# Inner functions and multipliers of Cauchy type integrals

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## § 1. Introduction

This paper deals with the space $\mathfrak{M}$ of all multipliers of the Cauchy type integrals in the unit disc $\mathbf{D}=\{z\in\mathbf{C}\colon |z|<1\}$ in the complex plane $\mathbf{C}$. To be more precise let $\mathbf{M}(\mathbf{T})$ be the Banach space of all finite Borel measures on $\mathbf{T}$ with the usual variation norm and let $\mathfrak{K}\mu$ be the Cauchy transform of a measure $\mu$ in $\mathbf{M}(\mathbf{T})$:

$$\mathfrak{K}\mu(z) \overset{\text{def}}{=} \int_{\mathbf{T}} \frac{d\mu(t)}{t-z}, \quad |z|<1.$$

The space of the Cauchy type integrals is the Banach space $K\overset{\text{def}}{=}\{f\colon \exists\mu\in\mathbf{M}(\mathbf{T}),\ f=\mathfrak{K}\mu\}$ with a natural norm

$$\|f\|_K \overset{\text{def}}{=} \inf\{\|\mu\|\colon f=\mathfrak{K}\mu,\ \mu\in\mathbf{M}(\mathbf{T})\}.$$

It is easy now to define the space of multipliers mentioned above in a correct way. Let $\mathfrak{M}$ be the class of holomorphic functions $\varphi$ in $\mathbf{D}$ satisfying

$$\|\varphi\|_{\mathfrak{M}} \overset{\text{def}}{=} \sup\{\|\varphi f\|_K\colon \|f\|_K \leqq 1\} < +\infty.$$

It is clear that $\mathfrak{M}$ is the Banach algebra with the norm $\|\cdot\|_m$ and it is easy to check that

$$\|\varphi\|_\infty \overset{\text{def}}{=} \sup\{|\varphi(z)|\colon z\in\mathbf{D}\} \leqq \|\varphi\|_{\mathfrak{M}}.$$

Therefore the identity map imbeds $\mathfrak{M}$ continuously in the algebra $\mathbf{H}^\infty$ of all uniformly bounded holomorphic functions in $\mathbf{D}$.

The study of the space $\mathfrak{M}$ was started in the papers of V. P. Havin [1], [2] and was continued in [3] and [4]. It turned out that the elements of $\mathfrak{M}$ have a surprising collection of properties. For example, radial limits $\lim_{r\to 1-0}\varphi(r\zeta)$ of any multiplier $\varphi$ exist everywhere on $\mathbf{T}$ and partial sums of the Taylor series of $\varphi$ are bounded uniformly in $\mathbf{D}$ ([3], [4]).

The purpose of this paper is to describe inner functions in $\mathfrak{M}$ (see the Theorem 1 below). We prove that the Blaschke products with the sequence of zeros satisfying the Frostman condition are the only possible inner functions in $\mathfrak{M}$. This description leads in a natural way to the description of all families of rational fractions $\left(\frac{1-|a_n|^2}{1-\bar{a}_n z}\right)_{n\geq 1}$, $|a_n|<1$, which form a symmetric basis in the closure of their linear span in $\mathfrak{M}$ (Theorem 2). At last, an application to the pointwise convergence of Fourier series of bounded functions of the first Baire class on $\mathbf{T}$ is given.

The space $\mathfrak{M}$ is interesting not only because of its importance for the study of Cauchy type integrals. It is also the chief ingredient of the description of Toeplitz operators bounded on the disc algebra $C_A$. Let us remind some definitions. Let $\mathbf{P}_+$ be the orthogonal projection of $L^2(\mathbf{T})$ onto $\mathbf{H}^2$ and let $\mathbf{P}_-=\mathbf{I}-\mathbf{P}_+$, $\mathbf{I}$ being the identity operator. For $\varphi$ in $L^\infty(\mathbf{T})$, the Toeplitz operator with symbol $\varphi$ is the operator $\mathbf{T}_\varphi$ on $\mathbf{H}^2$ defined by $\mathbf{T}_\varphi h=\mathbf{P}_+\varphi h$ and the Hankel operator with the same symbol is defined by the formula $H_\varphi h=\mathbf{P}_-\varphi h$, $h\in\mathbf{H}^2$. Clearly

$$\varphi h = H_\varphi h+T_\varphi h, \quad h\in\mathbf{H}^2.$$

**Lemma 1.1.** *Let* $\varphi\in\mathbf{H}^\infty$. *Then the operator* $\mathbf{T}_{\bar{\varphi}}$ *is bounded on* $C_A$ *(or equivalently on* $\mathbf{H}^\infty$*) iff* $\varphi\in\mathfrak{M}$. *Moreover*

$$\|\varphi\|_{\mathfrak{M}} = \|T_{\bar{\varphi}}\|.$$

See [4] for the proof of the lemma. This proof follows from the formula for the natural duality between the spaces $C_A$ and $K$. We shall write this duality in an antilinear way

$$(f, h) \overset{\text{def}}{=} \lim_{r\to 1-0}\int_{\mathbf{T}} f(r\xi)\overline{h(\xi)}\,dm(\xi), \quad f\in K,\ h\in C_A.$$

Here $m$ denotes the usual normalized Lebesgue measure on $\mathbf{T}$. The description of Toeplitz operators $T_\varphi$ bounded on $C_A$ (or $\mathbf{H}^\infty$) is now a simple corollary of Lemma 1.1. Indeed, let $\overline{\mathfrak{M}}_0 \overset{\text{def}}{=} \{\bar{\varphi}\colon \varphi\in\mathfrak{M},\ \hat{\varphi}(0)=0\}$ and let

$$\overline{\mathfrak{M}}_0+C_A \overset{\text{def}}{=} \{\varphi\in L^\infty(\mathbf{T})\colon \mathbf{P}_-\varphi\in\overline{\mathfrak{M}}_0,\ \mathbf{P}_+\varphi\in C_A\},$$

$$\overline{\mathfrak{M}}_0+\mathbf{H}^\infty \overset{\text{def}}{=} \{\varphi\in L^\infty(\mathbf{T})\colon \mathbf{P}_-\varphi\in\overline{\mathfrak{M}}_0,\ \mathbf{P}_+\varphi\in H^\infty\}.$$

Then $T_\varphi$ is bounded on $C_A$ (or $\mathbf{H}^\infty$) iff $\varphi\in\overline{\mathfrak{M}}_0+C_A$ ($\varphi\in\overline{\mathfrak{M}}_0+H^\infty$). It is curious that both spaces $\overline{\mathfrak{M}}_0+C_A$ and $\overline{\mathfrak{M}}_0+H^\infty$ are algebras. To see this it is obviously sufficient to prove that $\bar{\varphi}f\in\overline{\mathfrak{M}}_0+H^\infty$ for any $\varphi\in\mathfrak{M}_0$ and $f\in\mathbf{H}^\infty$. It follows from Lemma 1.1 that $\mathbf{P}_+\bar{\varphi}f\in\mathbf{H}^\infty$ and the formula $\mathbf{P}_+(\mathbf{P}_-\bar{\varphi}f)g=\mathbf{P}_+gH_{\bar{\varphi}}f=\mathbf{P}_+(g\bar{\varphi}f-gT_{\bar{\varphi}}f)=\mathbf{P}_+\bar{\varphi}gf-gT_{\bar{\varphi}}f\in\mathbf{H}^\infty$, $g\in\mathbf{H}^\infty$, implies the inclusion $\mathbf{P}_-\bar{\varphi}f\in\overline{\mathfrak{M}}_0$. With the norm

$$\|\varphi\| \overset{\text{def}}{=} \|\overline{\mathbf{P}_-\bar{\varphi}}\|_{\mathfrak{M}}+\|\mathbf{P}_+\varphi\|_\infty$$

the space $\overline{\mathfrak{M}}_0+\mathbf{H}^\infty$ becomes a Banach space closed under the pointwise multiplication of functions on $\mathbf{T}$. Therefore there is an equivalent norm on $\mathfrak{M}_0+\mathbf{H}^\infty$ such that $\mathfrak{M}_0+\mathbf{H}^\infty$ is a Banach algebra with this norm. It certainly is not a uniform subalgebra of $L^\infty(\mathbf{T})$.

