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Differentiability properties of Orlicz–Sobolev functions

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1. Introduction and main results

In this paper we are concerned with the pointwise behaviour of functions in certain classes of weakly differentiable functions.

The ancestor of all modern results dealing with pointwise properties of non-smooth functions is certainly the Lebesgue differentiation theorem, which asserts that if Ω is an open subset of $\mathbf{R}^n$, $n \geq 1$, and u is a locally integrable function in Ω, then $\lim_{r\to 0^+} \fint_{B_r(x)} u(y)\,dy$ exists and is finite for a.e. $x \in \Omega$, and

$$\lim_{r\to 0^+} \fint_{B_r(x)} |u(y)-u(x)|\,dy = 0 \tag{1.1}$$

for a.e. $x \in \Omega$. Here, $B_r(x)$ is the ball centered at x and having radius r, and $\fint_E u(y)\,dy$ stands for $(1/|E|)\int_E u(y)\,dy$, when E is a measurable set with finite Lebesgue measure $|E|$. A point x where (1.1) holds will be called a *Lebesgue point* for u, or a point of *approximate continuity* for u. (This terminology is borrowed from [AFP], where a comparison with a slightly weaker definition of approximate continuity due to Federer can also be found.) The function defined as the limit of the averages of u at those points where such a limit exists, and 0 elsewhere, is usually referred to as the precise representative of u. Henceforth, we shall assume that every locally integrable function is precisely represented.

It has been long known that Sobolev functions fulfill (1.1) in a stronger sense, in that the exceptional set of those points where (1.1) does not hold is considerably smaller. The size of this exceptional set can be properly estimated through the notion of capacity. Indeed, one of the fundamental fine properties of Sobolev functions tells us that any element u from the Sobolev space $W^{k,p}_{\mathrm{loc}}(\Omega)$ of functions endowed with kth order weak derivatives in $L^p_{\mathrm{loc}}(\Omega)$ satisfies

$$\lim_{r\to 0^+} \fint_{B_r(x)} |u(y)-u(x)|^p\,dy = 0 \tag{1.2}$$

for every x in Ω outside a set of $C_{k,p}$-capacity zero (see e.g. [AH] and [Z]).

The theory of fine properties of Sobolev functions is strictly related to an analogous theory for the Bessel potential spaces $L^p_\alpha(\mathbf{R}^n)$ (and also for the Riesz potential spaces), consisting of those functions which are the Bessel potential of order $\alpha\in\mathbf{R}$ of some function in $L^p(\mathbf{R}^n)$; we refer to [AH] and [A1] for a comprehensive treatment of this topic. In fact, inasmuch as $W^{k,p}(\mathbf{R}^n)=L^p_k(\mathbf{R}^n)$ for every $p\in(1,\infty)$ and every integer $k\geq 1$, any property of Sobolev spaces can be derived as a special case of a corresponding property of potential spaces.

Fine properties of functions from potential spaces $L^A_\alpha(\mathbf{R}^n)$, where the role of the Lebesgue space $L^p(\mathbf{R}^n)$ is more generally played by the Orlicz space $L^A(\mathbf{R}^n)$ associated with a Young function A, were studied in [CS] in terms of Orlicz capacities. Results in a similar spirit for functions from first-order Orlicz–Sobolev spaces $W^{1,A}_{\mathrm{loc}}(\Omega)$ are contained in [MSZ]. Related results can also be found in [AHS2]. Let us emphasize that, unlike the classical Sobolev spaces, Orlicz–Sobolev spaces and Orlicz potential spaces do not agree in general, unless the defining Young function satisfies additional assumptions. Let us also mention that a theory of capacity in Orlicz spaces requires some regularity of the involved Young function (see e.g. [AB], [CS] and [AHS1]).

All the above-mentioned results in the Orlicz space setting deal with the Lebesgue points of functions, i.e. with their (approximate) continuity properties. Our aim here is to further investigate the regularity of Orlicz–Sobolev functions and to initiate a study of their differentiability properties.

A classical theorem by Rademacher [R] states that any locally Lipschitz continuous function in an open subset Ω of $\mathbf{R}^n$ (and hence any function from the Sobolev space $W^{1,\infty}_{\mathrm{loc}}(\Omega)$) is differentiable at a.e. point in Ω, and that its classical gradient agrees with its weak gradient a.e. in Ω. An extension of this result ensures that the same conclusion remains true even for functions from the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega)$, as long as $p>n$ (see e.g. [EG, Section 6.2, Theorem 1] and [MZ, Theorem 1.72]). Counterexamples show that exponents $p\leq n$ cannot be allowed; in fact, for these values of p, functions from $W^{1,p}_{\mathrm{loc}}(\Omega)$ need not even be continuous at any point of Ω, nor bounded in a neighborhood of any point of Ω ([S1, Chapter V, 6.3]).

When functions from $W^{1,p}_{\mathrm{loc}}(\Omega)$, with $1\leq p<n$, are taken into account, a substitute for these results holds, provided that the notion of classical differentiability is replaced by that of differentiability in L^{p^*} sense, where $p^*=np/(n-p)$, the Sobolev conjugate of p. Precisely, if $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ for some $p\in[1,n)$, then

$$
(1.3)\qquad \lim_{r\to 0^+} \fint_{B_r(x)} \left| \frac{u(y)-u(x)-\langle \nabla u(x), y-x\rangle}{r} \right|^{p^*} dy = 0,
$$

for a.e. $x\in\Omega$ ([EG, Section 6.1.2, Theorem 2], [Z, Theorem 3.4.2]). Here, $\langle\cdot,\cdot\rangle$

denotes scalar product, and ∇u is the gradient of u. In fact, any such function u enjoys a slightly stronger property: it is a.e. approximately differentiable in L^{p^*}, in the sense that

$$\lim_{r\to 0^+} \fint_{B_r(x)} \left| \frac{u(y)-u(x)-\langle \nabla u(x), y-x\rangle}{|y-x|} \right|^{p^*} dy = 0, \tag{1.4}$$

for a.e. $x\in\Omega$.

The borderline Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega)$ has been considered by D. Adams in [A1], [A2], where, in the spirit of Trudinger's embedding theorem, it is shown that, for every function u from this space, a constant σ exists such that

$$\lim_{r\to 0^+} \fint_{B_r(x)} \left(\exp\left(\frac{|u(y)-u(x)-\langle \nabla u(x), y-x\rangle|}{\sigma |y-x|} \right)^{n'} - 1 \right) dy = 0 \tag{1.5}$$

for a.e. $x\in\Omega$, where $n'=n/(n-1)$.

Properties (1.2) and (1.4) are special cases of general theorems for the Sobolev spaces $W^{k,p}_{\mathrm{loc}}(\Omega)$, where the set of points of approximate differentiability of order $m\in[0,k]$ with $(k-m)p<n$ is shown to be the complement in Ω of a set of $C_{k-m,p}$-capacity zero. These theorems go back to [FZ] for the case where $k=1$, and to [BaZ], [Me] and [CFR] for general $k\geq 1$. Another notion of differentiability of functions, resting upon the concept of thin set, is that of fine differentiability (see [M1] and [M2]). Fine differentiability properties of Sobolev functions follow via potential theory techniques. In particular, both approximate differentiability and fine differentiability of Sobolev functions can be recovered as a consequence of the ultra-fine differentiability property of potentials proved in [A2].

In the present paper we focus on first order Orlicz–Sobolev spaces, and we establish optimal theorems of Rademacher type and of approximate differentiability type in this setting. In particular, we provide a unified framework for the classical results (1.3)–(1.5). Moreover, in their strongest form, our conclusions also slightly sharpen these results.

Our proofs rely upon Sobolev spaces techniques, and are related to a method presented in [S1] and [EG] for ordinary Sobolev spaces. Hence, apart from the Lebesgue differentiation theorem (1.1), tools from harmonic analysis or potential theory are avoided. This is of basic importance, since, as already pointed out, Orlicz–Sobolev spaces and Orlicz potential spaces are not equivalent in general. Of course, a study of differentiability properties of functions from Orlicz potential spaces would be of independent interest, but this goes beyond the scope of this paper.

We begin our discussion on exhibiting a sharp assumption on A ensuring the a.e. differentiability of functions from $W^{1,A}_{\mathrm{loc}}(\Omega)$, which extends the condition $p>n$ for the ordinary Sobolev spaces $W^{1,p}_{\mathrm{loc}}(\Omega)$. The relevant assumption is that A grows so fast at infinity that

$$\int^{+\infty} \left(\frac{t}{A(t)}\right)^{1/(n-1)} dt < +\infty. \tag{1.6}$$

Condition (1.6) is known to be necessary and sufficient for $W^{1,A}_{\mathrm{loc}}(\Omega)$ to be continuously embedded into the space of locally bounded ([M, Par. 5.4], [T1], [C1]) and also continuous ([C1]) functions. Here, we prove the following result.

