C_{l+2}$ for $l \geq 0$, the following relations are equivalent:

$$[N_{k+1}'', B_{k+1}''] \subset S_k'', \tag{12.14}$$

$$[N_{k+l+1}'', B_{k+l+1}''] \subset S_{k+l}'', \quad \text{for all } l \geq 0, \tag{12.15}$$

$$[N_{l+1}', B_{l+1}'] \subset S_l', \quad \text{for all } l \geq 1, \tag{12.16}$$

$$[N_{l+1}, B_{l+1}] \subset S_l, \quad \text{for all } l \geq 1, \tag{12.17}$$

$$[N_{l+1}, B_{l+1}^{\#}] \subset S_l, \quad \text{for all } l \geq 1. \tag{12.18}$$

Indeed, (12.14) and (12.15) are equivalent by Lemma 1.4 and the equivalence of (12.15) and (12.16) follows from the commutativity of (1.36); by (12.5), we see that (12.16) and (12.17) are equivalent. Finally, (12.7) implies that (12.18) is a consequence of (12.17). Therefore, according to Lemma 1.5, we have

$$[\tilde{\mathcal{R}}_{k+1}'', \mathcal{N}_k''] \subset \mathcal{N}_k'' \tag{12.19}$$

if and only if

$$[\tilde{\mathcal{R}}_{l+1}, \mathcal{N}_l] \subset \mathcal{N}_l, \quad \text{for all } l \geq 1. \tag{12.20}$$

If either (12.19) or (12.20) holds, then by (12.7)

$$[R_{l+1}^{\#}, N_{l+1}] \subset N_l, \quad \text{for all } l \geq 1,$$

and hence by Lemma 1.5

$$[\tilde{\mathcal{R}}_{l+1}^{\#}, \mathcal{N}_l] \subset \mathcal{N}_l, \quad \text{for all } l \geq 1.$$

Let us summarize some of the above results in

THEOREM 12.1. *Assume that Y is connected. Let $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ be a formally transitive and formally integrable Lie equation and let $k \geq k_1$. Then there exist a connected differentiable manifold X, a surjective submersion $\varrho: X \to Y$, a formally integrable Lie equation $C_1 \subset J_1(V)$, and for each formally integrable Lie equation $N''_k \subset R''_k$ a formally integrable ϱ-projectable Lie equation $N_1 \subset J_1(T; \varrho)$ such that:*

(i) *$\pi_0 \tilde{N}_1$ is an integrable sub-bundle W of T and $N_1 \subset J_1(W; \varrho)$;*

(ii) *N_1 is a prolongation of N''_k and the sequence*

$$0 \longrightarrow N^0_{\infty, a} \longrightarrow N_{\infty, a} \xrightarrow{\pi_k \circ \varrho} N''_{k, \varrho(a)} \longrightarrow 0$$

is exact, where $N^0_{\infty, a}$ is the kernel of $\pi_0: N_{\infty, a} \to J_0(T)_a$, for $a \in X$;

(iii) *the Lie equation $R_1 \subset J_1(T; \varrho)$ corresponding to R''_k is formally transitive and $N_1 \subset R_1$; if $a \in X$ and $L'' = R''_{\infty, \varrho(a)}$, $L''^0 = R''^0_{\infty, \varrho(a)}$, then*

$$\varrho: (R_{\infty, a}, R^0_{\infty, a}) \to (L'', D^k_{L''} L''^0)$$

is an isomorphism of pairs of topological Lie algebras;

(iv) *$\pi_0: C_1 \to J_0(V)$ is bijective and for $l \geq 0$*

$$[R_{l+1}, C_{l+1}] = 0, \quad R_{l+1} \cap C_{l+1} = 0,$$

$$[R_\infty, C_\infty] = 0, \quad R_\infty \cap C_\infty = 0;$$

(v) *$N^\#_2 = N_2 + C_2$ is a formally integrable Lie equation and $R^\#_2 = R_2 + C_2$ is a formally transitive and formally integrable Lie equation in $J_2(T; \varrho)$ with*

$$R^\#_\infty = R_\infty + C_\infty;$$

(vi) *if B''_k, $S''_k \subset R''_k$ are formally integrable Lie equations satisfying*

$$[N''_{k+1}, B''_{k+1}] \subset S''_k,$$

then the corresponding Lie equations B_1, $S_1 \subset R_1$ satisfy

$$[N_{l+1}, B^\#_{l+1}] \subset S_l,$$

for all $l \geq 1$, where $B^\#_2 = B_2 + C_2$;