**Definition.** *Let* $a=(a_n)_{n\geqq 1}$ *be a sequence (finite or infinite) of points of the unit disc* $\mathbf{D}$ *satisfying the Blaschke condition*

$$\sum_{n\geqq 1}(1-|a_n|^2)<+\infty, \tag{B}$$

*and let* $B^a$ *denote the corresponding Blaschke product*

$$B^a(z)\overset{\mathrm{def}}{=}\prod_{n=1}^{\infty}\frac{|a_n|}{a_n}\cdot\frac{a_n-z}{1-\bar a_n z}.$$

*A sequence* $a$ *is named the Frostman sequence (briefly* $a\in(F)$*) if*

$$\sup_{\zeta\in\mathbf{T}}\sum_{n=1}^{\infty}\frac{1-|a_n|^2}{|\zeta-a_n|}<+\infty. \tag{F}$$

**Theorem 1.** *Let* $I$ *be an inner function. Then* $I\in\mathfrak{M}$ *iff* $I$ *is a Blaschke product* $B^a$ *and* $a\in(F)$.

*Remark.* The sufficiency of the condition $a\in(F)$ for the inclusion $B^a\in\mathfrak{M}$ had been proved for the first time in [3] and was later proved independently in [5]. The second fact we shall use below is that the inclusion $B^a\in\mathfrak{M}$ implies $a\in(F)$ if $B^a$ is an interpolating Blaschke product [4]. For the sake of completeness of the exposition we shall give simple proofs for both of them.

Theorem 1 may be compared with the theorem describing inner functions in the multiplier space of Cauchy type integrals with uniformly bounded densities on $\mathbf{T}$. It was proved independently and by different tools in [6] and [7]. For $p$, $1\leqq p\leqq\infty$, the Banach space $L^p(\mathbf{T})$ of all functions on $\mathbf{T}$ summable with the power $p$ is imbedded in $\mathbf{M}(\mathbf{T})$ in a natural way: $f\to f\,dm$. Let $\mathfrak{M}(\mathfrak{K}L^p)$ be the space of all multipliers of the space $\mathfrak{K}L^p$. The following formulae hold

$$\mathfrak{M}(\mathfrak{K}L^1)=\mathfrak{M}(\mathfrak{K}\mathbf{M}(\mathbf{T}))\overset{\mathrm{def}}{=}\mathfrak{M},$$

$$\mathfrak{M}(\mathfrak{K}L^\infty)=\mathfrak{M}(\mathfrak{K}C(\mathbf{T}))\overset{\mathrm{def}}{=}\mathfrak{M}^\infty,$$

$$\mathfrak{M}(\mathfrak{K}L^p)=\mathbf{H}^\infty,\quad 1<p<\infty.$$

**Theorem** (see [6], [7]). *Let* $I$ *be an inner function and let* $I\in\mathfrak{M}^\infty$. *Then* $I$ *is a finite Blaschke product.*

Theorem 1 is also connected with the well-known Frostman theorem.

**Frostman theorem** (see [8], p. 54—55). *Let* $\zeta\in\mathbf{T}$. The following assertions are equivalent:

1. *the Blaschke product* $B^a$ *and all its subproducts have radial limits at* $\zeta$ *and moduli of all these limits are equal to one;*

2. $\sum_n \frac{1-|a_n|^2}{|\zeta-a_n|} < +\infty$.

To formulate our second theorem we recall some definitions of the unconditional bases theory. Let $\sigma(\Lambda)$ denote the set of all permutations of a set $\Lambda$.

**Definition.** *An unconditional basis* $(e_\lambda)_{\lambda\in\Lambda}$ *in a Banach space* $X$ *is said to be symmetric if and only if there is a positive constant* $C$ *such that*

$$\sup_{\sigma\in\sigma(\Lambda)} \left\|\sum_{\lambda\in\Lambda} \alpha_{\sigma\lambda} e_\lambda\right\|_X \leqq C\cdot\left\|\sum_{\lambda\in\Lambda}\alpha_\lambda e_\lambda\right\|_X$$

*for every complex function* $\lambda\to\alpha_\lambda$ *with a finite support in* $\Lambda$. *The coefficient space* $I(\Lambda)$ *of an unconditional basis* $(e_\lambda)_{\lambda\in\Lambda}$ *is defined to be the space of all families* $(\alpha_\lambda)_{\lambda\in\Lambda}$ *of complex numbers satisfying*

$$\|\alpha\|_{I(\lambda)} \overset{\text{def}}{=} \left\|\sum_{\lambda\in\Lambda}\alpha_\lambda e_\lambda\right\| X < +\infty.$$

The definitions of the Stolz domain and of the separated sequence used in the statement of the theorem may be found in § 3 of the paper.

**Theorem 2.** *A family* $\left(\frac{1-|\lambda|^2}{1-\bar\lambda z}\right)_{\lambda\in\Lambda} \subset \Lambda\subset \mathbf{D}$ *of the rational fractions forms a symmetric basis in the closure of their linear span in* $\mathfrak{M}$ *if and only if one of two following possibilities occurs.*

1. *There is a separated Frostman sequence* $a=(a_n)_{n\geqq 1}$ *such that* $\Lambda=\{a_n\colon n\geqq 1\}$.

2. *The set* $\Lambda$ *can be covered by a finite number of Stolz domains and* $\Lambda=\{\alpha_n\colon n\geqq 1\}$ *for a separated sequence* $a$. *The coefficient space* $I(\Lambda)$ *coincides with* $c_0(\Lambda)$ *if the first of the mentioned possibilities takes place and* $I(\Lambda)=l^1(\Lambda)$ *if the second one is occured.*

Let us remark that the family $\left(\frac{1-|a_n|^2}{1-\bar a_n z}\right)_{n\geqq 1}$ forms an unconditional basis in its linear span in $\mathbf{H}^\infty$ iff $a=(a_n)_{n\geqq 1}$ is separated Frostman sequence. In this case the coefficient space is $c_0$ (see § 5).

The methods of this article are applicable to the construction of some examples if discontinuous functions on the unit circle $\mathbf{T}$ with a good behavior of their Fourier series.

**Theorem 3.** *Let* $E$ *be a closed nowhere dense subset of the circle* $\mathbf{T}$. *Then there is a Blaschke product* $B$ *with the following list of properties.*

1. The infinite product the function $B$ is defined by convergence at every point of the closed disc.

2. The set of the discontinuity points of $B$ is precisely the set $E$.

3. The Fourier series of $B$ converges everywhere on $\mathbf{T}$ and $\hat{B}(n)=O\left(\frac{1}{n}\right)$, $\hat{B}(n)$ being the $n$-th Fourier coefficient of $B$.

The work is presented in six sections. In § 2 the necessary information about the space $\mathfrak{M}$ is collected. In § 3 relations of the Frostman condition with the interpolation theory in $\mathbf{H}^\infty$ are analyzed and, at last, in §§ 4—6 the proofs of our theorems are given.

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## § 2. The multipliers of the Cauchy type integrals

In this section we describe some auxiliary results about the space $\mathfrak{M}$ needed in what follows. Details may be found in [3], [4].