Theorem 1.1. *Let Ω be an open subset of $\mathbf{R}^n$ and let A be a Young function satisfying* (1.6)*. If $u\in W^{1,A}_{\mathrm{loc}}(\Omega)$, then u is differentiable a.e. in Ω and its classical gradient agrees with its weak gradient a.e. in Ω.*

Remark 1.2. A theorem by Stein [S2] ensures that any weakly differentiable function whose gradient is in the Lorentz space $L^{n,1}_{\mathrm{loc}}(\Omega)$ (a space strictly contained in $L^n_{\mathrm{loc}}(\Omega)$) is a.e. differentiable in Ω. Since

$$L^{n,1}_{\mathrm{loc}}(\Omega) = \bigcup_{A \text{ satisfies } (1.6)} L^A_{\mathrm{loc}}(\Omega),$$

(see [KKM]), then Theorem 1.1 turns out to be equivalent to the result of [S2]. We present here a direct proof of Theorem 1.1, which, in particular, provides an alternative approach to Stein's theorem.

Assume now that

$$\int^{+\infty} \left(\frac{t}{A(t)}\right)^{1/(n-1)} dt = +\infty. \tag{1.7}$$

With condition (1.7) in force, the a.e. differentiability of functions from $W^{1,A}_{\mathrm{loc}}(\Omega)$ is not guaranteed anymore, as demonstrated by the following proposition.

Proposition 1.3. *Let Ω be an open subset of $\mathbf{R}^n$ and let A be a Young function satisfying* (1.7)*. Then there exists $u\in W^{1,A}_{\mathrm{loc}}(\Omega)$ such that u is nowhere differentiable in Ω.*

On the other hand, under (1.7) results in the spirit of (1.3), (1.4) and (1.5) can be shown to hold, where the role of the function t^{p^*}, or $e^{t^{n'}}-1$, respectively, is played by the Sobolev conjugate A_n of A defined by

$$A_n(t) = A \circ H_n^{-1}(t) \quad \text{for } t \ge 0, \tag{1.8}$$

where H_n^{-1} is the (generalized) left-continuous inverse of the function $H_n: [0,\infty)\to [0,\infty)$ given by

$$(1.9)\qquad H_n(r)=\left(\int_0^r \left(\frac{t}{A(t)}\right)^{1/(n-1)} dt\right)^{1/n'} \quad \text{for } r\geq 0.$$

Obviously, for H_n, and hence A_n, to be well defined, A has to fulfill

$$(1.10)\qquad \int_0 \left(\frac{t}{A(t)}\right)^{1/(n-1)} dt<+\infty.$$

However, this is by no means a restriction, since $W^{1,A}_{\mathrm{loc}}(\Omega)$ is unchanged if A is modified near 0; thus, we may always assume that (1.10) is satisfied on replacing, if necessary, A by an equivalent Young function near infinity. Under customary regularity conditions on Ω, $L^{A_n}(\Omega)$ turns out to be the smallest Orlicz space into which $W^{1,A}(\Omega)$ is continuously embedded ([C3]; see also [C2] for an equivalent formulation). A corresponding Sobolev–Poincaré type inequality suitable for our applications is recalled in Theorem 2.1, Section 2. The relevant results about L^{A_n} differentiability are contained in the next two theorems. In the statements, $\not{L}^{A_n}(B_r(x))$ denotes an averaged Luxemburg norm—see Subsection 2.1.

Theorem 1.4. *Let Ω be an open subset of $\mathbf{R}^n$ and let A be a Young function satisfying* (1.7) *and* (1.10)*. Assume that $u\in W^{1,A}_{\mathrm{loc}}(\Omega)$. Then, for every $\sigma>0$,*

$$(1.11)\qquad \lim_{r\to 0^+} \fint_{B_r(x)} A_n\left(\frac{|u(y)-u(x)-\langle\nabla u(x), y-x\rangle|}{\sigma r}\right) dy=0$$

for a.e. $x\in\Omega$, and hence

$$(1.12)\qquad \lim_{r\to 0^+} \left\|\frac{u(\,\cdot\,)-u(x)-\langle\nabla u(x),\,\cdot-x\rangle}{r}\right\|_{\not{L}^{A_n}(B_r(x))}=0$$

for a.e. $x\in\Omega$.

Theorem 1.5. *Under the same assumptions as in Theorem 1.4, for every $\sigma>0$,*

$$(1.13)\qquad \lim_{r\to 0^+} \fint_{B_r(x)} A_n\left(\frac{|u(y)-u(x)-\langle\nabla u(x), y-x\rangle|}{\sigma |y-x|}\right) dy=0$$

for a.e. $x\in\Omega$, and hence

$$(1.14)\qquad \lim_{r\to 0^+} \left\|\frac{u(\,\cdot\,)-u(x)-\langle\nabla u(x),\cdot-x\rangle}{|\cdot-x|}\right\|_{\not{L}^{A_n}(B_r(x))}=0$$

for a.e. $x\in\Omega$.

Since $A_n(t)$ is equivalent to t^{p^*} when $A(t)=t^p$ for some $p\in[1,n)$, Theorems 1.4 and 1.5 recover (1.3) and (1.4), respectively. In the following corollary, Theorems 1.1 and 1.5 are applied to the borderline spaces $W^{1,A}_{\mathrm{loc}}(\Omega)$, with $A(t)=t^n\log^\alpha(e+t)$, which have attracted much attention in recent years (see e.g. [FLS], [EGO], [CP] and [AHS2]). Obviously, the important special case (1.5) is included here.

Corollary 1.6. *Let Ω be an open subset of $\mathbf{R}^n$ and let $A(t)=t^n\log^\alpha(e+t)$, with $\alpha\geq 0$. Let $u\in W^{1,A}_{\mathrm{loc}}(\Omega)$.*

If $\alpha<n-1$, then, for every $\sigma>0$,

$$\lim_{r\to 0^+} \fint_{B_r(x)} \left(\exp\left(\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{\sigma|y-x|}\right)^{n/(n-1-\alpha)}-1\right)dy=0 \tag{1.15}$$

for a.e. $x\in\Omega$, and hence

$$\lim_{r\to 0^+}\left\|\frac{u(\,\cdot\,)-u(x)-\langle\nabla u(x),\cdot-x\rangle}{|\cdot-x|}\right\|_{L^{\exp^{n/(n-1-\alpha)}}(B_r(x))}=0 \tag{1.16}$$

for a.e. $x\in\Omega$. Here, $L^{\exp^q}(B_r(x))$ stands for the Orlicz space associated with the Young function $e^{s^q}-1$ with $q\geq 1$.

If $\alpha=n-1$, then, for every $\sigma>0$,

$$\lim_{r\to 0^+} \fint_{B_r(x)} \left(\exp\left(\exp\left(\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{\sigma|y-x|}\right)^{n'}\right)-e\right)dy=0 \tag{1.17}$$

for a.e. $x\in\Omega$, and hence

$$\lim_{r\to 0^+}\left\|\frac{u(\,\cdot\,)-u(x)-\langle\nabla u(x),\cdot-x\rangle}{|\cdot-x|}\right\|_{L^{\exp\exp^{n'}}(B_r(x))}=0 \tag{1.18}$$

for a.e. $x\in\Omega$. Here, $L^{\exp\exp^{n'}}(B_r(x))$ stands for the Orlicz space associated with the Young function $\exp(\exp(s^{n'}))-e$.

If $\alpha>n-1$, then u is differentiable a.e. in Ω.

We now address the problem of whether conclusions (1.3)–(1.5), and, more generally, (1.11), can be somewhat sharpened. Consider first (1.3), involving the norm in $L^{p^*}(\Omega)$, the Lebesgue space into which $W^{1,p}(\Omega)$ is continuously embedded when Ω is a sufficiently smooth open subset of $\mathbf{R}^n$. This embedding is optimal, as long as Lebesgue (and also Orlicz) range spaces are allowed, but it can be improved if Lorentz spaces are employed. Actually, $W^{1,p}(\Omega)$ is continuously embedded into the Lorentz space $L^{p^*,p}(\Omega)$, a space strictly contained in $L^{p^*}(\Omega)$, whenever $1\leq p<n$.