(vii) *if $[\tilde{R}''_{k+1}, \mathcal{N}''_k] \subset \mathcal{N}''_k$, then $[\tilde{R}^\#_{l+1}, \mathcal{N}_l] \subset \mathcal{N}_l$ for all $l \geq 1$.*

The following result is an immediate consequence of Theorem 6.1 of [12]:

THEOREM 12.2. *Let L be a transitive Lie algebra and I a closed ideal of L. Then there is a nested sequence*

$$I = I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_k = 0 \tag{12.21}$$

of closed ideals of L such that, for each j, where $0 \leqslant j \leqslant k-1$, either I_j/I_{j+1} is abelian or there are no closed ideals of L properly contained between I_j and I_{j+1}.

We say that a sequence (12.21) satisfying the conditions described in Theorem 12.2 is a Jordan-Hölder sequence for (L, I) and that it is of length k. We define $l(L, I)$ to be the minimum of the lengths of Jordan-Hölder sequences for (L, I).

Let L be a transitive Lie algebra and L^0 a fundamental subalgebra of L. Following [10], we say that a closed ideal I of L is defined by a foliation in (L, L^0) if the only ideal I' of L satisfying

$$I \subset I' \subset I + L^0$$

is I itself. If L^k denotes the fundamental subalgebra $D_L^k L^0$ of L, then, according to Proposition 10.1 of [10], for any closed ideal I of L there is an integer $m \geqslant 0$ such that I is defined by a foliation in (L, L^m).

THEOREM 12.3. *Let L be a transitive Lie algebra, L^0 a fundamental subalgebra of L and A, B closed subalgebras of L. Assume that A is abelian and that*

$$L = L^0 + A + B,$$
$$[A, B] = 0.$$

Let I, J be closed ideals of L; suppose that $[B, I] = 0$. Then there exist a transitive Lie algebra $L^\#$, a fundamental subalgebra $L^{\#0}$ of $L^\#$, closed subalgebras $A^\#$, $B^\#$ of $L^\#$, a closed ideal $J^\#$ of $L^\#$ and, if $L'' = L^\#/J^\#$, monomorphisms $i: L \to L^\#$, $j: L/J \to L''$ of transitive Lie algebras such that:

(i) *$i(L)$ is a closed ideal of $L^\#$ and*

$$L^\# = i(L) + L^{\#0}, \tag{12.22}$$

$$L^\# = L^{\#0} + A^\# + B^\#, \tag{12.23}$$

$$i(J) = i(L) \cap J^\#, \tag{12.24}$$

$$[A^\#, A^\#] = 0, \quad [A^\#, B^\#] = 0, \tag{12.25}$$

$$[B^\#, i(I)] = 0 \tag{12.26}$$

and such that the diagram

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[d] & L^\# \arrow[d] \\ L/J \arrow[r, "j"] & L'' \end{tikzcd} \tag{12.27}$$

whose vertical arrows are the natural projections, is commutative;

(ii) *$J^{\#}$ is defined by a foliation in $(L^{\#}, L^{\#0})$, the image L''^0 of $L^{\#0}$ in L'' is a fundamental subalgebra of L'', the images A'', B'' of $A^{\#}$, $B^{\#}$ in L'' are closed subalgebras of L'' and $j(L/J)$ is a closed ideal of L'', and*

$$L'' = j(L/J) + L''^0, \tag{12.28}$$

$$L'' = L''^0 + A'' + B'', \tag{12.29}$$

$$[A'', A''] = 0, \quad [A'', B''] = 0, \tag{12.30}$$

$$[B'', j(I/J)] = 0; \tag{12.31}$$

(iii) *if I' is a closed ideal of L, then $i(I')$ is a closed ideal of $L^{\#}$ and*

$$l(L^{\#}, i(I')) = l(L, I'), \tag{12.32}$$

and we have an isomorphism of graded Lie algebras

$$H^*(L, I') \to H^*(L^{\#}, i(I')),$$

and an isomorphism of cohomology

$$\tilde{H}^1(L, I') \to \tilde{H}^1(L^{\#}, i(I'));$$

(iv) *if I' is a closed ideal of L containing J, then $j(I'/J)$ is a closed ideal of L'' and*