**2.1.** Let $\delta_\zeta$ be a unit mass at the point $\zeta$ of $\mathbf{T}$. The convex hull of the set $\{\delta_\zeta\colon \zeta\in\mathbf{T}\}$ is a weak-star dense subset of the unit ball in $\mathbf{M}(\mathbf{T})$. This, together with the obvious identity $(\zeta-z)^{-1}=\mathfrak{K}\delta_\zeta(z)$, implies that

$$\|\varphi\|_{\mathfrak{M}} = \sup\left\{\left\|\frac{\varphi(z)}{\zeta-z}\right\|_K\colon \zeta\in\mathbf{T}\right\}.$$

**2.2.** It follows from Lemma 1.1 that $\|z^n\|_{\mathfrak{M}}=\|T_{\bar{z}^n}\|$. Therefore it is easy to check that

$$\tfrac{1}{6}\log n \leqq \|z^n\|_{\mathfrak{M}} \leqq 1+\log n, \quad n\in\{2, 3, \dots\}.$$

**2.3.** The formula for the norm $\|\varphi\|_{\mathfrak{M}}$ mentioned above entails the inequality

$$\|f\|_{\mathfrak{M}} \leqq \|f\|_\infty+\sup_{\zeta\in\mathbf{T}}\int_{\mathbf{T}}\left|\frac{f(t)-f(\zeta)}{t-\zeta}\right| dm(t)$$

for every $f$ in $C_A$. Let us assume the function $f$ to be differentiable in the closed disc. The integral in the right-hand side of the inequality can be divided into two parts corresponding to non-overlapping arcs $\Gamma_1=\{t\in\mathbf{T}\colon |\arg t-\arg\zeta|<\pi/n\}$ and $\Gamma_2=\mathbf{T}\setminus\Gamma_1$, $n$ being a positive integer. Simple estimates show that the following inequality is valid

$$\|f\|_{\mathfrak{M}} \leqq \frac{1}{n}\|f'\|_\infty+(\log ne)\cdot\|f\|_\infty. \tag{1}$$

**Lemma 2.4.** *Let* $\alpha\in\mathbf{T}$, *let* $\alpha\in\mathbf{D}$, *and let* $\gamma(z)\overset{\text{def}}{=}\alpha\dfrac{a-z}{1-\bar{a}z}$ *denote a conformal automorphism of the disc. Then the following inequalities hold*

$$4^{-1}\cdot\|\varphi\|_{\mathfrak{M}}\leqq\|\varphi\circ\gamma\|_{\mathfrak{M}}\leqq 4\cdot\|\varphi\|_{\mathfrak{M}}.$$

*Proof.* It is sufficient to test the left-hand inequality only. Formula 2.1 shows that

$$\|\varphi\|_{\mathfrak{M}}=\sup\left\{\left|\int_{\mathbf{T}}\frac{\varphi(z)}{1-\eta z}\,\overline{h(z)}\,dz\right|:\ |\eta|=1,\ \|h\|_\infty\leqq 1,\ h(0)=0\right\},$$

and the change of variables $z\to\gamma(z)$ implies

$$\int_{\mathbf{T}}\frac{\varphi(z)}{1-\eta z}\,\overline{h(z)}\,dz=\int_{\mathbf{T}}\frac{\varphi\circ\gamma(z)}{1-\eta\gamma(z)}\cdot\gamma'(z)\left\{\overline{h(\gamma(z))}-\overline{h(\gamma(0))}\right\}dz.$$

It may be assumed without loss of generality that $\alpha=1$. Then

$$\frac{\gamma'(z)}{1-\eta\gamma(z)}=\frac{\bar{\eta}}{\left(\dfrac{\eta+\bar{a}}{1+a\eta}\right)^{-1}-z}-\frac{\bar{\eta}}{1/\bar{a}-z}$$

and therefore $\left\|(1-\eta\gamma(z))^{-1}\cdot\gamma'(z)\right\|_k\leqq 2$. To finish the proof it remains to remark that $\|h\circ\gamma-h\circ\gamma(0)\|_\infty\leqq 2\|h\|_\infty$. ●

**2.5.** The division theorem. Let $\varphi$ belong to $\mathfrak{M}$ and $I$ be an inner function dividing $\varphi$ (i.e. $\varphi\cdot I^{-1}\in\mathbf{H}^\infty$). Then $\varphi\cdot I^{-1}\in\mathfrak{M}$ and moreover

$$\|\varphi\cdot I^{-1}\|_{\mathfrak{M}}\leqq\|\varphi\|_{\mathfrak{M}}.$$

Here is a simple proof of this theorem due to P. Nikolov [7]:

$$\|\varphi\cdot I^{-1}\|_{\mathfrak{M}}=\|T_{\bar{\varphi}I}\|=\sup\{\mathbf{P}_+\bar{\varphi}Ih\|_\infty:\ h\in\mathbf{H}^\infty,\ \|h\|_\infty\leqq 1\}$$
$$\leqq\sup\{\|\mathbf{P}_+\bar{\varphi}h\|_\infty:\ \|h\|_\infty\leqq 1\}=\|\varphi\|_{\mathfrak{M}}.\quad\bullet$$

## § 3. The Frostman condition and the interpolation in $\mathbf{H}^\infty$

We begin with some definitions of the interpolation theory. The sequence $(a_n)_{n\geqq 1}$ of pairwise distinct points in $\mathbf{D}$ is said to be an interpolating sequence for $\mathbf{H}^\infty$ iff for every bounded sequence $(x_n)_{n\geqq 1}$ there is a function $f$ in $\mathbf{H}^\infty$ satisfying

$$x_n=f(a_n),\quad n\in\{1,2,\ldots\}.\tag{1}$$

It is well known that (1) can be reformulated in purely geometric terms. To do this let $a$ be a sequence of points in $\mathbf{D}$ and let

$$\mu_a=\sum\nolimits_{n\geqq 1}(1-|a_n|^2)\delta_{a_n}.$$

Let $\mathbf{D}(\zeta, r)$ denote the disc $\{z\in\mathbf{C}\colon |z-\zeta|<r\}$. A non-negative measure $\mu$ in $\mathbf{D}$ is called a Carleson measure if

$$\gamma(\mu) \overset{\text{def}}{=} \sup\left\{r^{-1}\cdot\mu(D(\zeta, r))\colon \zeta\in\mathbf{T},\ r>0\right\} < +\infty.$$

The pseudo-euclidean distance $\varrho(a, b)$ between points $a$ and $b$ of $\mathbf{D}$ is defined by the formula

$$\varrho(a, b) = \left|\frac{a-b}{1-\bar{a}b}\right|.$$

The sequence $a=(a_n)_{n\geqq 1}$ is named separated if

$$S(a) = \inf_{n\neq m} \varrho(a_n, a_m) > 0.$$

We denote by $B_n^a$ the Blaschke product with the zero set $\{a_k\colon k\neq n\}$.

**Carleson interpolation theorem.** *The following conditions on $a$ are equivalent:*

1) *$a$ is an interpolating sequence;*

2) $\delta(a)=\inf_n |B_n^a(a_n)|>0$;

3) *$a$ is separated and $\mu_a$ is a Carleson measure. If $a$ is the interpolating sequence then $\delta(a)\geqq\exp\{-\mathrm{Const.}\ \gamma(\mu_a)\cdot s^{-2}(a)\}$ and there is a solution $f$ of (1) in $\mathbf{H}^\infty$ such that*

$$\|f\|_\infty \leqq \mathrm{Const.}\left(1+\delta(a)^{-1}\cdot\log\delta^{-1}(a)\right).$$

See, for example, [9] and [10] for the proof. The references on the original publication may be found there also.

A sequence $a$ satisfying

$$\sup_{\zeta\in\mathbf{D}} \mathrm{Card}\{n\colon \varrho(a_n, \zeta) < \varepsilon\} < +\infty \tag{2}$$

for some positive $\varepsilon$ is a disjoint union of a finite family of separated ones. This fact is of course well-known. Nevertheless we are going to prove it because of its importance for the proof of Theorem 1 and because we have failed to find a reference.

**Lemma 3.1.** *Let $X$ be a metric space endowed with a metric $d$, let $D(x,\varepsilon)\overset{\text{def}}{=}\{y\in X\colon d(x, y)<\varepsilon\}$ and let $E$ be a subset of $X$ satisfying*

$$n = \sup_{x\in E} \mathrm{Card}\left(E\cap D(x,\varepsilon)\right) < +\infty.$$

*Then there is a finite partition $(E_k)_{k=1}^n$ of $E$ such that*

$$\inf\{d(x, y)\colon x, y\in E_k,\ x\neq y\} \geqq 2^{-n}\cdot\varepsilon$$

*for every $k$, $k\in\{1, 2, \dots, n\}$.*

*Proof.* Let $(D(x, \varepsilon/2))_{x\in A_1}$ be a maximal family of pairwise disjoint balls such that $A_1 \subset E$ and

$$\operatorname{Card}(E \cap D(x, \varepsilon/2)) = n.$$

The existence of the family is a simple corollary of Zorn's lemma.