Moreover, $L^{p^*,p}(\Omega)$ is known to be the smallest rearrangement invariant range space for Sobolev embeddings of $W^{1,p}(\Omega)$ ([CP], [EKP]). Thus, one may expect that a result in the spirit of (1.3) holds with the L^{p^*}-norm replaced by the $L^{p^*,p}$-norm. This is indeed the case, and it follows as a special instance of Theorem 1.7 below. This theorem relies on a version for Orlicz–Sobolev spaces $W^{1,A}(\Omega)$, recently proved in [C4], of the Sobolev embedding into Lorentz spaces, which involves certain spaces of Orlicz–Lorentz type and yields the best possible rearrangement invariant range space—see Theorem 2.3, Subsection 2.3. Here, a key role is played by the Young function $D_{A,n}$ associated with any Young function A satisfying (1.10) as follows. Let $a\colon[0,+\infty)\to[0,+\infty]$ be the non-decreasing left-continuous function such that

$$A(t)=\int_0^t a(r)\,dr \quad \text{for } t\ge 0,$$

and let $d\colon[0,+\infty)\to[0,+\infty)$ be the left-continuous function whose (generalized) left-continuous inverse obeys

$$d^{-1}(s)=\left(\int_{a^{-1}(s)}^{\infty}\left(\int_0^t\left(\frac{1}{a(r)}\right)^{1/(n-1)}dr\right)^{-n}\frac{dt}{a(t)^{n'}}\right)^{1/(1-n)} \quad \text{for } s\ge 0.$$

Then

$$D_{A,n}(t)=\int_0^t d(r)\,dr \quad \text{for } t\ge 0. \tag{1.19}$$

Let us notice that A always dominates $D_{A,n}$, and is in fact equivalent to $D_{A,n}$ if and only if $A(t)$ is strictly below t^n (see [C4, Proposition 5.2] for a precise statement of this fact). For instance, if $A(t)=t^p$ with $1\le p<n$, then $D_{A,n}(t)$ is equivalent to t^p, but if $A(t)$ is equivalent to t^n near infinity (and satisfies (1.10)), then $D_{A,n}(t)$ is equivalent to $t^n\log^{-n}(e+t)$ near infinity.

Theorem 1.7. *Let Ω be an open subset of $\mathbf{R}^n$ and let A be a Young function satisfying* (1.7) *and* (1.10). *Let $u\in W^{1,A}(\Omega)$. Then there exists a constant $\eta>0$ such that*

$$\lim_{r\to 0^+} \fint_0^{|B_r(x)|} D_{A,n}(\eta s^{-1/n}(u(\cdot)-u(x)-\langle\nabla u(x),\cdot-x\rangle)^*(s))\,ds=0 \tag{1.20}$$

for a.e. $x\in\Omega$. Here "$*$" stands for the decreasing rearrangement.

Remark 1.8. Theorem 1.7 strengthens Theorem 1.4, since a constant k, depending only on n, exists such that, for any measurable subset Ω of $\mathbf{R}^n$ and any Young function A satisfying (1.7) and (1.10),

$$\int_\Omega A_n\left(\frac{k|f(x)|}{4(\int_0^\infty D_{A,n}(r^{-1/n}f^*(r))\,dr)^{1/n}}\right)dx\le\int_0^\infty D_{A,n}(r^{-1/n}f^*(r))\,dr \tag{1.21}$$

for every measurable function f in Ω. Thus, in particular, Theorem 1.4 could be deduced from Theorem 1.7, via (1.21). We are not going to prove inequality (1.21) for a general A; let us just mention that it can be established by the methods of [C4, Theorem 4.1]. Instead, we illustrate the situation in the classical setting where $A(s)=s^p$ with $p\in[1,n)$. Let $u\in W^{1,p}(\Omega)$. Then (1.20) yields

$$\lim_{r\to 0^+} \|u(\cdot)-u(x)-\langle\nabla u(x),\cdot - x\rangle\|_{\text{Ł}^{p^*,p}(B_r(x))}=0 \tag{1.22}$$

for a.e. $x\in\Omega$. Since a constant c, depending only on p and n, exists such that

$$\|f\|_{\text{Ł}^{p^*}(B_r(x))}\le cr\|f\|_{\text{Ł}^{p^*,p}(B_r(x))}$$

for every $f\in L^{p^*,p}(B_r(x))$, then (1.22) implies (1.13). Equation (1.22) contains however more accurate information, due to the strict inclusion of $L^{p^*,p}(B_r(x))$ into $L^{p^*}(B_r(x))$.

Let us also notice that, when $A(t)$ is equivalent to t^n near infinity, and hence $D_{A,n}(t)$ is equivalent to $t^n\log^{-n}(e+t)$ near infinity, then (1.20) is related to the embedding of [BW] and [H].

The remaining part of the paper is organized as follows. Section 2 contains the necessary background from the theory of Orlicz spaces and, more generally, rearrangement invariant spaces, as well as some preliminary results about Sobolev–Poincaré type inequalities in Orlicz–Sobolev spaces. Proofs of the results stated above are presented in Section 3.

2. Background and preliminary results

2.1. Rearrangements and rearrangement invariant spaces

Let Ω be a measurable subset of $\mathbf{R}^n$. Given any real-valued measurable function u in Ω, we denote by $u^*\colon[0,+\infty)\to[0,+\infty]$ its *decreasing rearrangement*, defined as

$$u^*(s)=\sup\{t>0:|\{x\in\Omega:|u(x)|>t\}|>s\}\quad\text{for } s\ge 0.$$

It is easily checked that u^* is non-increasing and right-continuous in $[0,+\infty)$, and that u^* and u are equidistributed. Note that the support of u^* is contained in $[0,|\Omega|]$. The function u^{**} is defined by $u^{**}=(1/s)\int_0^s u^*(r)\,dr$ for $s>0$.

The *signed decreasing rearrangement* u° of u is the function from $[0,|\Omega|]$ into $\mathbf{R}$ given by

$$u^\circ(s)=\sup\{t\in\mathbf{R}:|\{x\in\Omega:u(x)>t\}|>s\}\quad\text{for } s\in[0,|\Omega|].$$

A *rearrangement invariant space*— briefly, an *r.i. space*—$X(\Omega)$ is a Banach function space whose norm $\|\cdot\|_{X(\Omega)}$ satisfies

$$\|v\|_{X(\Omega)} = \|u\|_{X(\Omega)}, \quad \text{if } u^* = v^*. \tag{2.1}$$

The *associate space* $X'(\Omega)$ of $X(\Omega)$ is the r.i. space defined as

$$X'(\Omega) = \Big\{ v : v \text{ is a real-valued measurable function in } \Omega \text{ and } \int_\Omega |uv|\, dx < \infty \text{ for all } u \in X(\Omega) \Big\}$$

and is endowed with the norm

$$\|v\|_{X'(\Omega)} = \sup_{u \neq 0} \frac{\int_\Omega |uv|\, dx}{\|u\|_{X(\Omega)}}. \tag{2.2}$$

As a consequence, the Hölder type inequality

$$\int_\Omega |uv|\, dx \leq \|u\|_{X(\Omega)} \|v\|_{X'(\Omega)}, \tag{2.3}$$

holds for every $u \in X(\Omega)$ and $v \in X'(\Omega)$.

The *fundamental function* φ_X of $X(\Omega)$ is defined in $[0, |\Omega|]$ as

$$\varphi_X(t) = \|\chi_E\|_{X(\Omega)} \quad \text{for } t \in [0, |\Omega|],$$

where E is any measurable subset of Ω such that $|E| = t$. The fundamental functions of $X(\Omega)$ and of $X'(\Omega)$ are related by the equality

$$\varphi_X(t)\varphi_{X'}(t) = t \quad \text{for } t \in [0, |\Omega|]. \tag{2.4}$$

We refer to [BS] for more details on these topics.

2.2. Orlicz, Lorentz and Orlicz–Lorentz spaces

Let Ω be a measurable subset of $\mathbf{R}^n$, and let A be a Young function, as defined in Section 1. Then the *Orlicz space* $L^A(\Omega)$ is the set of all real-valued measurable functions u in Ω such that

$$\|u\|_{L^A(\Omega)} = \inf\Big\{ \lambda > 0 : \int_\Omega A\Big(\frac{|u(x)|}{\lambda}\Big)\, dx \leq 1 \Big\} \tag{2.5}$$

is finite. The expression $\|\cdot\|_{L^A(\Omega)}$ is called the *Luxemburg norm*; clearly, $L^A(\Omega)$ is an r.i. space equipped with this norm. The space $L^A_{\mathrm{loc}}(\Omega)$ is defined as the set of those functions which belong to $L^A(\Omega')$ for every compact subset Ω' of Ω. When $|\Omega|<\infty$, we define the averaged norm $\|\cdot\|_{\not L^A(\Omega)}$ as in (2.5) with $\int_\Omega$ replaced by $\fint_\Omega$. Notice that the Lebesgue spaces $L^p(\Omega)$ are recovered as special instances of Orlicz spaces with $A(s)=s^p$, if $1\le p<\infty$, and with $A(s)=0$ for $0\le s\le 1$ and $A(s)=+\infty$ for $s>1$, if $p=+\infty$. In both cases, $\|\cdot\|_{L^A(\Omega)}$ agrees with the usual norm in $L^p(\Omega)$. Notice also that

$$\|\chi_E\|_{L^A(\Omega)} = \frac{1}{A^{-1}\left(\frac{1}{|E|}\right)}, \tag{2.6}$$

for every subset E of Ω having finite measure. Hereafter, A^{-1} denotes the (generalized) right-continuous inverse of A.

The *associate space* of $L^A(\Omega)$ is, up to equivalent norms, $L^{\tilde A}(\Omega)$, where $\tilde A$ is the Young conjugate of A defined as $\tilde A(s)=\sup\{rs-A(r):r\ge 0\}$ for $s\ge 0$. In fact, one has

$$\|v\|_{L^{\tilde A}(\Omega)} \le \|v\|_{(L^A)'(\Omega)} \le 2\|v\|_{L^{\tilde A}(\Omega)} \tag{2.7}$$

for every $v\in L^{\tilde A}(\Omega)$.