$$l(L'', j(I'/J)) = l(L/J, I'/J), \tag{12.33}$$

and we have an isomorphism of graded Lie algebras

$$H^*(L/J, I'/J) \to H^*(L'', j(I'/J))$$

and an isomorphism of cohomology

$$\tilde{H}^1(L/J, I'/J) \to \tilde{H}^1(L'', j(I'/J)).$$

Proof. Let Y be a simply connected analytic manifold, $y \in Y$ and let $R''_{k_1} \subset J_{k_1}(T_Y; Y)$ be a formally transitive and formally integrable analytic Lie equation, $N''_{k_1}, M''_{k_1}, S''_{k_1}, B''_{k_1} \subset R''_{k_1}$ formally integrable analytic Lie equations and $\psi: L \to R''_{\infty, y}$ an isomorphism of transitive Lie algebras such that

$$\psi(L^0) = R''^0_{\infty, y}, \quad \psi(I) = N''_{\infty, y}, \quad \psi(J) = M''_{\infty, y}, \quad [\tilde{R}''_{k_1+1}, \mathcal{N}''_{k_1}] \subset \mathcal{N}''_{k_1}, \quad [\tilde{R}''_{k_1+1}, \mathcal{M}''_{k_1}] \subset \mathcal{M}''_{k_1},$$
$$[S_{k_1+1}, S_{k_1+1}] = 0, \quad [S_{k_1+1}, B_{k_1+1}] = 0, \quad [B_{k_1+1}, N_{k_1+1}] = 0$$

and

$$R''_\infty = R''^0_\infty + S_\infty + B_\infty. \tag{12.34}$$

All these objects other than M''_{k_1} satisfying these conditions are given to us by Theorem 11.4, while the existence of M''_{k_1} follows from Theorem 10.1 of [10]. Let $k \geq k_1$ be an integer such that J is defined by a foliation in (L, L^k), where $L^k = D^k_L L^0$; the kernel of $\pi_k: R''_{\infty, y} \to R''_{k, y}$

is equal to $\psi(L^k)$. We now apply Theorem 12.1 to $R''_{k_1}\subset J_{k_1}(T_Y;\,Y)$ and obtain a connected differentiable manifold X, a surjective submersion $\varrho\colon X\to Y$, formally integrable Lie equations

$$R_1\subset J_1(T;\varrho),\quad C_1\subset J_0(V),$$
$$N_1\subset R_1,\quad M_1\subset R_1,$$
$$S_1\subset R_1,\quad B_1\subset R_1,$$
$$R_2^{\#}=R_2+C_2,\quad B_2^{\#}=B_2+C_2,$$

such that R_1, $R_2^{\#}$ are formally transitive, $\pi_0\colon C_1\to J_0(V)$ is an isomorphism,

$$R_\infty^{\#}=R_\infty+C_\infty,\tag{12.35}$$

$$[R_\infty, C_\infty]=0,\tag{12.36}$$

$$[\tilde{R}_{l+1}^{\#},\mathcal{N}_l]\subset\mathcal{N}_l,\quad[\tilde{R}_{l+1}^{\#},\mathcal{M}_l]\subset\mathcal{M}_l,\tag{12.37}$$

$$[S_l, S_l]=0,\quad [S_l, B_l]=0,\tag{12.38}$$

$$[B_{l+1}^{\#}, N_{l+1}]=0,\tag{12.39}$$

for all $l\geqslant 1$, and

$$\varrho(S_{l,a})=S''_{l,\varrho(a)},\quad \varrho(B_{l,a})=B''_{l,\varrho(a)},\tag{12.40}$$

for all $l\geqslant k$ and $a\in X$ and

$$\psi^{-1}\circ\varrho\colon (R_{\infty,x}, R^0_{\infty,x})\to(L, L^k)\tag{12.41}$$

is an isomorphism of pairs of topological Lie algebras and

$$\varrho\colon N_{\infty,x}\to N''_{\infty,y},$$
$$\varrho\colon M_{\infty,x}\to M''_{\infty,y}$$

are isomorphisms of Lie algebras for all $x\in\varrho^{-1}(y)$. Fix $x\in X$ with $\varrho(x)=y$; set

$$L^{\#}=R^{\#}_{\infty,x},\quad L^{\#0}=R^{\#0}_{\infty,x},\quad A^{\#}=S_{\infty,x},\quad B^{\#}=B^{\#}_{\infty,x},$$

and let $i\colon L\to L^{\#}$ be the composition

$$L\xrightarrow{\ \psi\ }R''_{\infty,y}\xrightarrow{\ \varrho^{-1}\ }R_{\infty,x}\longrightarrow R^{\#}_{\infty,x}.$$