It follows from the triangle inequality that

$$G \cap D(x, \varepsilon/2) = \emptyset \tag{3}$$

for $G = \bigcup_{x\in A_1} D(x, \varepsilon/2)$ and for every $x$ in $E \setminus G$. This implies the inequality $d(x, y) \geqq \varepsilon/2$ for every $x$ in $E \setminus G$ and for every $y$ in $G \cap E$. Another consequence of (3) is that

$$\operatorname{Card}((E \cap G) \cap D(x, \varepsilon/2)) \leqq n-1$$

for $x$ in $E \setminus G$. Indeed, the opposite inequality contradicts the assumption of the maximality of the family $(D(x, \varepsilon/2))_{x\in A_1}$ if it holds for some $x$ in $E \setminus G$.

We see now that our construction can be proceeded by induction. We get after the $n$ step induction procedure the family $A_1, A_2, \ldots, A_n$ of subsets in $E$ satisfying the following conditions:

(a) $E \subset \bigcup_{k=1}^{n} \bigcup_{x\in A_k} D(x, \varepsilon \cdot 2^{-k})$;

(b) $\operatorname{Card}(E \cap D(x, \varepsilon \cdot 2^{-k})) \leqq n-k$ for every $x$ in $A_k$;

(c) the distance between different balls of the family

$$\mathfrak{F} \overset{\text{def}}{=} \{D(x, \varepsilon \cdot 2^{-k}) \colon x \in A_k,\ k \in \{1, \ldots, n\}\}$$

is more than $2^{-n}\varepsilon$. Let now $E_1$ be a subset of $E$ which has at most one point in common with every ball of the family $\mathfrak{F}$. The set $E_2$ is then a subset of $E \setminus E_1$ with the same property. The induction completes the construction of the partition $(E_k)_{k=1}^{n}$.

**Corollary 3.2.** *Suppose a sequence a satisfies* (2) *with some positive constant* $\varepsilon$. *Then a is the disjoint union of a finite number of separated sequences.*

*Proof.* We may assume without loss of generality that $a_n \neq a_m$ if $n \neq m$. To finish the proof it is sufficient to apply Lemma 3.1. to the metric space $\mathbf{D}$ with the non-euclidean metric $d(z, w) = \log \dfrac{1+\varrho(z, w)}{1-\varrho(z, w)}$ and to the set $E = \{a_n \colon n = 1, 2, \ldots\}$.

**Corollary 3.3.** *Let* $\mu_a$ *be a Carleson measure. Then the sequence a is a finite union of the interpolating sequences.*

The proof follows immediately from the corollary. See [11] for other proof.

**Lemma 3.4.** *Let a be a Frostman sequence in* $\mathbf{D}$. *Then* $\mu_a$ *is a Carleson measure.*

The proof hinges on a simple inequality:

$$\mu_a(D(\zeta, r)) = r \cdot \sum_{|a_n-\zeta|\leqq r} \frac{1-|a_n|^2}{r} \leqq \sigma_P(a) \cdot r. \quad \bullet$$

Let $0<\alpha<\pi/2$ and let $\zeta\in\mathbf{T}$. Let us recall that the Stolz domain $\Omega_\alpha(\zeta)$ in $\mathbf{D}$ is the interior of the convex hull of $\zeta$ and the circle $\{z\in\mathbf{C}\colon |z|=\sin\alpha\}$.

**Lemma 3.5.** *Let $a$ be a Frostman sequence in* $\mathbf{D}$. *Then*

$$\operatorname{Card}\{n\colon a_n\in\Omega_\alpha(\zeta)\} \leqq (1-\sin\alpha)^{-1}\sigma_F(a)$$

*for every Stolz domain $\Omega_\alpha(\zeta)$.*

*Proof.* It follows from the identity

$$|z|^2 = 1+|z-\zeta|^2-2|z-\zeta|\cos\theta$$

that

$$\frac{1-|z|^2}{|z-\zeta|^2} = 2\cos\theta-|z-\zeta|.$$

The length of the chord of $\mathbf{T}$ passing through the points $\zeta$ and $z$ obviously equals to $2\cos\theta$ and the radius of the circular part of $\partial\Omega_\alpha(\zeta)$ equals to $\sin\alpha$. Therefore $2\cos\theta-|z-\zeta|\geqq 1-\sin\alpha$. $\bullet$

**Lemma 3.6.** *Let $a=(a_n)_{n\geqq 1}$ be a sequence in $\mathbf{C}_+ \overset{\text{def}}{=} \{z\colon \operatorname{Im} z>0\}$ satisfying*

$$c(\alpha) = \sup_{t\in\mathbf{R}} \operatorname{Card}\{a_n\in\Omega_\alpha(t)\} < +\infty. \tag{4}$$

*Then $\mu_a$ is a Carleson measure.*

*Remark.* A similar lemma holds for the case of the unit disc.

*Proof.* Let $J_n$ denote the interval $(\operatorname{Re} a_n-\operatorname{Im} a_n\cdot\operatorname{tg}\alpha,\ \operatorname{Re} a_n+\operatorname{Im} a_n\cdot\operatorname{tg}\alpha)$ (see Fig. 1):

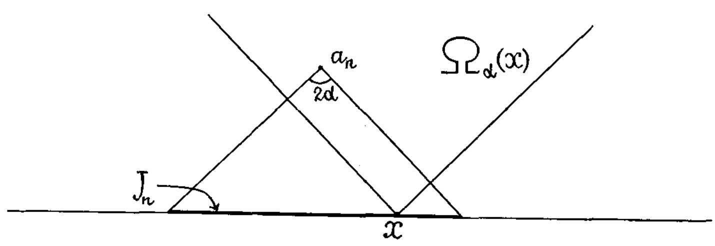


Fig. 1

The condition (4) implies that

$$\operatorname{Card}\{n\colon\ x\in J_n\}\leqq c(\alpha).$$

Let now $\mathbf{1}_n$ denote the indicator function of the set $J_n$ and let $t$ be a point in $\mathbf{R}$. Then

$$\mu_a(D(t,r))\leqq\frac{1}{2\operatorname{tg}\alpha}\cdot\sum\nolimits_{J_k\subset(t-r/\cos\alpha,\ t+r/\cos\alpha)}mJ_k$$

$$=\frac{1}{2\operatorname{tg}\alpha}\int_{t-r/\cos\alpha}^{t+r/\cos\alpha}\sum\nolimits_{k=1}^{\infty}\mathbf{1}_k\,dt\leqq\frac{1}{2\operatorname{tg}\alpha}\cdot c(\alpha)\cdot\frac{2r}{\cos\alpha}=c(\alpha)\cdot\frac{r}{\sin\alpha}.\quad\bullet$$

To finish with the Frostman condition let us remark that $\sigma_F(a)$ coincides with the norm of the embedding operator of the space $K$ into $L^1(\mu_a)$.

We shall need also the following lemma concerning conditions $(C)$ and $(R)$.

**Lemma 3.7.** *Let $a$ — be a sequence in a Stolz domain $\Omega_\alpha(\zeta)$. Then the conditions $(C)$ and $(R)$ are equivalent. Moreover the constant $\delta(a)$ depends on $\alpha$ and $s(a)$ only.*

See [9] for the proof.

## § 4. The proof of Theorem 1

**4.1.** We shall prove at first that $B^a\in\mathfrak{M}\Rightarrow a\in(F)$ assuming $a$ is interpolating sequence. The proof is based on the following lemma.

**Lemma 4.1.** *Let $a=(a_n)_{n\geqq1}$ be a sequence in $\mathbf{D}$, let $(x_n)_{n\geqq1}$ be a sequence satisfying $\sum_{n\geqq1}|x_n|(1-|a_n|)<+\infty$, and let $\varphi$ be a function in $\mathbf{H}^\infty$ such that*

$$\varphi(z)=\sum\nolimits_{n\geqq1}x_n\cdot\frac{1-|a_n|^2}{1-\bar a_n z},\quad |z|<1.$$

*Then the following inequalities hold*

1°. $\displaystyle\|\varphi\|_{\mathfrak{M}}\leqq\|\varphi\|_\infty+\sup_{\zeta\in\mathbf{T}}\sum\nolimits_{n\geqq1}|x_n|\cdot|a_n|\cdot\frac{1-|a_n|^2}{|1-\bar a_n\zeta|}$;

2°. $\displaystyle\sup_{\zeta\in\mathbf{T}}\sum\nolimits_n|x_n|\cdot|a_n|\cdot\frac{1-|a_n|^2}{|1-\bar a_n\zeta|}\leqq\text{Const.}\left(\delta(a)^{-1}\log\delta^{-1}(a)+1\right)\cdot\|\varphi\|_{\mathfrak{M}}$.