A function A is said to dominate another function D near infinity if positive constants k and s_∞ exist such that $D(s)\le A(ks)$ for $s\ge s_\infty$. If this inequality holds for every $s\ge 0$, then A is said to dominate D globally. The functions A and D are called equivalent near infinity [resp. globally equivalent] if they dominate each other near infinity [globally]. If A and D are Young functions, then the inclusion $L^A(\Omega)\subseteq L^D(\Omega)$ holds if and only if either $|\Omega|=\infty$ and A dominates D globally, or $|\Omega|<\infty$ and A dominates D near infinity. Hence,

$$L^A_{\mathrm{loc}}(\Omega) \subseteq L^D_{\mathrm{loc}}(\Omega) \quad \text{if and only if } A \text{ dominates } D \text{ near infinity.} \tag{2.8}$$

Lorentz spaces represent another example of r.i. spaces. Recall that, if either $1<p<\infty$ and $1\le q\le\infty$, or $p=q=\infty$, the Lorentz space $L^{p,q}(\Omega)$ is the space of real-valued measurable functions u in G such that the quantity

$$\|u\|_{L^{p,q}(\Omega)} = \|s^{1/p-1/q}u^*(s)\|_{L^q(0,|\Omega|)} \tag{2.9}$$

is finite. Such a quantity is a norm in $L^{p,q}(\Omega)$ if $q\le p$. In general, it can be turned into an equivalent norm after replacing u^* by u^{**} in the right-hand side of (2.9).

The averaged norm $\|\cdot\|_{\mathcal{L}^{p,q}(\Omega)}$ is defined accordingly, with $\|\cdot\|_{L^q(0,|\Omega|)}$ replaced by $\|\cdot\|_{\mathcal{L}^q(0,|\Omega|)}$ in (2.9).

Various notions of *Orlicz–Lorentz spaces* have been introduced in the literature, in the attempt of providing a unified framework for Orlicz and Lorentz spaces. Here, we need to work with spaces from a family of Orlicz–Lorentz spaces considered in [C4] and defined as follows. Given any $q\in(1,\infty]$ and any Young function D satisfying $\int^\infty (D(t)/t^{1+q})\,dt<\infty$ (if $q<\infty$), we call $L(q,D)(\Omega)$ the r.i. space of those real-valued measurable functions u on Ω such that the norm $\|u\|_{L(q,D)(\Omega)}=\|s^{-1/q}u^*(s)\|_{L^D(0,|\Omega|)}$ is finite. Plainly, the Orlicz spaces $L^A(\Omega)$ and the Lorentz spaces $L^{p,q}(\Omega)$, with $q\le p$, are recovered as special cases of the spaces $L(q,D)(\Omega)$.

2.3. Orlicz–Sobolev spaces

Let Ω be an open subset of $\mathbf{R}^n$ and let A be a Young function. The *Orlicz–Sobolev space* $W^{1,A}(\Omega)$ is defined as

$$W^{1,A}(\Omega)=\{u:u\in L^A(\Omega),\ u \text{ is weakly differentiable in } \Omega \text{ and } |\nabla u|\in L^A(\Omega)\}.$$

The space $W^{1,A}_{\mathrm{loc}}(\Omega)$ is defined accordingly. Furthermore, we denote by $W^{1,A}_0(\Omega)$ the subspace of $W^{1,A}(\Omega)$ of those functions u which vanish on $\partial\Omega$, in the sense that the continuation of u outside Ω by 0 is a weakly differentiable function in $\mathbf{R}^n$. A Sobolev–Poincaré type inequality with sharp Orlicz range norm is given by the following result.

Theorem 2.1. *Let B be any ball in $\mathbf{R}^n$. Let A be a Young function satisfying* (1.10) *and let A_n be the Sobolev conjugate of A defined by* (1.8). *Then a constant $k_1(n)$, depending only on n, exists such that*

$$\|u-u_B\|_{L^{A_n}(B)}\le k_1(n)\big\|\,|\nabla u|\,\big\|_{L^A(B)} \tag{2.10}$$

for every weakly differentiable function u in B such that $|\nabla u|\in L^A(B)$. Here, $u_B=\fint_B u(x)\,dx$, the mean value of u over B.

A proof of inequality (2.10) is given in [C2, Theorem 2] (for a much larger class of ground domains Ω), with A_n replaced by the Young function $\hat{A}_n$ given by

$$\begin{aligned} \hat{A}_n(s)&=\int_0^s r^{n'-1}(\phi_n^{-1}(r^{n'}))^{n'}\,dr \quad \text{for } s\ge 0,\\ \phi_n(s)&=n'\int_0^s \frac{\tilde{A}(t)}{t^{1+n'}}\,dt \qquad\qquad \text{for } s\ge 0. \end{aligned} \tag{2.11}$$

Thus, Theorem 2.1 follows from this result and the next lemma, a combination of [C4, Lemma 2.4] and [C3, Lemma 2].

Lemma 2.2. *Let A be a Young function. Then*

$$\int^{+\infty}\left(\frac{t}{A(t)}\right)^{1/(n-1)} dt<+\infty \quad \text{if and only if} \quad \int^{+\infty}\frac{\tilde{A}(t)}{t^{1+n'}}\,dt<+\infty \tag{2.12}$$

and

$$\int_0\left(\frac{t}{A(t)}\right)^{1/(n-1)} dt<+\infty \quad \text{if and only if} \quad \int_0\frac{\tilde{A}(t)}{t^{1+n'}}\,dt<+\infty. \tag{2.13}$$

If (1.10) holds, then the Young functions A_n and $\hat{A}_n$ defined by (1.8) and (2.11), respectively, are globally equivalent with equivalence constants depending only on n. Moreover, the function given by

$$\bar{A}_n(s)=(s\phi_n^{-1}(s^{n'}))^{n'} \quad \text{for } s\geq 0$$

satisfies

$$\bar{A}_n^{-1}(1/s)=\|(\,\cdot\,)^{-1/n'}\chi_{(s,\infty)}(\,\cdot\,)\|_{L^{\tilde{A}}(0,\infty)} \quad \text{for } s>0, \tag{2.14}$$

and is globally equivalent to A_n and to $\hat{A}_n$, with equivalence constants depending only on n.

Theorem 2.1 is a key tool in our proof of Theorem 1.4. The proof of Theorem 1.7 requires the stronger Sobolev–Poincaré inequality, involving norms of Orlicz–Lorentz type defined in Subsection 2.1, contained in the next result.

Theorem 2.3. *Let B be any ball in $\mathbf{R}^n$. Let A be a Young function satisfying (1.10) and let $D_{A,n}$ be the Young function associated with A and n as in (1.19). Then there exists a constant $k_2(n)$, depending only on n, such that*

$$\|u-u_B\|_{L(n,D_{A,n})(B)}\leq k_2(n)\|\,|\nabla u|\,\|_{L^A(B)} \tag{2.15}$$

for every weakly differentiable function u in B such that $|\nabla u|\in L^A(B)$.

A version of (2.15), with $\|u-u_B\|_{L(n,D_{A,n})(B)}$ replaced by $\|u\|_{L(n,D_{A,n})(B)}$ and for functions $u\in W_0^{1,A}(B)$, is established in [C4] via symmetrization and interpolation techniques. The proof of Theorem 2.3 follows an analogous scheme. However, the first part of the proof, whose task is to reduce (2.15) to a one-dimensional inequality, is more delicate, due to the fact that functions are taken into account which do not necessarily vanish on ∂B. The symmetrization argument in this case rests upon a form of the Pólya–Szegő principle (see e.g. [C1]), which tells us that if

A is a Young function and $u \in W^{1,A}(B)$, then u° is locally absolutely continuous in $(0,|B|)$, and a constant $k_3(n)$, depending only on n, exists such that

$$\left\| k_3(n)\min^{1/n'}\{(\,\cdot\,),|B|-(\,\cdot\,)\}\left(-\frac{du^\circ}{ds}(\,\cdot\,)\right)\right\|_{L^A(0,|B|)} \le \|\,|\nabla u|\,\|_{L^A(B)}. \tag{2.16}$$

Proof. We have

$$\|u-u_B\|_{L(n,D_{A,n})(B)} = \|u^\circ - u_B\|_{L(n,D_{A,n})(0,|B|)}. \tag{2.17}$$