Thus $i(L)$ is a closed ideal of $L^{\#}$ by (12.36) and

$$i(L^k)=L^{\#0}\cap i(L),$$
$$i(I)=N_{\infty,x},\quad i(J)=M_{\infty,x}.$$

Since R_1 is formally transitive, we have (12.22). From (12.34) and (12.40), it follows that

$$\pi_0S_1+\pi_0B_1+\pi_0C_1=J_0(T)$$

and hence that (12.23) holds. From (12.38) and (12.39), we deduce (12.25) and (12.26) respectively. If W denotes the integrable sub-bundle $\pi_0 \tilde{M}_1$ of T, then, since $M_1 \subset J_1(W)$, we have

$$M_{\infty,x} \subset R_{\infty,x} \cap J_\infty(W)_x.$$

By Proposition 5.4 of [9], $R_1^{\#} = \pi_1 R_2^{\#}$ is a Lie equation and $R_{l+1}^{\#} \subset (R_1^{\#})_{+l}$. Now (12.37) implies that

$$[\tilde{R}_1^{\#}, J_0(\mathcal{W})] \subset J_0(\mathcal{W}),$$

and hence by Lemma 10.5 of [10] that

$$[\tilde{R}_{l+1}^{\#}, J_l(\mathcal{W})] \subset J_l(\mathcal{W}), \quad \text{for all } l \geq 0,$$

and that $R_{\infty,x} \cap J_\infty(W)_x$ and

$$J^{\#} = L^{\#} \cap J_\infty(W)_x$$

are closed ideals of $R_{\infty,x}$ and $L^{\#}$ respectively. Clearly

$$R_{\infty,x} \cap J_\infty(W)_x \subset M_{\infty,x} + R^0_{\infty,x}.$$

Since J is defined by a foliation in (L, L^k) and (12.41) is an isomorphism, $M_{\infty,x}$ is defined by a foliation in $(R_{\infty,x}, R^0_{\infty,x})$ and so

$$M_{\infty,x} = R_{\infty,x} \cap J_\infty(W)_x,$$

and (12.24) holds. Thus j is a monomorphism of transitive Lie algebras and diagram (12.27) commutes, completing the proof of (i). Since

$$J_0(W)_x = \pi_0 M_{\infty,x} \subset \pi_0 J^{\#} \subset J_0(W)_x,$$

we have $\pi_0 J^{\#} = J_0(W)_x$ and, by Proposition 10.3, (iii) of [10], the closed ideal $J^{\#}$ of $L^{\#}$ is defined by a foliation in $(L^{\#}, L^{\#0})$. By Proposition 10.2 of [10], L''^0 is a fundamental subalgebra of L'' and the relations (12.28)–(12.31) follow from (12.22), (12.23), (12.25) and (12.26), and so (ii) holds. Since $i(L)$ is a closed subalgebra of $L^{\#}$ and (12.36) holds, if I' is a closed ideal of L, then $i(I')$ is a closed ideal of $L^{\#}$ and the image of $i(I')$ in L'' is therefore a closed ideal of L''. Conversely, a closed ideal of $L^{\#}$ contained in $i(L)$ is necessarily a closed ideal of $i(L)$ and its image in L'', which is a closed ideal of L'' contained in $j(L/J)$, is also a closed ideal of $j(L/J)$. The equalities (12.32) and (12.33) follow directly from the last remarks. As (12.22) and (12.28) hold, the isomorphisms of (iii) and (iv) are given to us by Theorem 13.2 of [10] and Theorem 10.5, completing the proof of the theorem.