The lemma has been proved in [4]. We shall give its short proof for the sake of completeness of exposition, but let us stop for a moment to explain why this lemma implies the assertion stated at the beginning of the paragraph. If $a$ is an interpolating sequence the Blaschke product $B^a$ is the sum of the simple fractions:

$$B^a(z)=\frac{1}{\overline{B(0)}}-\sum\nolimits_{n\geqq1}\frac{1}{\overline{B_n(a_n)}}\cdot\frac{1}{|a_n|}\cdot\frac{1-|a_n|^2}{(1-\bar a_n z)}.\tag{5}$$

The required statement follows now from 2° with $x_n=(\overline{B_n(a_n)}\cdot|a_n|)^{-1}$, $n\in\{1,2,\ldots\}$. Conversely, if $\sigma_F(a)<+\infty$ then it follows from 1° and (5) that

$$\|B^a\|_{\mathfrak{M}}\leqq 1+\delta(a)^{-1}\cdot\sigma_P(a).$$

The sufficiency part of Theorem 1 follows from this inequality and from Lemma 3.4 and Corollary 3.3.

Let us remark at last that the family $((1-|a_n|^2)\cdot(1-\bar a_n z)^{-1})_{n\geqq 1}$ forms an unconditional basis in its closed span in $\mathfrak{M}$ and an unconditional basis in the weak-star closure of this span (we mean weak-star topology of $\mathfrak{M}$) if $a$ is an interpolating sequence. These facts are easy corollaries of Lemma 4.1. In this connection the following formula is useful: $\left\|\frac{1-|\lambda|^2}{1-\bar\lambda z}\right\|_{\mathfrak{M}}=(1+|\lambda|)^2$, $\lambda\in\mathbf{D}$. The coefficient space of every weak-star unconditional basis formed by the rational fractions

$$((1-|a_n|^2)(1-\bar a_n z)^{-1})_{n\geqq 1}$$

in their weak-star closed linear span in $\mathfrak{M}$ is isometric to the Banach space of all complex sequences satisfying

$$\sup_{\zeta\in\mathbf{T}}\sum\nolimits_{n\geqq 1}|x_n|\cdot\frac{1-|a_n|^2}{|1-\bar a_n\zeta|}<+\infty. \tag{6}$$

The analogous coefficient space for the unconditional basis is the closure of the family of all finite sequences $x=(x_n)_{n\geqq 1}$ in that norm.

*The proof of Lemma 4.1.* It is easy to verify that

$$H_{(1-\lambda\bar\zeta)^{-1}}f=\frac{\lambda\bar\zeta}{1-\lambda\bar\zeta}\cdot f(\lambda)$$

for every $\lambda$ in $\mathbf{D}$ and for every $f$ in $\mathbf{H}^\infty$. If follows from this formula that

$$H_{\bar\varphi}f(\zeta)=\sum\nolimits_n a_n\cdot\bar x_n f(a_n)\cdot\bar\zeta\cdot\frac{1-|a_n|^2}{(1-\bar a_n\zeta)}. \tag{7}$$

The proof of 1° is now finished by the following inequalities

$$\|\varphi\|_{\mathfrak{M}}=\|T_{\bar\varphi}\|\leqq\|\varphi\|_\infty+\sup_{\|f\|_\infty\leqq 1}\|H_{\bar\varphi}f\|_\infty\leqq\|\varphi\|_\infty+\sup_{\zeta\in\mathbf{T}}\sum\nolimits_n|x_n|\cdot|a_n|\frac{1-|a_n|^2}{|1-a_n\bar\zeta|}.$$

To prove 2° we use (7) and Carleson interpolation theorem:

$$\sup_{\zeta\in\mathbf{T}}\sum\nolimits_n|a_n|\cdot|x_n|\cdot\frac{1-|a_n|^2}{|1-a_n\bar\zeta|}=\sup_{\zeta\in\mathbf{T}}\sup_{\|(\varepsilon_n)\|_{l^\infty}\leqq 1}\left|\sum\nolimits_n a_n\bar x_n\varepsilon_n\bar\zeta\frac{1-|a_n|^2}{1-a_n\bar\zeta}\right|$$

$$\leqq\|H_{\bar\varphi}\|\cdot\text{Const.}\,(\delta(a)^{-1}\log\delta(a)^{-1}+1)$$

$$\leqq\text{Const.}\,(\delta(a)^{-1}\log\delta(a)^{-1}+1)(\|\varphi\|_\infty+\|\varphi\|_{\mathfrak{M}}).\quad\bullet$$

**4.2.** The second step of the proof is to prove that the sequence $a$ is the union of a finite number of separated ones if $B^a \in \mathfrak{M}$.

**Lemma 4.2.** *Let* $0<\varepsilon<10^{-6}$, *let* $n>27$ *be an integer and let* $B$ *be a finite Blaschke product with* $n$ *zeros (counted with their multiplicities) in the disc* $\{z\in \mathbf{C}\colon |z|<\varepsilon\}$, *and non vanishing outside this disc. Then*

$$\tfrac{1}{6}\log n \leqq 11 n\varepsilon \log n + \|B\|_{\mathfrak{M}}.$$

*Proof.* It follows from 2.2 and 2.3 that

$$\tfrac{1}{6}\log n \leqq \|B(1)z^n\|_{\mathfrak{M}} \leqq \|B(1)z^n - B\|_{\mathfrak{M}} + \|B\|_{\mathfrak{M}}$$

$$\leqq \frac{1}{n}\|n\cdot B(1)z^{n-1} - B'\|_\infty + \|B(1)z^n - B\|_\infty \cdot \log ne + \|B\|_{\mathfrak{M}}.$$

Let $a_1, \dots, a_n$ be the full list of the zeros of the Blaschke product $B$. Then for every $z$ in $\mathbf{T}$ we have

$$n\cdot B(1)\cdot z^{n-1} - \frac{B'}{B}\cdot B = nz^{n-1}B(1) - z^{-1}B\cdot \sum_{k=1}^{n} \frac{1-|a_k|^2}{|a_k - z|^2}$$

$$= nz^{-1}\{B(1)z^n - B\} + z^{-1}B\left\{n - \sum_{k=1}^{n}\frac{1-|a_k|^2}{|a_k-z|^2}\right\}.$$

Therefore

$$\|nB(1)z^{n-1} - B'\|_\infty \leqq n\|B(1)z^n - B\|_\infty + \sum_{k=1}^{n}\left\|\frac{1-|a_k|^2}{|a_k-z|^2} - 1\right\|_\infty$$

$$\leqq n\|B(1)z^n - B\|_\infty + \frac{2n\varepsilon}{1-\varepsilon}.$$

To estimate the norm $\|B(1)z^n - B\|_\infty$ observe that $B = b_1\cdot\ldots\cdot b_n$, $b_k$ being the Blaschke factor corresponding to the point $a_k$, $k\in\{1, \dots, n\}$. Then the triangle inequality gives

$$\|B(1)z^n - B\|_\infty \leqq \sum_{k=1}^{n}\|zb_k(1) - b_k\|_\infty.$$

It is easy to check that

$$|zb_k(1) - b_k| = \left|\frac{a_k - 1}{1-\bar a_k}z - \frac{a_k - z}{1-\bar a_k z}\right| \leqq \frac{6\varepsilon}{(1-\varepsilon)^2}.$$

The last two inequalities together imply that

$$\|B(1)z^n - B\|_\infty \leqq \frac{6n\varepsilon}{(1-\varepsilon)^2}.$$

Hence

$$\|B(1)z^n - B\|_{\mathfrak{M}} \leqq \frac{2\varepsilon}{1-\varepsilon} + \|B(1)z^n - B\|_\infty \log(e^2 n)$$

$$\leqq \frac{2\varepsilon}{1-\varepsilon} + \frac{6n\varepsilon}{(1-\varepsilon)^2}\log(e^2 n) \leqq 11 n\varepsilon \log n,$$

where $\varepsilon \in (0, 10^{-6})$ and $n > 27$. ●

**Lemma 4.3.** *Let $B$ be a Blaschke product in $\mathfrak{M}$. Then every disc in* **D** *with pseudo-euclidean radius* $\exp(-100 \cdot \|B\|_{\mathfrak{M}})$ *contains not more than* $\exp(100 \cdot \|B\|_{\mathfrak{M}})$ *zeros of $B$ (counted with their multiplicities).*