By the triangle inequality and the very definition of the norm in $L(n,D_{A,n})(0,|B|)$,

$$\begin{aligned}\|u^\circ - u_B\|_{L(n,D_{A,n})(0,|B|)} &\le \|(u^\circ - u_B)\chi_{(0,|B|/2)}\|_{L(n,D_{A,n})(0,|B|)}\\ &\quad+\|(u^\circ - u_B)\chi_{(|B|/2,|B|)}\|_{L(n,D_{A,n})(0,|B|)}\\ &= \|(\,\cdot\,)^{-1/n}((u^\circ - u_B)\chi_{(0,|B|/2)})^*(\,\cdot\,)\|_{L^{D_{A,n}}(0,|B|)}\\ &\quad+\|(\,\cdot\,)^{-1/n}((u^\circ - u_B)\chi_{(|B|/2,|B|)})^*(\,\cdot\,)\|_{L^{D_{A,n}}(0,|B|)}.\end{aligned} \tag{2.18}$$

Since u° is locally absolutely continuous in $(0,|B|)$, it is easily verified that

$$u^\circ(s) - u_B = \int_0^{|B|}\left(\chi_{(s,|B|)}(r) - \frac{r}{|B|}\right)\left(-\frac{du^\circ}{dr}\right)dr \quad \text{for } s\in(0,|B|). \tag{2.19}$$

Let $\varphi(s)$ be the right-hand side of (2.19). The function φ is non-increasing in $\left(0,\tfrac12|B|\right)$ and non-decreasing in $\left(\tfrac12|B|,|B|\right)$. Hence,

$$\begin{aligned}(\varphi\chi_{(0,|B|/2)})^*(s) &= \varphi(s)\chi_{(0,|B|/2)}(s),\\ (\varphi\chi_{(|B|/2,|B|)})^*(s) &= \varphi(|B|-s)\chi_{(0,|B|/2)}(s)\end{aligned} \tag{2.20}$$

for $s\ge 0$. From (2.18)–(2.20) we infer, after a change of variable, that

$$\begin{aligned}&\|u^\circ - u_B\|_{L(n,D_{A,n})(0,|B|)}\\ &\quad\le \left\|(\,\cdot\,)^{-1/n}\chi_{(0,|B|/2)}(\,\cdot\,)\int_0^{|B|}\left|\chi_{(\,\cdot\,,|B|)}(r)-\frac{r}{|B|}\right|\left(-\frac{du^\circ}{dr}(r)\right)dr\right\|_{L^{D_{A,n}}(0,|B|)}\\ &\qquad+\left\|(\,\cdot\,)^{-1/n}\chi_{(0,|B|/2)}(\,\cdot\,)\int_0^{|B|}\left|\chi_{(\,\cdot\,,|B|)}(r)-\frac{r}{|B|}\right|\left(-\frac{du^\circ}{dr}(|B|-r)\right)dr\right\|_{L^{D_{A,n}}(0,|B|)}.\end{aligned} \tag{2.21}$$

Now, define the linear operator T, at a locally integrable function ψ on $(0,|B|)$, as

$$T\psi(s) = s^{-1/n}\chi_{(0,|B|/2)}(s)\int_0^{|B|} r^{-1/n'}\left|\chi_{(s,|B|)}(r)-\frac{r}{|B|}\right|\psi(r)\,dr \quad \text{for } s\in(0,|B|).$$

If we prove that a constant c, depending only on n, exists such that

$$\|T\psi\|_{L^{D_{A,n}}(0,|B|)} \leq c\|\psi\|_{L^A(0,|B|)} \tag{2.22}$$

for every $\psi \in L^A(0,|B|)$, then we deduce from (2.21) and from the Pólya–Szegő principle (2.16) that

$$\begin{aligned} \|u^\circ - u_B\|_{L(n,D_{A,n})(0,|B|)} &\leq c\left\|(\,\cdot\,)^{1/n'}\left(-\frac{du^\circ}{ds}(\,\cdot\,)\right)\right\|_{L^A(0,|B|)} \\ &\quad + c\left\|(\,\cdot\,)^{1/n'}\left(-\frac{du^\circ}{ds}(\,\cdot\,)\right)\right\|_{L^A(0,|B|)} \\ &\leq 2ck_3(n)\|\,|\nabla u|\,\|_{L^A(B)}. \end{aligned} \tag{2.23}$$

Hence, (2.15) follows with $k_2(n) = 2ck_3(n)$. As for (2.22), it is not difficult to show that constants c_1 and c_2, depending only on n, exist such that

$$\|T\psi\|_{L^1(0,|B|)} \leq c_1 \|\psi\|_{L^1(0,|B|)} \tag{2.24}$$

for $\psi \in L^1(0,|B|)$, and

$$\|T\psi\|_{L^{n,\infty}(0,|B|)} \leq c_2 \|\psi\|_{L^{n,1}(0,|B|)} \tag{2.25}$$

for $\psi \in L^{n,1}(0,|B|)$. Thus, by the interpolation theorem [C4, Theorem 3.1], inequality (2.22) holds with $c = \max\{c_1, c_2\}$. □

3. Proof of the main results

Our approach exploits some recent developments in the theory of Orlicz–Sobolev spaces, as well as techniques employed in [S1] and [EG] for ordinary Sobolev spaces. An underlying idea is to make use of the Lebesgue differentiation theorem applied to the gradient of a weakly differentiable function. In this connection, a basic result in the present setting is contained in the following lemma.

Lemma 3.1. *Let Ω be a measurable subset of $\mathbf{R}^n$ and let A be a finite-valued Young function. Let $f \in L^A(\Omega)$. Then there exists $\lambda > 0$ such that*

$$\lim_{r\to 0^+} \fint_{B_r(x)} A\left(\frac{|f(y)-f(x)|}{\lambda}\right) dy = 0 \tag{3.1}$$

for a.e. $x \in \Omega$.

A proof of Lemma 3.1 closely follows that of [EG, Corollary 1, Section 1.7.1]. We include it for completeness. Let us emphasize that this proof, as well as the proofs of the other results of this paper, does not rely upon approximation arguments. This is of fundamental importance, since any such approximation argument would require a Δ_2 condition on the Young function A, an assumption which we never make in this paper.

Proof. Since $f \in L^A$, there exists $\lambda>0$ such that $\int_\Omega A(4|f(y)|/\lambda)\,dy<+\infty$. Consequently, for every $t\in\mathbf{R}$ and for every subset E of Ω having finite measure,

$$\begin{aligned}\int_E A\left(\frac{2|f(y)-t|}{\lambda}\right)dy &\le \int_E A\left(\frac{2|f(y)|+2|t|}{\lambda}\right)dy\\ &\le \frac{1}{2}\int_E A\left(4\frac{|f(y)|}{\lambda}\right)dy+\frac{1}{2}\int_E A\left(4\frac{|t|}{\lambda}\right)dy\\ &= \frac{1}{2}\int_E A\left(4\frac{|f(y)|}{\lambda}\right)dy+\frac{1}{2}A\left(4\frac{|t|}{\lambda}\right)|E|\\ &< +\infty.\end{aligned}$$

Thus, $A(2|f(y)-t|/\lambda)\in L^1_{\mathrm{loc}}(\Omega)$ for every $t\in\mathbf{R}$. Let $\{t_i\}_{i\in\mathbf{N}}$ be any dense sequence in $\mathbf{R}$. By the Lebesgue differentiation theorem (1.1), a family $\{N_i\}_{i\in\mathbf{N}}$ of subsets of Ω exists such that, for every $i\in\mathbf{N}$, $|N_i|=0$ and

$$\lim_{r\to0^+}\fint_{B_r(x)} A\left(\frac{2|f(y)-t_i|}{\lambda}\right)dy = A\left(\frac{2|f(x)-t_i|}{\lambda}\right)$$

for every $x\in\Omega\setminus N_i$. Hence, by setting $N=\bigcup_{i\in\mathbf{N}} N_i$, we have that $|N|=0$ and

$$\lim_{r\to0^+}\fint_{B_r(x)} A\left(\frac{2|f(y)-t_i|}{\lambda}\right)dy = A\left(\frac{2|f(x)-t_i|}{\lambda}\right) \tag{3.2}$$

for every $x\in\Omega\setminus N$. Now, fix any $x\in\Omega\setminus N$ and any $\varepsilon>0$, and choose $t_i\in\mathbf{R}$ such that

$$A\left(\frac{2|f(x)-t_i|}{\lambda}\right)<\varepsilon. \tag{3.3}$$

Thus, by (3.2) and (3.3),

$$\begin{aligned}\limsup_{r\to0^+}\fint_{B_r(x)} A\left(\frac{|f(y)-f(x)|}{\lambda}\right)dy &\le \limsup_{r\to0^+}\fint_{B_r(x)} A\left(\frac{|f(y)-t_i|}{\lambda}+\frac{|f(x)-t_i|}{\lambda}\right)dy\\ &\le \frac{1}{2}\left[\lim_{r\to0^+}\fint_{B_r(x)} A\left(\frac{2|f(y)-t_i|}{\lambda}\right)dy\right.\\ &\qquad\left.+A\left(\frac{2|f(x)-t_i|}{\lambda}\right)\right]\\ &= A\left(\frac{2|f(x)-t_i|}{\lambda}\right)\\ &<\varepsilon.\end{aligned}$$