13. The integrability problem

We now summarize some implications of the preceding sections of this paper relating to the integrability problem (vanishing of the non-linear cohomology), and we begin by listing the following three conjectures:

CONJECTURE I. *Let L be a transitive Lie algebra and I a non-abelian minimal closed ideal of L. Then $H^j(L, I)=0$ for $j>0$ and $\tilde{H}^1(L, I)=0$.*

CONJECTURE II. *Let L be a transitive Lie algebra and I a closed ideal of L. Let*

$$I = I_0 \supset I_1 \supset \dots \supset I_k = 0$$

be a Jordan-Hölder sequence for (L, I). If for each j for which I_j/I_{j+1} is abelian, where $0 \leqslant j \leqslant k-1$, we have

$$H^1(L/I_{j+1}, I_j/I_{j+1}) = 0,$$

then

$$H^1(L, I) = 0 \quad and \quad \tilde{H}^1(L, I) = 0.$$

CONJECTURE III. *Let L be a transitive Lie algebra and I a closed ideal of L. If there exist a fundamental subalgebra L^0 of L, closed subalgebras A, B of L such that A is abelian and*

$$L = L^0 + A + B,$$

$$[A, B] = 0, \quad [B, I] = 0,$$

then $H^j(L, I)=0$ for $j>0$ and $\tilde{H}^1(L, I)=0$.

We have:

THEOREM 13.1. *Conjecture* I *implies Conjecture* II.

THEOREM 13.2. *Conjecture* I *implies Conjecture* III.

Moreover, we shall sketch a method, based on the work of Guillemin [12], for proving Conjecture I. Before doing this or proving Theorems 13.1 and 13.2, we list some consequences of Conjecture III.

(a) *Let L be a transitive Lie algebra. If there exist a fundamental subalgebra L^0 of L and an abelian subalgebra A of L such that*

$$L = L^0 \oplus A,$$

then $H^j(L)=0$, $\tilde{H}^1(L)=0$ and $H^j(L, I)=0$, $\tilde{H}^1(L, I)=0$ for every closed ideal I of L and all $j>0$.

(b) *Assume that* X *is connected. Let* $R_k \subset J_k(T)$ *be a formally transitive and formally integrable Lie equation and* $N_k \subset R_k$ *a formally integrable Lie equation such that*

$$[\tilde{\mathcal{R}}_{k+1}, \mathcal{N}_k] \subset \mathcal{N}_k.$$

Let $x \in X$; *if there is a fundamental subalgebra* L^0 *of* $R_{\infty,x}$ *and an abelian subalgebra* A *of* $R_{\infty,x}$ *such that*

$$R_{\infty,x} = L^0 \oplus A,$$

then R_k, N_k *are integrable differential equations and*

$$H^j(N_k) = 0, \quad H^j(R_k) = 0, \quad \tilde{H}^1(N_k)_a = 0, \quad \tilde{H}^1(R_k)_a = 0,$$

for $j > 0$ *and all* $a \in X$. *If* $N_{\infty,x}$ *is abelian, then* N_k *is an abelian Lie equation and the structure of* N_k *is given by Theorem* 11.1.

Assertion (a) is obtained from Conjecture III by setting $B = 0$. The assertions of (b) concerning cohomology follow from (a) and Theorem 10.4, (ii). By Lemma 10.3, (ii) of [10], $\pi_0 N_k$ is a vector bundle; therefore, if N_k is abelian, the hypotheses of Theorem 11.1 hold for N_k.

From (a), we infer in particular that the integrability problem is solved for all Lie pseudogroups acting on $\mathbf{R}^n$ which contain the translations, a fortiori for all flat pseudogroups. Even if one were interested in proving only this result, one would be forced, by the necessity of performing prolongations, to introduce the subalgebra B, as is seen from § 12. In fact, as has been noted in the Introduction, under prolongation the subalgebra B, even if it is assumed initially to be zero, reappears and contains a subalgebra corresponding to transformations along the fibers of a principal bundle. Moreover, under prolongation the transitive Lie algebra L corresponds to a closed ideal of a transitive Lie algebra and hence, in studying the cohomology of transitive Lie algebras, one is forced to consider the cohomology of closed ideals of transitive Lie algebras.