*Proof.* Let $\Delta = \left\{z \in \mathbf{C}: \left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| < \varepsilon\right\}$ be the disc with the non-euclidean center $\alpha$ and with the pseudo-euclidean radius $\varepsilon$, and let $\gamma$ be a conformal automorphism of **D** mapping $\Delta$ onto the disc $\{z \in \mathbf{C}: |z| < \varepsilon\}$. Then it follows by Lemma 2.4. that $B \circ \gamma^{-1} \in \mathfrak{M}$ and

$$\|B \circ \gamma^{-1}\|_{\mathfrak{M}} \leqq 4 \cdot \|B\|_{\mathfrak{M}}.$$

It is clear that the number $N$ of the zeros of $B \circ \gamma^{-1}$ in $\{z \in \mathbf{C}: |z| < \varepsilon\}$ is equal to one the Blaschke product $B$ has in $\Delta$. Let now $\varepsilon = \exp(-100\|B\|_{\mathfrak{M}})$. The inequality $1 = \|B\|_\infty \leqq \|B\|_{\mathfrak{M}}$ shows that $\varepsilon < 10^{-6}$. Suppose now that $N > 1/\varepsilon$.

Let $B^*$ be a Blaschke product corresponding to $n$ zeros of $B \circ \gamma^{-1}$ in the disc $D(0, \varepsilon)$, $n$ being the integer equal to $[1/99\varepsilon]$.

We have by the division theorem (see 2.5) that

$$\|B^*\|_{\mathfrak{M}} \leqq \|B \circ \gamma^{-1}\|_{\mathfrak{M}} \leqq 4 \cdot \|B\|_{\mathfrak{M}}.$$

Lemma 4.1 can be applied to the Blaschke product $B^*$ now. It follows that

$$\tfrac{1}{6}\log n \leqq 11 n\varepsilon \log n + \|B^*\|_{\mathfrak{M}} \leqq \tfrac{1}{9}\log n + 4 \cdot \|B\|_{\mathfrak{M}}$$

and therefore

$$\left[\frac{1}{99\varepsilon}\right] = n \leqq e^{72\|B\|_{\mathfrak{M}}}; \quad \frac{1}{100\varepsilon} < e^{72\|B\|_{\mathfrak{M}}}; \quad \varepsilon > e^{-100\|B\|_{\mathfrak{M}}}.$$

The last inequality contradicts our choice of the number $\varepsilon$.

Lemma 4.3. together with Corollary 3.2. show that the zero sequence of every Blaschke product in $\mathfrak{M}$ is a finite union of separated sequences.

**4.3.** We are prepared now for the proof of the implication $B^a \in \mathfrak{M} \Rightarrow a \in (F)$. If $B^a \in \mathfrak{M}$ then it follows from 4.2. that the sequence $a$ is a finite union of separated ones. Therefore we may assume without loss of generality (by the division theorem) that $a$ is a separated sequence.

Let $\Omega_{\pi/4}(\zeta)$ be a Stolz domain in $\mathbf{D}$ and let $B^*$ be the Blaschke product with the zero set $\{a_n:\ a_n\in\Omega_{\pi/4}(\zeta)\}$. It follows from Lemma 3.7. that the sequence of the zeros of $B_a^*$ forms an interpolating sequence and therefore 4.1 showes that

$$\sigma_F(a^*) \leqq \text{Const.}\left(1+\delta(a^*)^{-1}\log\delta^{-1}(a^*)\right)\cdot\|B^a\|_{\mathfrak{M}}.$$

$a^*$ being the zero sequence of $B^*$. We apply Lemma 3.5 now and see that

$$\operatorname{Card}\{n:\ a_n\in\Omega_{\pi/4}(\zeta)\} \leqq \left(1-1/\sqrt{2}\right)^{-1}\cdot\sigma_F(a^*) < +\infty.$$

Then Lemma 3.6 implies that $\mu_a$ is a Carleson measure and the second application of 4.1 gives us the desired inclusion $a\in(F)$. Moreover it is easy to see that there is an increasing function $\Phi$, defined on the half-line $\mathbf{R}_+ \stackrel{\text{def}}{=} \{x\in\mathbf{R}:\ x\geqq 0\}$ satisfying

$$\sigma_F(a) \leqq \Phi(\|B^a\|_{\mathfrak{M}}). \tag{8}$$

**4.4.** All we have now to prove is that no singular inner function belongs to $\mathfrak{M}$. Suppose that this were not the case. Then we could find a nontrivial singular inner function

$$I(z) = \exp\left\{-\int_{\mathbf{T}}\frac{\zeta+z}{\zeta-z}\,d\mu(\zeta)\right\}$$

in the space $\mathfrak{M}$. By the Frostman theorem (see [8], p. 58) it is possible to find a sequence $(\alpha_n)_{n\geqq 0}$ tending to zero such that every function $\Pi_n=\dfrac{I-\alpha_n}{1-\bar{\alpha}_n I}$ is a Blaschke product. We may assume that $|\alpha_n|\cdot\|I\|_{\mathfrak{M}}<2^{-1}$ for every $n$ in $\{1, 2, \ldots\}$ and the simple computations with the Taylor series show that

$$\|I-\Pi_n\|_\infty \leqq \|I-\Pi_n\|_{\mathfrak{M}} < 4\cdot|\alpha_n|\cdot\|I\|^2_{\mathfrak{M}}.$$

To simplify the notation let $\sigma_F(B) \stackrel{\text{def}}{=} \sigma_F(a)$ for $B=B^a$. Then it follows from (8) that

$$\sup_n \sigma_F(\Pi_n) < +\infty \quad\text{and}\quad \lim_n \|I-\Pi_n\|_\infty = 0. \tag{9}$$

Let $\mathfrak{B}$ denote the set of all Blaschke product. It is convenient to include in $\mathfrak{B}$, the function $\mathbf{1}$ equal to 1 identically, assuming that $\mathbf{1}$ is the Blaschke product with the empty zero set.

**Lemma 4.4.** *The set* $\mathfrak{B}_c \stackrel{\text{def}}{=} \{B\in\mathfrak{B}:\ \sigma_F(B)\leqq c\}$ *is the weak-star closed subset of* $\mathbf{H}^\infty$ *for every* $c$, $c>0$.

*Remark 1.* The lemma together with (9) implies that the assumption $I\in\mathfrak{M}$ leads to the contradiction.

*Remark 2.* Caratheodory has proved [12] that $\mathfrak{B}$ is a dense subset of the unit ball of $\mathbf{H}^\infty$ in the weak-star topology.

*The proof of Lemma 4.4.* Let $(B_n)_{n\geqq 1}$ be a sequence in $\mathfrak{B}_c$ and let $\lim_n B_n = B\cdot S\cdot F$ in the weak-star topology of $\mathbf{H}^\infty$. Here $B$ denotes a Blaschke product, $S$ denotes the singular function with the singular measure $\gamma_S$ and $F$ denotes an outer function. It is convenient to consider two measures $\nu_n$ and $\gamma_n$ determined by the zero sequence $(a_k^n)_{k\geqq 1}$ of $B_n$:

$$\nu_n \overset{\text{def}}{=} \sum_{k\geqq 1} \delta_{a_k^n}, \quad \gamma_n \overset{\text{def}}{=} \sum_{k\geqq 1} (1-|a_k^n|)\delta_{a_k^n}.$$