Hence, (3.1) follows, owing to the arbitrariness of ε. $\square$

Proof of Theorem 1.1. Thanks to (2.8), after replacing, if necessary, A with another Young function still satisfying (1.6), we may assume that A is finite-valued. Let $v\in W^{1,A}_{\mathrm{loc}}(\Omega)$. By (1.6) and by [C1, Theorem 1b], v is a continuous function. Fix any $x\in\Omega$ and any $r>0$ such that $B_r(x)\Subset\Omega$. Plainly,

$$|v(y)-v(x)|\leq \fint_{B_r(x)} (|v(y)-v(z)|+|v(x)-v(z)|)\,dz \tag{3.4}$$

for $y\in B_r(x)$. By [EG, Lemma 1, Section 4.5.2] (which is stated for $v\in C^1(B_r(x))$, but continues to hold even if v is any continuous function from $W^{1,1}(B_r(x))$), a constant $c_0(n)$, depending only on n, exists such that the right-hand side of (3.4) does not exceed

$$c_0(n)\biggl(\int_{B_r(x)} |\nabla v(z)|\,|z-y|^{1-n}\,dz+\int_{B_r(x)} |\nabla v(z)|\,|z-x|^{1-n}\,dz\biggr).$$

Hence,

$$\begin{aligned}\frac{|v(y)-v(x)|}{|B_r(x)|}&\leq c_0(n)\biggl(\fint_{B_r(x)} |\nabla v(z)|\,|z-y|^{1-n}\,dz+\fint_{B_r(x)} |\nabla v(z)|\,|z-x|^{1-n}\,dz\biggr)\\ &\leq 2c_0(n)\bigl(\|\,|\cdot-y|^{1-n}\|_{\mathcal{L}^{\tilde A}(B_r(x))}\\ &\qquad+\|\,|\cdot-x|^{1-n}\|_{\mathcal{L}^{\tilde A}(B_r(x))}\bigr)\|\,|\nabla v|\,\|_{\mathcal{L}^{A}(B_r(x))}.\end{aligned} \tag{3.5}$$

After denoting the measure of the unit ball in $\mathbf{R}^n$ by ω_n, we have

$$\|\,|\cdot-y|^{1-n}\|_{\mathcal{L}^{\tilde A}(B_r(x))}\leq\|\,|\cdot-x|^{1-n}\|_{\mathcal{L}^{\tilde A}(B_r(x))}=\omega_n^{1/n'}\|(\,\cdot\,)^{-1/n'}\|_{\mathcal{L}^{\tilde A}(0,|B_r(x)|)}, \tag{3.6}$$

where the inequality is a consequence of the Hardy–Littlewood inequality (see e.g. [BS, Theorem 2.2, Chapter 2]) and the equality is due to the fact that

$$(|\cdot-x|^{1-n})^*(s)=\omega_n^{1/n'}s^{-1/n'}$$

for $s>0$. Combining (3.5) and (3.6) yields

$$\frac{|v(y)-v(x)|}{|B_r(x)|}\leq 4c_0(n)\omega_n^{1/n'}\|(\,\cdot\,)^{-1/n'}\|_{\mathcal{L}^{\tilde A}(0,|B_r(x)|)}\|\,|\nabla v|\,\|_{\mathcal{L}^{A}(B_r(x))} \tag{3.7}$$

for $y\in B_r(x)$. By setting

$$F(t)=n't^{n'}\int_t^\infty \frac{\tilde A(\tau)}{\tau^{1+n'}}\,d\tau\quad\text{for } t>0, \tag{3.8}$$

we have

$$\frac{1}{|B_r(x)|}\int_0^{|B_r(x)|}\tilde{A}\left(\frac{s^{-1/n'}}{\lambda}\right)ds = F\left(\frac{1}{\lambda|B_r(x)|^{1/n'}}\right) \quad \text{for } \lambda>0. \tag{3.9}$$

Notice that, by (2.12), $F(t)<+\infty$ for every $t>0$; moreover, F strictly increases from 0 to $+\infty$ as t goes from 0 to $+\infty$. Hence,

$$\|(\,\cdot\,)^{-1/n'}\|_{\text{Ł}^{\tilde{A}}(0,|B_r(x)|)} = \frac{1}{|B_r(x)|^{1/n'}F^{-1}(1)}. \tag{3.10}$$

From (3.7) and (3.10) one gets

$$|v(y)-v(x)| \le \frac{4c_0(n)\omega_n r}{F^{-1}(1)}\|\,|\nabla v|\,\|_{\text{Ł}^A(B_r(x))} \tag{3.11}$$

for $y\in B_r(x)$. Now, let λ be any positive number such that $\int_{B_r(x)} A(|\nabla v|/\lambda)\,dy<+\infty$. Let us set $M=\fint_{B_r(x)} A(|\nabla v|/\lambda)\,dy$ and

$$A_M(s)=\frac{A(s)}{M} \quad \text{for } s\ge 0. \tag{3.12}$$

By the definition of the averaged Luxemburg norm, we have $\|\,|\nabla v|/\lambda\|_{\text{Ł}^{A_M}(B_r(x))}\le 1$. Moreover, if we let F_M be the function defined as in (3.8) with A replaced by A_M, then $F_M(t)=(1/M)F(tM)$ for $t>0$, whence $F_M^{-1}(s)=(1/M)F^{-1}(Ms)$ for $s>0$. Consequently, setting

$$L(s)=\frac{s}{F^{-1}(s)} \quad \text{for } s>0$$

and applying (3.11) with v replaced by $(1/\lambda)(u(y)-u(x)-\langle\nabla u(x),y-x\rangle)$ and with A replaced by A_M yield

$$\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{\lambda r} \le 4c_0(n)\omega_n L\left(\fint_{B_r(x)} A\left(\frac{|\nabla u(y)-\nabla u(x)|}{\lambda}\right)dy\right) \tag{3.13}$$

for $y\in B_r(x)$.

An application of the l'Hospital rule shows that $\lim_{t\to 0}F(t)/t=0$, whence $\lim_{s\to 0}L(s)=0$. By Lemma 3.1, for any fixed open set $\Omega'\Subset\Omega$, the averaged integral on the right-hand side of (3.13) converges to 0 as r goes to 0 for a.e. $x\in\Omega'$ if λ is sufficiently large. Thus, (3.13) applied to $r=|y-x|$, ensures that

$$\lim_{y\to 0}\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{|y-x|}=0 \tag{3.14}$$

for a.e. $x\in\Omega'$. Hence, the conclusion follows. □

Proof of Proposition 1.3. When A satisfies (1.7), as a consequence of [C1, Theorems 1a and 1b] and of Lemma 2.2, $W^{1,A}(B_1(0))$ is not continuously embedded into $L^\infty(B_1(0))$. In particular, an inspection of the proofs of those theorems reveals that a sequence of nonnegative spherically symmetric functions $\{u_k\}_{k\in\mathbf{N}}$ can be chosen in such a way that $u_k\in W_0^{1,A}(B_1(0))$, $\int_{B_1(0)} A(|\nabla u_k|)\,dx\le 1$ and $\operatorname{ess\,sup} u_k = u_k(0)\ge 4^k$. Let us still denote by u_k the continuation by 0 of u_k outside $B_1(0)$. Thus, u_k is weakly differentiable in the whole of $\mathbf{R}^n$. Let $\{x_k\}_{k\in\mathbf{N}}$ be the sequence of points in $\mathbf{R}^n$ with rational coordinates, and let $u\colon\mathbf{R}^n\to[0,+\infty)$ be the function defined as $u(x)=\sum_{k=1}^\infty (1/2^k)u_k(x-x_k)$ for $x\in\mathbf{R}^n$.

Since $\operatorname{ess\,sup} u_k(x_k)\ge(1/2^k)u_k(0)\ge 2^k$, u is not essentially bounded in any neighborhood of any point of $B_1(0)$. Hence, u is not equivalent to any function which is differentiable at some point of $B_1(0)$. On the other hand, inasmuch as $\int_{B_1(0)} A(|u_k|)\,dx\le\int_{B_1(0)} A(|\nabla u_k|)\,dx$ for every k (see [T2, Lemma 3]),

$$\begin{aligned}\int_{B_1(0)} A(|u(x)|)\,dx &\le \int_{B_1(0)} A\biggl(\sum_{k=1}^\infty \frac{1}{2^k}|u_k(x-x_k)|\biggr)\,dx\\ &\le \sum_{k=1}^\infty \frac{1}{2^k}\int_{B_1(0)} A(|u_k(x-x_k)|)\,dx \le \sum_{k=1}^\infty \frac{1}{2^k}=1,\end{aligned}$$

where the second inequality is due to the convexity of A. Thus, $u\in L^A(B_1(0))$, and since

$$\begin{aligned}\int_{B_1(0)} A\biggl(\sum_{k=1}^\infty \frac{1}{2^k}|\nabla u_k(x-x_k)|\biggr)\,dx &\le \sum_{k=1}^\infty \frac{1}{2^k}\int_{B_1(0)} A(|\nabla u_k(x-x_k)|)\,dx\\ &\le \sum_{k=1}^\infty \frac{1}{2^k}=1,\end{aligned}$$

u is easily seen to be a weakly differentiable function in $B_1(0)$ with

$$\int_{B_1(0)} A(|\nabla u(x)|)\,dx\le 1.$$

Hence, $u\in W^{1,A}(B_1(0))$. $\square$

The proofs of Theorems 1.4 and 1.7 require the next lemma, containing a weak version of (1.11).