Proof of Theorem 13.1. Considering the natural epimorphisms $\phi_j: L/I_{j+1} \to L/I_j$ and the exact sequences of ideals of L/I_{j+1} and L/I_j

$$0 \longrightarrow I_j/I_{j+1} \longrightarrow I/I_{j+1} \xrightarrow{\phi_j} I/I_j \longrightarrow 0,$$

for $0 \leqslant j \leqslant k-1$, by repeated applications of Theorem 13.1, (iii) of [10], we see that $H^1(L, I) = 0$ if $H^1(L/I_{j+1}, I_j/I_{j+1}) = 0$ for $0 \leqslant j \leqslant k-1$, and of Theorem 10.4, (iii) that $\tilde{H}^1(L, I) = 0$ if $\tilde{H}^1(L/I_{j+1}, I_j/I_{j+1}) = 0$ for $0 \leqslant j \leqslant k-1$. Since I_j/I_{j+1} is either a non-abelian minimal closed ideal or an abelian closed ideal of L/I_{j+1}, we have $H^1(L/I_{j+1}, I_j/I_{j+1}) = 0$ and $\tilde{H}^1(L/I_{j+1}, I_j/I_{j+1}) = 0$ according to Conjecture I or our hypothesis and Theorem 11.3.

Proof of Theorem 13.2. We prove III by induction on $l(L, I)$. If $l(L, I)=0$, then $I=0$ and the result is trivially true. Let $k\geqslant 1$; assume that Conjecture III holds for all closed ideals I of transitive Lie algebras L satisfying the conditions of Conjecture III with $l(L, I)<k$. Suppose that I is a closed ideal of a transitive Lie algebra L with $l(L, I)=k$ satisfying the conditions of Conjecture III. Consider a Jordan-Hölder sequence (12.21) for (L, I) of length k. Set $J=I_{k-1}$; then $H^j(L, J)=0$ for $j>0$ and $\tilde{H}^1(L, J)=0$ by Theorem 11.5 or Conjecture I according to whether J is an abelian ideal or a non-abelian minimal closed ideal of L. Clearly we have $l(L/J, I/J)=k-1$. Considering the exact sequence of ideals of L and L/J

$$0\to J\to I\to I/J\to 0,$$

by Theorem 13.1, (iii) of [10], we see that $H^j(L, I)=0$ if and only if $H^j(L/J, I/J)=0$ for $j>0$, and by Theorem 10.4, (iii) that $\tilde{H}^1(L, I)=0$ if $\tilde{H}^1(L/J, I/J)=0$. We now consider the objects obtained by applying Theorem 12.3 to L, L^0, A, B, I, J; by Theorem 12.3, (iv), we have isomorphisms

$$H^j(L/J, I/J)\to H^j(L'', j(I/J)),$$

$$\tilde{H}^1(L/J, I/J)\to \tilde{H}^1(L'', j(I/J))$$

for $j\geqslant 0$ and

$$l(L'', j(I/J))=l(L/J, I/J)=k-1.$$

By Theorem 12.3, (ii), the transitive Lie algebra L'' and its closed ideal $j(I/J)$ satisfy the conditions of Conjecture III, so that

$$H^j(L'', j(I/J))=0,\quad \tilde{H}^1(L'', j(I/J))=0,$$

for $j>0$, by our induction hypothesis. Therefore

$$H^j(L/J, I/J)=0,\quad \tilde{H}^1(L/J, I/J)=0$$

which implies that the conjecture holds for the closed ideal I of L.

Outline of a proof of Conjecture I. We begin by recalling briefly required algebraic facts, most of which are contained in Guillemin's paper [12]. The main result to be used is Guillemin's structure theorem; it essentially reduces the structure of non-abelian minimal closed ideals of (real) transitive Lie algebras to the determination of simple, non-abelian transitive Lie algebras (over the real numbers) and all of these are known.

Let E, F be linearly compact topological vector spaces over $\mathbf{R}$, whose topological duals we denote by E^*, F^*. We define $E\hat{\otimes} F$ to be the linearly compact topological vector space which is the topological dual of the algebraic tensor product $E^*\otimes F^*$ endowed with the discrete topology. We then have a natural mapping

$$E\otimes F\to E\hat{\otimes} F.$$

Let L be a transitive Lie algebra and I a non-abelian minimal closed ideal of L. Then, according to Proposition 7.1 of [12], I possesses a unique maximal closed ideal J of I. Thus $R = I/J$ is a simple transitive Lie algebra, i.e., it possesses no non-trivial ideals (see [12], Proposition 4.3).