Let $(\nu, \gamma)$ be the same pair of measures for the Blaschke product $B$. It follows from the maximum principle for the subharmonic functions that

$$\int_{\mathbf{D}} d\gamma_n \leqq \sup_{z\in\mathbf{T}} \int_{\mathbf{D}} \frac{d\gamma_n(\zeta)}{|1-z\bar{\zeta}|} \leqq C < +\infty$$

and, in particular, that $B\cdot F\cdot S\not\equiv 0$. Let $\gamma^*$ be a limit point for $(\gamma_n)_{n\geqq 1}$ in the weak-star topology of the space of all bounded measures in the closed disc $\{|z|\leqq 1\}$. We observe at first that

$$\gamma^* = \gamma + \gamma_s + \log|F|\, dm \tag{10}$$

This fact was proved in [13], but for the reader's convenience we shall give here its simple proof. Let $G(\xi, z)$ denote the Green function $\log\left|\frac{1-\bar{\xi}z}{\xi-z}\right|$ for the disc $\mathbf{D}$. The Poisson kernel $P_z(\zeta) = \frac{1-|z|^2}{|1-\bar{\zeta}z|^2}$, $\zeta\in\mathbf{T}$, $z\in\mathbf{D}$ is equal to the normal derivative of $G$ and therefore for every $\zeta$ in $\mathbf{T}$ and for every $z$ in $\mathbf{D}$ we have

$$\lim_{\xi\to\zeta} \frac{G(\xi, z)}{1-|\xi|} = P_z(\zeta).$$

This equality and the equality $\gamma^* = (*)-\lim_n \gamma_n$ imply that

$$\lim_n \int_{\mathbf{D}} G(\xi, z)\, d\nu_n(\xi) = \lim_n \int_{\mathbf{D}} \frac{G(\xi, z)}{1-|\xi|}\, d\gamma_n(\xi)$$
$$= \int_{\mathbf{D}} \frac{G(\xi, z)}{1-|\xi|}\, d\gamma^* + \int_{\mathbf{T}} P_z(\zeta)\, d\gamma^*.$$

for every point $z$ in $\mathbf{D}$ satisfying $\gamma\{z\}=0$. It is clear that $\log|B_n(z)|^{-1} = \int_{\mathbf{D}} G(\xi, z)\, d\nu_n$ and that

$$\lim_n \log|B_n(z)|^{-1} = \log|B(z)|^{-1} + \log|S(z)|^{-1} + \log|F(z)|^{-1}.$$

Therefore we get the identity

$$\int_{\mathbf{D}} G(\xi, z)\, d\nu(\xi) + \int_{\mathbf{T}} P_z(\zeta)\{d\gamma_s(\zeta) + \log|F|\, dm(\zeta)\}$$
$$= \int \frac{G(\xi, z)}{1-|\xi|}\, d\gamma^*(\xi) + \int P_z(\zeta)\, d\nu^*(\zeta)$$

holding for every non-zero point $z$ for $B$ in $\mathbf{D}$. The application of the Laplace operator to the both sides of the above identity shows that $\gamma^*|\mathbf{D}=\gamma$ and then (10) follows from the uniqueness of the Poisson integral.

Now the weak convergence of the sequence $(\gamma_n)$ to $\gamma^*$ implies that

$$\sup_{z\in\mathbf{T}}\int_{\{|\zeta|\leqq 1\}}\frac{1+|\zeta|}{|1-\bar{\zeta}z|}\,d\gamma^*(\zeta)\leqq C$$

and therefore $\sigma_F(B)\leqq C$. It is clear that $\int_{\mathbf{T}}\frac{dm(z)}{|1-z\bar{\xi}|}=+\infty$ for every $\zeta$ in $\mathbf{T}$. This shows, obviously, that $\gamma_S+\log|F|\,dm\equiv 0$. Therefore $S\equiv F\equiv\mathbf{1}$ and the set $\mathfrak{B}_c$ is weak-star closed.

## § 5. The proof of Theorem 2

The proof depends on two geometric lemmas.

**Lemma 5.1.** [9] *Let $a=(a_n)_{n\geqq 1}$ be a sequence of points of $\mathbf{D}\setminus\{0\}$ and let $(\varepsilon_n)_{n\geqq 1}$ be a sequence of points of the interval* (0, 1) *such that the following conditions hold:*

a) $\sum_{n\geqq 1}(1-|a_n|)\varepsilon_n^{-1}<+\infty$; b) $D\left(\frac{a_n}{|a_n|},\,\varepsilon_n\right)\cap D\left(\frac{a_m}{|a_m|},\,\varepsilon_m\right)=\emptyset$ for $n\neq m$.

*Then $\sigma_F(a)<+\infty$ and $a$ is separated.*

**Lemma 5.2.** *Let $\Lambda$ be a subset of $\mathbf{D}$. Then the set $\Lambda$ can be covered by a finite number of Stolz domains if and only if $\sigma_F(a)=+\infty$ for every infinite sequence $a$ of points of $\Lambda$.*

*Proof.* The necessity of the condition $\sigma_F(a)=+\infty$ is obvious. To prove thə sufficiency let us assume that it is impossible to cover the set $\Lambda$ by a finite number of Stolz domains. The two cases are possible: the set $\mathbf{T}\cap\operatorname{Clos}\Lambda$ is infinite and $\operatorname{Card}(\mathbf{T}\cap\operatorname{Clos}\Lambda)<+\infty$. If the first of these possibilities takes place then there is a sequence $(\xi_n)_{n\geqq 1}$ in $\mathbf{T}\cap\operatorname{Clos}\Lambda$ and a point $\xi_0$ in $\mathbf{T}$ such that

$$|\xi-\xi_{n+1}|\leqq\tfrac{1}{2}|\xi_0-\xi_n|,\quad n\in\{1,2\,,\ldots$$

Given such a point $\xi_n$, we may find a point $\lambda_n$ in $\Lambda$ satisfying

$$|\lambda_n-\xi_n|\leqq 2^{-n}\cdot|\xi_0-\xi_n|,\quad n\in\{1,2,\ldots\}.$$

If to put now $a_n=\lambda_n$, $\varepsilon_n=|\xi_0-\xi_n|\cdot 16^{-1}$, $n\in\{1,2,\ldots\}$ then Lemma 5.1 shows that $\sigma_F(\lambda)<+\infty$.

We turn to the case $\operatorname{Card}(\mathbf{T}\cap \operatorname{Clos}\Lambda)<+\infty$. Then there exists a point $\xi_0\in\mathbf{T}\cap\operatorname{Clos}\Lambda$ with the property $(D(\xi_0,\varepsilon)\cap\Lambda)\cap(\mathbf{D}\setminus\Omega_\alpha(\xi_0))\neq\emptyset$ for every positive $\varepsilon$ and for every number $\alpha$ in $(0,\pi/2)$. This implies the existence of the sequence $(\lambda_n)_{n\geqq 1}$ in $\Lambda$ satisfying

$$1-|\lambda_n|\leqq 2^{-n}\cdot|\xi_0-\lambda_n|,\quad |\xi_0-\lambda_{n+1}|\leqq 2^{-1}\cdot|\xi_0-\lambda_n|,\quad n\in\{1,2,\ldots\}.$$

The second reference on Lemma 5.1 (with $\varepsilon_n=4^{-1}\cdot|\xi_0-\lambda_n|$) finishes the proof.

Let $\left(\frac{1-|\lambda|}{1+|\lambda|}\cdot\frac{1}{1-\bar\lambda z}\right)_{\lambda\in\Lambda}$ be a family of rational fractions in the unit sphere of the space $\mathfrak{M}$ and suppose it forms an unconditional basis in its closed linear span in $\mathfrak{M}$. Then the set $\Lambda$ is separated. The proof of this assertion is the main task of the following lemma.