Lemma 3.2. *Let Ω be an open subset of $\mathbf{R}^n$ and let A be a Young function. Assume that $u\in W^{1,A}_{\mathrm{loc}}(\Omega)$. Then*

(3.15)
$$\fint_{B_r(x)} A\biggl(\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{r}\biggr)\,dy \le \sup_{0\le s\le 1}\fint_{B_{sr}(x)} A(|\nabla u(y)-\nabla u(x)|)\,dy$$

for a.e. $x\in\Omega$ and for every $r>0$ such that $B_r(x)\Subset\Omega$.

Proof. Let $u\in W^{1,A}_{\mathrm{loc}}(\Omega)$. Hence, in particular, $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$. Thus, for a.e. $x\in\Omega$, u is absolutely continuous on a.e. ray issued from x (see e.g. [Z, Chapter 3, Exercise 3.15]). Fix any such x and let r be a positive number such that $B_r(x)\Subset\Omega$. Then, for a.e. $y\in B_r(x)$, the function $s\mapsto u(x+s(y-x))$ is absolutely continuous in $[0,1]$. Moreover,

$$\frac{d}{ds}u(x+s(y-x))=\langle\nabla u(x+s(y-x)),y-x\rangle \tag{3.16}$$

for a.e. $s\in[0,1]$ (see e.g. [AFP, Theorem 3.108]). Consequently,

$$u(y)-u(x)-\langle\nabla u(x),y-x\rangle=\int_0^1\langle\nabla u(x+s(y-x))-\nabla u(x),y-x\rangle\,ds \tag{3.17}$$

for a.e. $y\in B_r(x)$. Hence,

$$\begin{aligned}
&\fint_{B_r(x)} A\biggl(\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{r}\biggr)\,dy\\
&\qquad\le\fint_{B_r(x)} A\biggl(\int_0^1\frac{1}{r}|\nabla u(x+s(y-x))-\nabla u(x)|\,|y-x|\,ds\biggr)\,dy\\
&\qquad\le\fint_{B_r(x)}\int_0^1 A\biggl(\frac{1}{r}|\nabla u(x+s(y-x))-\nabla u(x)|\,|y-x|\biggr)\,ds\,dy\\
&\qquad=\int_0^1\fint_{B_{sr}(x)} A\biggl(\frac{1}{sr}|\nabla u(z)-\nabla u(x)|\,|z-x|\biggr)\,dz\,ds\\
&\qquad\le\int_0^1\fint_{B_{sr}(x)} A(|\nabla u(z)-\nabla u(x)|)\,dz\,ds\\
&\qquad\le\sup_{0\le s\le 1}\fint_{B_{sr}(x)} A(|\nabla u(z)-\nabla u(x)|)\,dz,
\end{aligned}\tag{3.18}$$

where the second inequality holds by Jensen's inequality. $\square$

Proof of Theorem 1.4. Fix any open set $\Omega'\Subset\Omega$. Let $x\in\Omega'$ and $r>0$ be such that $B_r(x)\Subset\Omega'$. Let $v\in W^{1,A}_{\mathrm{loc}}(\Omega)$. Then

$$
\begin{aligned}
\|v\|_{L^{A_n}(B_r(x))} &\le \|v-v_{B_r(x)}\|_{L^{A_n}(B_r(x))}+\|v_{B_r(x)}\|_{L^{A_n}(B_r(x))}\\
&\le k_1(n)\|\,|\nabla v|\,\|_{L^A(B_r(x))}+\|v_{B_r(x)}\|_{L^{A_n}(B_r(x))}\\
&\le k_1(n)\|\,|\nabla v|\,\|_{L^A(B_r(x))}+\frac{1}{|B_r(x)|}\int_{B_r(x)}|v|\,dx\|1\|_{L^{A_n}(B_r(x))}\\
&\le k_1(n)\|\,|\nabla v|\,\|_{L^A(B_r(x))}\\
&\quad+\frac{2}{|B_r(x)|}\|v\|_{L^A(B_r(x))}\|1\|_{L^{\tilde{A}}(B_r(x))}\|1\|_{L^{A_n}(B_r(x))}\\
&= k_1(n)\|\,|\nabla v|\,\|_{L^A(B_r(x))}\\
&\quad+\frac{2}{|B_r(x)|}\|v\|_{L^A(B_r(x))}\frac{1}{\tilde{A}^{-1}\left(\frac{1}{|B_r(x)|}\right)}\frac{1}{A_n^{-1}\left(\frac{1}{|B_r(x)|}\right)}.
\end{aligned}
\tag{3.19}
$$

Notice that the second inequality holds thanks to the Sobolev–Poincaré inequality (2.10) and that the last equality follows from (2.6). [C2, Inequality (3.26)] and Lemma 2.2 ensure that a constant $c_1(n)$, depending only on n, exists such that

$$
\frac{1}{t^{1/n'}\tilde{A}^{-1}\left(\frac{1}{t}\right)A_n^{-1}\left(\frac{1}{t}\right)}\le c_1(n)\quad\text{for } t>0.
\tag{3.20}
$$

Combining (3.19) and (3.20) yields

$$
\|v\|_{L^{A_n}(B_r(x))}\le k_1(n)\|\,|\nabla v|\,\|_{L^A(B_r(x))}+2c_1(n)\omega_n^{-1/n}\left\|\frac{v}{r}\right\|_{L^A(B_r(x))}.
\tag{3.21}
$$

Assume for a moment that the quantity

$$
\int_{B_r(x)}A(|\nabla v|)\,dy+\int_{B_r(x)}A\left(\frac{|v|}{r}\right)dy
\tag{3.22}
$$

is finite, and call it M. For such a choice of M, define A_M as in (3.12), and observe that, if $(A_M)_n$ is the function associated with A_M as in (1.8), then

$$
(A_M)_n(t)=\frac{1}{M}A_n\left(\frac{t}{M^{1/n}}\right)\quad\text{for } t\ge 0.
\tag{3.23}
$$

Since

$$
\|\,|\nabla v|\,\|_{L^{A_M}(B_r(x))}\le 1\quad\text{and}\quad\left\|\frac{v}{r}\right\|_{L^{A_M}(B_r(x))}\le 1,
$$

then, after replacing A by A_M in (3.21), one gets

$$\|v\|_{L^{(A_M)_n}(B_r(x))} \le c_2(n), \tag{3.24}$$

where $c_2(n) = k_1(n) + 2c(n)\omega_n^{-1/n}$. From (3.23) and (3.24) we deduce that

$$\begin{aligned} &\fint_{B_r(x)} A_n\left(\frac{|v(y)|}{c_3(n) r \left(\fint_{B_r(x)} A(|\nabla v|)\,dz + \fint_{B_r(x)} A\left(\frac{|v|}{r}\right) dz\right)^{1/n}}\right) dy \\ &\qquad \le \fint_{B_r(x)} A(|\nabla v|)\,dy + \fint_{B_r(x)} A\left(\frac{|v|}{r}\right) dy, \end{aligned} \tag{3.25}$$

where $c_3(n) = c_2(n)\omega_n^{1/n}$. Obviously, inequality (3.25) continues to hold even if the expression (3.22) is infinite. Applying (3.25) with

$$v(\,\cdot\,) = \frac{1}{\lambda}(u(\,\cdot\,) - u(x) - \langle \nabla u(x),\, \cdot - x\rangle)$$

and $\lambda > 0$, and setting

$$N = \fint_{B_r(x)} A\left(\frac{|\nabla u(z) - \nabla u(x)|}{\lambda}\right) dz + \fint_{B_r(x)} A\left(\frac{|u(z) - u(x) - \langle \nabla u(x), y - x\rangle|}{\lambda r}\right) dz$$

yield

$$\fint_{B_r(x)} A_n\left(\frac{|u(y) - u(x) - \langle \nabla u(x), y - x\rangle|}{\lambda c_3(n) r N^{1/n}}\right) dy \le N. \tag{3.26}$$

Define

$$\phi(r,x) = 2 \sup_{0 \le s \le 1} \fint_{B_{sr}(x)} A\left(\frac{|\nabla u(y) - \nabla u(x)|}{\lambda}\right) dy. \tag{3.27}$$

Then we get from (3.26) and from Lemma 3.2,

$$\fint_{B_r(x)} A_n\left(\frac{|u(y) - u(x) - \langle \nabla u(x), y - x\rangle|}{\lambda c_3(n) r \phi(r,x)^{1/n}}\right) dy \le \phi(r,x) \tag{3.28}$$

for a.e. $x \in \Omega'$. By Lemma 3.1, a number λ exists such that $\lim_{r\to 0} \phi(r,x) = 0$ for a.e. $x \in \Omega'$. Fix any such x. Given any $\varepsilon > 0$, there exists r_ε such that $\phi(r,x) < \varepsilon$ if $0 < r < r_\varepsilon$. Thus, by (3.28),

$$\fint_{B_r(x)} A_n\left(\frac{|u(y) - u(x) - \langle \nabla u(x), y - x\rangle|}{\lambda c_3(n) r \varepsilon^{1/n}}\right) dy < \varepsilon \tag{3.29}$$

if $0<r<r_\varepsilon$. Choosing $\varepsilon<(\sigma/\lambda c_3(n))^n$ in (3.29) yields (1.11).