We have decomposed our outline into six statements which we now list. Each of these statements requires a proof; after each statement, we indicate briefly a basis on which a proof of it depends.

(1) The Lie algebra Der (R) of continuous derivations of R is a transitive Lie algebra and R can be identified with a closed ideal of finite codimension of Der (R). Moreover, Der (R) possesses a fundamental subalgebra $\mathrm{Der}^0(R)$ such that $R^0 = R \cap \mathrm{Der}^0(R)$ is a fundamental subalgebra of R and

$$\mathrm{Der}\,(R) = R + \mathrm{Der}^0\,(R).$$

We remark that, in the case of a finite-dimensional, simple Lie algebra R, we have Der $(R) = R$.

A proof of (1) depends on the classification of infinite, simple transitive Lie algebras.

(2) The commutator ring K_R of R (i.e., the algebra of linear mappings $R \to R$ which commute with all the mappings ad ξ: $R \to R$ with $\xi \in R$) is equal to $\mathbf{R}$ or $\mathbf{C}$. Furthermore, Der (R) is a K_R-algebra and R is a K_R-subalgebra of Der (R).

By Proposition 4.4 of [12], K_R is a field which is a finite algebraic extension of $\mathbf{R}$; hence K_R is contained in the complex numbers $\mathbf{C}$. A proof that K_R is equal to $\mathbf{R}$ or $\mathbf{C}$ depends on the classification of infinite, simple transitive Lie algebras. For simplicity, we shall henceforth assume that $K_R = \mathbf{R}$.

Before stating (3), we recall some results which are known (and therefore require no proofs). Let N be the normalizer of J in L. By Proposition 6.2 of [12], N is open in L and is therefore of finite codimension in L. Let $U = (L/N)^*$ and let F be the ring of formal power series on the vector space U. If F^0 is the unique maximal ideal of F, the powers F^i of F^0 are the elements of a fundamental system of neighborhoods of 0 for the Krull topology on F. The ring F endowed with this topology is a linearly compact, topological vector space.

Let Der (F) be the Lie algebra of continuous derivations of F and let $\mathrm{Der}^i\,(F)$ be the subalgebra of Der (F) consisting of all elements u of Der (F) satisfying $u(F^0) \subset F^i$. Then $\{\mathrm{Der}^i\,(F)\}$ is a fundamental system of neighborhoods of 0 for a topology on Der (F) and, endowed with this topology, Der (F) is a transitive Lie algebra and $\mathrm{Der}^0\,(F)$ is a fundamental subalgebra of Der (F). Let Y be a differentiable manifold whose dimension is equal to that of U and let $y \in Y$; then (Der (F), $\mathrm{Der}^0\,(F)$) and $(J_\infty(T_Y;\, Y)_y,\, J^0_\infty(T_Y;\, Y)_y)$ are isomorphic pairs of topological Lie algebras.

Since R and $\operatorname{Der}(R)$ are Lie algebras and F is an associative algebra, the tensor products $R\otimes F$ and $\operatorname{Der}(R)\otimes F$ are Lie algebras. There are unique structures of topological Lie algebras on $R\hat{\otimes} F$ and $\operatorname{Der}(R)\hat{\otimes} F$ such that the mappings

$$R\otimes F\to R\hat{\otimes} F,\ \operatorname{Der}(R)\otimes F\to \operatorname{Der}(R)\hat{\otimes} F$$

are homomorphisms of Lie algebras.

(3) The Lie algebra $\operatorname{Der}(R\hat{\otimes} F)$ of continuous derivations of $R\hat{\otimes} F$ is a transitive Lie algebra; $\operatorname{Der}(F)$ can be identified with a closed subalgebra of $\operatorname{Der}(R\hat{\otimes} F)$ and $\operatorname{Der}(R)\hat{\otimes} F$ with a closed ideal of $\operatorname{Der}(R\hat{\otimes} F)$. Moreover

$$\operatorname{Der}(R\hat{\otimes} F)=(\operatorname{Der}(R)\hat{\otimes} F)\oplus \operatorname{Der}(F) \tag{13.1}$$

and

$$\operatorname{Der}^0(R\hat{\otimes} F)=(\operatorname{Der}^0(R)\hat{\otimes} F+\operatorname{Der}(R)\hat{\otimes} F^0)\oplus \operatorname{Der}^0(F) \tag{13.2}$$

is a fundamental subalgebra of $\operatorname{Der}(R\hat{\otimes} F)$. Furthermore, $R\hat{\otimes} F$ can be identified with a closed ideal of $\operatorname{Der}(R\hat{\otimes} F)$.