**Lemma 5.3.** *Let $E$ be a Banach space of holomorphic functions in $\mathbf{D}$ and suppose the following conditions are fulfilled:*

1° $z^n\in E,\quad n\in\{0,1,\ldots\}$ *and* $\varlimsup_n \|z^n\|_E^{1/n}=1$;

2° *the evaluation functionals $f\to f(\lambda)$ are bounded on $E$ for every $\lambda$ in $\mathbf{D}$;*

3° $\gamma\overset{\text{def}}{=}\sup_{\lambda\in\mathbf{D},\,\|f\|_E\leqq 1}\left\|\left(\frac{\lambda-z}{1-\bar\lambda z}\right)f\right\|_E<+\infty.$

*If the family $(\alpha_\lambda\cdot(1-\bar\lambda z)^{-1})_{\lambda\in\Lambda}$ $(\alpha_\lambda\overset{\text{def}}{=}\|(1-\bar\lambda z)^{-1}\|_E^{-1},\ \lambda\in\Lambda\subset\mathbf{D})$ of rational fractions is separated in $E$, that is*

$$\inf\left\{\left\|\frac{\alpha_\lambda}{1-\bar\lambda z}-\frac{\alpha_\xi}{1-\bar\xi z}\right\|_E:\ \lambda,\xi\in\Lambda,\ \lambda\neq\xi\right\}>0$$

*then*

$$\inf\left\{\left|\frac{\lambda-\xi}{1-\bar\lambda\xi}\right|:\ \lambda,\xi\in\Lambda,\ \lambda\neq\xi\right\}>0.$$

*Proof.* We may assume without loss of generality that polynomials are dense in $E$. The separation condition of the family $\left(\frac{\alpha_\lambda}{1-\bar\lambda z}\right)_{\lambda\in\Lambda}$ implies the existence of the family $(\Phi_\lambda)_{\lambda\in\Lambda}$ of functionals on $E$ with the properties:

$$l\overset{\text{def}}{=}\sup_{\lambda\in\Lambda}\|\Phi_\lambda\|_{E^*}<+\infty;\quad \Phi_\lambda\left(\frac{\alpha_\xi}{1-\bar\xi z}\right)=\begin{cases}1, & \xi=\lambda\\ 0, & \xi\neq\lambda,\ \lambda,\xi\in\Lambda.\end{cases}$$

The holomorphic function $\mathfrak{R}\Phi_\lambda(t)\overset{\text{def}}{=}\Phi_\lambda((1-\bar t z)^{-1})$ is obviously equal to zero at the point $\xi$ $(\xi\in\Lambda,\ \xi\neq\lambda)$. Accordingly, we have by the condition 3°

$$\mathfrak{R}\Phi_\lambda\frac{1-\bar\xi z}{z-\xi}=\mathfrak{R}\varphi_{\lambda,\xi};\quad \|\varphi_{\lambda,\xi}\|\leqq\gamma\cdot\|\Phi_\lambda\|\leqq\gamma\cdot l<+\infty.$$

It remains only to remark that

$$\left|\frac{1-\bar{\xi}\lambda}{\lambda-\xi}\right| = \alpha_\lambda \cdot |\mathfrak{R}\varphi_{\lambda,\xi}(\lambda)| \leqq \alpha_\lambda \|\varphi_{\lambda,\xi}\| \cdot \|(1-\bar{\lambda}z)^{-1}\| = \|\varphi_{\lambda,\xi}\| \leqq \gamma \cdot 1. \quad \bullet$$

We have now all what is needed for the proof of Theorem 2. The sufficiency is evident in view of Lemma 4.1. If now the family $(\alpha_\lambda \cdot (1-\bar{\lambda}z)^{-1})_{\lambda \in \Lambda}$ forms a symmetric basis in its closed linear span in $\mathfrak{M}$ then it forms an unconditional basis as well. It follows from the Lemma 5.3 that the set $\Lambda$ is separated. The coefficient space $I(\Lambda)$ is the closure of the set of all finite sequences $x$, $x \in l^\infty(\Lambda)$ in the norm

$$\sup_{\sigma \in \sigma} \sup_{\xi \in \Pi} \sum_{\lambda \in \Lambda} |x_{\sigma\lambda}| \cdot \frac{1-|\lambda|^2}{|1-\bar{\xi}\lambda|} < +\infty.$$

If it is possible to cover the set $\Lambda$ by the finite number of Stolz domains then the case 2 takes place and therefore $I(\Lambda) = l^1(\Lambda)$. If it is not the case then by Lemma 5.2 there is an infinite subset $\Lambda_\infty$ of $\Lambda$ satisfying the Frostman condition. This implies the equality $I(\Lambda_\infty) = c_0(\Lambda_\infty)$. The desired assertion $I(\Lambda) = c_0(\Lambda)$ follows now from the symmetry condition.

We have compared in the introduction Theorem 2 with one unpublished result of S. A. Vinogradov (see however [4]). Now we shall give a sketch of its proof.

**Theorem 5.4.** *The family* $((1-|a_n|^2)(1-\bar{a}_n z)^{-1})_{n \geqq 1}$ *forms an unconditional basis in its linear closed span in* $\mathbf{H}^\infty$ *iff* $a$ *is a separated Frostman sequence.*

*Proof.* The sufficiency of the condition of the theorem may be proved with the help of techniques we have used for the proof of Lemma 4.1. If now $((1-|a_n|^2)(1-\bar{a}_n z)^{-1})_{n \geqq 1}$ is an unconditional basis then the standard duality arguments show that $a$ is an interpolating sequence. The coefficient space of the basis is described by (6). It is dear that the function $B^a - B^a(0)^{-1}$ is bounded and (5) shows that its coefficients in our basis are bounded from below. It follows now from (6) that $a$ is a Frostman sequence.

## § 6. The proof of Theorem 3

Let $(\Delta_n)_{n \geqq 1}$ denote the sequence of the complementary arcs of the closed nowhere dense subset $E$ of the circle $\mathbf{T}$ numbered so that their lengths decrease. Let $\xi_n$ be the center of the arc $\Delta_n$ and let $a_n \overset{\text{def}}{=} \left(1 - \frac{m\Delta_n}{2^n}\right)\xi_n$, $n \in \{1, 2, \ldots\}$. We shall prove that the Blaschke product $B^a$ satisfies all conditions of the theorem. It follows from Lemma 5.1 that $\sigma_F(a) < +\infty$ and therefore $B^a \in \mathfrak{M}$. On the other hand it is clear that

$$\frac{1-|a_{n+1}|}{1-|a_n|}=\frac{m\Delta_{n+1}}{m\Delta_n}\cdot\frac{1}{2}\leqq\frac{1}{2}$$

and therefore $\hat{B}^a(n)=O\left(\frac{1}{n}\right)$, $n\to+\infty$ (see [14]). We see that the partial sums of the Fourier series of $B^a$ are bounded uniformly on the circle $\mathbf{T}$. The condition $\sigma_F(a)<+\infty$ implies obviously the convergence of the Blaschke product at every point of the closed unit disc. The limit $\lim_{r\to 1-0} B^a(r\zeta)$ exists at every point $\zeta$ of the circle $\mathbf{T}$ (we use the inclusion $B^a\in\mathfrak{M}$) and is equal to $B^a(\zeta)$ as the proof of the Frostman theorem shows (see [8], p. 54—55). The Tauberian theorem of Littlewood ([15], p. 137) yields the formula

$$\lim_{n\to+\infty}\sum_{k=0}^{n}\hat{B}^a(k)\zeta^k=B(\zeta)$$

for every $\zeta$ in $\mathbf{T}$ $\left(\text{we recall that } \hat{B}^a(n)=O\left(\frac{1}{n}\right)\right)$. Clearly $B^a$ is analytic at the points of $\mathbf{T}\setminus E$ and because every point of $E$ is a limit point of the sequence $a$ it is clear also that $E$ is the set of all discontinuity points for $B^a|\mathbf{T}$.

**Corollary 5.1.** *Let $E$ be a set of the first category on* $\mathbf{T}$. *Then there is a function $f$ in $H^\infty$ such that*

1°. $f\in\mathfrak{M}$, $\hat{f}(n)=O\left(\frac{1}{n}\right)$, *the partial sums of the Fourier series of $f$ are uniformly bounded on* $\mathbf{T}$*;*

2°. *the Fourier series of $f$ converges everywhere on* $\mathbf{T}$*;*

3°. *every point of $E$ is the point of discontinuity for $f$.*

*Proof.* Let $(F_n)_{n\in\mathbf{Z}}$ be a sequence of nowhere dense closed subset of the circle $\mathbf{T}$ satisfying

$$F_k\cap F_n=\emptyset,\quad k\neq n;\quad E\subset\bigcup_{n\geqq 1}F_n.$$

Theorem 3 does the rest now. Indeed, let $B_n$ be the Blaschke product for the set $F_n$ constructed in the Theorem 3. Then

$$f\overset{\text{def}}{=}\sum_{n\geqq 1}\frac{1}{2^n}\cdot B_n\cdot\|B_n\|_m^{-1}$$

satisfies 1°—3°.

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