The decomposition (13.1) is analogous to Proposition 5.3 of [12] for the Lie algebra of all derivations of $R\hat{\otimes} F$; an argument similar to the proof of this proposition given in [12] is necessary.

(4) Let L'' be a closed subalgebra of $\operatorname{Der}(R\hat{\otimes} F)$ and M be the image of L'' under the projection of $\operatorname{Der}(R\hat{\otimes} F)$ onto $\operatorname{Der}(F)$ given by (13.1). If $R\hat{\otimes} F\subset L''$ and M is a transitive Lie algebra and if

$$\operatorname{Der}(F)=M+\operatorname{Der}^0(F),$$

then L'' is a transitive Lie algebra and

$$\operatorname{Der}(R\hat{\otimes} F)=L''+\operatorname{Der}^0(R\hat{\otimes} F). \tag{13.3}$$

A proof of (4) depends on (1) and (3).

(5) There is a continuous homomorphism of Lie algebras

$$\phi: L\to \operatorname{Der}(R\hat{\otimes} F)$$

such that $\phi(I)=R\hat{\otimes} F$ and (structure theorem)

$$\phi: I\to R\hat{\otimes} F \tag{13.4}$$

is an isomorphism and such that the composition of ϕ and the projection of $\operatorname{Der}(R\hat{\otimes} F)$ onto $\operatorname{Der}(F)$ given by (13.1) is a mapping $\lambda: L\to \operatorname{Der}(F)$ which takes N into $\operatorname{Der}^0(F)$, and the mapping

$$L/N\to \operatorname{Der}(F)/\operatorname{Der}^0(F) \tag{13.5}$$

induced by λ is an isomorphism.

A proof of (5) depends on arguments similar to those given in § 7 of [12].

(6) We have

$$H^j(\mathrm{Der}\,(R\hat{\otimes}F),\, R\hat{\otimes}F)=0 \quad \text{for } j>0 \quad \text{and} \quad \tilde{H}^1(\mathrm{Der}\,(R\hat{\otimes}F),\, R\hat{\otimes}F)=0. \tag{13.6}$$

Since the simple, infinite transitive Lie algebras are classified, by an explicit construction of formally integrable analytic Lie equations R_k, $N_k \subset J_k(T)$ on an analytic manifold X such that $N_k \subset R_k$ and R_k is formally transitive, $[\tilde{\mathcal{R}}_{k+1}, \mathcal{N}_k] \subset \mathcal{N}_k$, and such that the pairs of topological Lie algebras $(R_{\infty,x}, N_{\infty,x})$ and $(\mathrm{Der}\,(R\hat{\otimes}F), R\hat{\otimes}F)$ are isomorphic for all $x \in X$, a proof of (13.6) follows from Frobenius' or Darboux's theorem with parameters.

Finally, in order to deduce Conjecture I in the case $K_R=\mathbf{R}$, we see from (13.5) that $L''=\phi(L)$ satisfies the conditions of (4) and hence is a transitive Lie algebra satisfying (13.3). Therefore, by Theorem 13.2 of [10] and Theorem 10.5, we obtain isomorphisms

$$H^*(L'', R\hat{\otimes}F) \to H^*(\mathrm{Der}\,(R\hat{\otimes}F), R\hat{\otimes}F),$$

$$\tilde{H}^1(L'', R\hat{\otimes}F) \to \tilde{H}^1(\mathrm{Der}\,(R\hat{\otimes}F), R\hat{\otimes}F).$$

From (13.4), Corollary 13.1, (ii) of [10], and Theorem 10.4, (iv), we obtain isomorphisms

$$H^*(L, I) \to H^*(L'', R\hat{\otimes}F),$$

$$\tilde{H}^1(L, I) \to \tilde{H}^1(L'', R\hat{\otimes}F).$$

From (13.6) and the above isomorphisms, we obtain Conjecture I when $K_R=\mathbf{R}$. As for the case $K_R=\mathbf{C}$, the proof of the statement corresponding to (6) requires the Newlander-Nirenberg theorem.

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