Finally, given any $\sigma>0$, fix any x for which (1.11) holds. Then there exists $r_\sigma>0$ such that

$$\fint_{B_r(x)} A_n\left(\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{\sigma r}\right)dy\le 1$$

if $0<r<r_\sigma$. Hence,

$$\left\|\frac{u(\,\cdot\,)-u(x)-\langle\nabla u(x),\,\cdot-x\rangle}{r}\right\|_{\not L^{A_n}(B_r(x))}\le\sigma$$

if $0<r<r_\sigma$, and (1.12) follows. □

Proof of Theorem 1.5. The conclusion will be derived from Theorem 1.4, via a discretization argument (see e.g. [AFP, Example 3.16]). Fix any open set $\Omega'\Subset\Omega$. Let $x\in\Omega'$ and $r>0$ be such that $B_r(x)\Subset\Omega'$. Let $u\in W^{1,A}_{\mathrm{loc}}(\Omega)$ and let ϕ be the function, non-decreasing in r, defined by (3.27). Let $c_3(n)$ be the constant appearing in (3.28). Given any $\lambda>0$, we have

$$\begin{aligned}
&\fint_{B_r(x)} A_n\left(\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{2\lambda c_3(n)|y-x|\phi(r,x)^{1/n}}\right)dy\\
&\qquad=\sum_{i=0}^{\infty}\frac{1}{\omega_n r^n}\int_{B_{r2^{-i}}(x)\setminus B_{r2^{-i-1}}(x)} A_n\left(\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{2\lambda c_3(n)\,|y-x|\phi(r,x)^{1/n}}\right)dy\\
&\qquad\le\sum_{i=0}^{\infty}\frac{1}{\omega_n r^n}\int_{B_{r2^{-i}}(x)} A_n\left(\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{r2^{-i}\lambda c_3(n)\phi(r2^{-i},x)^{1/n}}\right)dy \qquad (3.30)\\
&\qquad=\sum_{i=0}^{\infty}2^{-in}\fint_{B_{r2^{-i}}(x)} A_n\left(\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{r2^{-i}\lambda c_3(n)\phi(r2^{-i},x)^{1/n}}\right)dy.
\end{aligned}$$

By inequality (3.28) applied with r replaced by $r2^{-i}$ for $i=0,1,\dots$, the last sum does not exceed $\sum_{i=0}^{\infty}2^{-in}\phi(r2^{-i},x)$ for a.e. $x\in\Omega'$, and this expression is smaller than or equal to $\sum_{i=0}^{\infty}2^{-in}\phi(r,x)=(2^n/(2^n-1))\phi(r,x)$. Thus,

$$\fint_{B_r(x)} A_n\left(\frac{|u(y)-u(x)-\langle\nabla u(x),y-x\rangle|}{2\lambda c_3(n)|y-x|\phi(r,x)^{1/n}}\right)dy\le\frac{2^n}{2^n-1}\phi(r,x) \qquad (3.31)$$

for a.e. $x\in\Omega'$. Starting from (3.31) instead of (3.28) and arguing as in the proof of Theorem 1.4 yield (1.13) and (1.14). □

Proof of Corollary 1.6. Assume first that $0\le\alpha<n-1$. Let $\mathcal{A}(t)$ be any Young function which is equivalent to $t^{n/(n-\alpha)}$ near 0 and is equivalent to $A(t)$ near infinity. Then $\mathcal{A}$ satisfies (1.7) and (1.10). Moreover, $W^{1,\mathcal{A}}_{\mathrm{loc}}(\Omega)=W^{1,A}_{\mathrm{loc}}(\Omega)$. Thus, $u\in W^{1,\mathcal{A}}_{\mathrm{loc}}(\Omega)$. It is easily verified that the function $\exp(t^{n/(n-1-\alpha)})-1$ and the function $\mathcal{A}_n(t)$ associated with $\mathcal{A}$ as in (1.8) are globally equivalent. Hence, (1.15) and (1.16) follow from (1.13) and (1.14), respectively.

Consider now the case where $\alpha=n-1$, and let $\mathcal{A}(t)$ be any Young function which is equivalent to t near 0 and to $A(t)$ near infinity. Since $\mathcal{A}_n(t)$ is globally equivalent to $\exp(\exp(t^{n'}))-e$, the conclusion follows as above.

Finally, if $\alpha>n-1$, then A fulfills (1.6). Hence, by Theorem 1.1, u is differentiable a.e. in Ω. $\square$

Our last task is the proof of Theorem 1.7. We shall need the following lemma.

Lemma 3.3. *Let A be a Young function satisfying* (1.10). *Then a positive constant $k_4(n)$, depending only on n, exists such that*

$$\|\chi_{(0,t)}\|_{L(n,D_{A,n})(0,\infty)}\le k_4(n)\frac{1}{A_n^{-1}\left(\frac{1}{t}\right)} \quad \textit{for } t>0. \tag{3.32}$$

Proof. By [C4, Inequality (3.1)] a constant c, depending only on n, exists such that

$$\left\|\int_{(\,\cdot\,)}^{\infty} r^{-1/n'}\psi(r)\,dr\right\|_{L(n,D_{A,n})(0,\infty)}\le c\|\psi\|_{L^A(0,\infty)} \tag{3.33}$$

for every $\psi\in L^A(0,,\infty)$. Thus,

$$\begin{aligned}
c&\ge\sup_{\psi\in L^A(0,\infty)}\frac{\|\int_{(\,\cdot\,)}^{\infty} r^{-1/n'}|\psi(r)|\,dr\|_{L(n,D_{A,n})(0,\infty)}}{\|\psi\|_{L^A(0,\infty)}}\\
&=\sup_{\psi\in L^A(0,\infty)}\ \sup_{\varphi\in L(n,D_{A,n})'(0,\infty)}\frac{\int_0^\infty\varphi^*(s)\int_s^\infty r^{-1/n'}|\psi(r)|\,dr\,ds}{\|\varphi\|_{L(n,D_{A,n})'(0,\infty)}\|\psi\|_{L^A(0,\infty)}}\\
&=\sup_{\varphi\in L(n,D_{A,n})'(0,\infty)}\ \sup_{\psi\in L^A(0,\infty)}\frac{\int_0^\infty|\psi(r)|r^{-1/n'}\int_0^r\varphi^*(s)\,ds\,dr}{\|\varphi\|_{L(n,D_{A,n})'(0,\infty)}\|\psi\|_{L^A(0,\infty)}}\\
&\ge\sup_{\varphi\in L(n,D_{A,n})'(0,\infty)}\frac{\|(\,\cdot\,)^{-1/n'}\int_0^{(\,\cdot\,)}\varphi^*(s)\,ds\|_{L^{\tilde{A}}(0,\infty)}}{\|\varphi\|_{L(n,D_{A,n})'(0,\infty)}}.
\end{aligned} \tag{3.34}$$

Notice that the first equality relies on (2.2), and the last inequality is due to (2.7). We have from (3.34) that

$$\left\| (\,\cdot\,)^{-1/n'} \int_0^{(\,\cdot\,)} \varphi^*(s)\,ds \right\|_{L^{\tilde{A}}(0,\infty)} \le c \|\varphi\|_{L(n,D_{A,n})'(0,\infty)} \tag{3.35}$$

for $\varphi \in L(n, D_{A,n})'(0,\infty)$. Given $t>0$, choose $\varphi(s)=\chi_{(0,t)}(s)$ and observe that

$$r^{-1/n'} \int_0^r \chi_{(0,t)}(s)\,ds = tr^{-1/n'}$$

if $r \ge t$. Thus, (3.35) yields

$$t\|(\,\cdot\,)^{-1/n'}\chi_{(t,\infty)}(\,\cdot\,)\|_{L^{\tilde{A}}(0,\infty)} \le c\|\chi_{(0,t)}\|_{L(n,D_{A,n})'(0,\infty)} \quad \text{for } t>0. \tag{3.36}$$

Hence, by Lemma 2.2, a constant c', depending only on n, exists such that

$$\frac{t}{\|\chi_{(0,t)}\|_{L(n,D_{A,n})'(0,\infty)}} \le \frac{c'}{A_n^{-1}\left(\dfrac{1}{t}\right)} \quad \text{for } t>0. \tag{3.37}$$

By (2.4), the left-hand side of (3.37) equals $\|\chi_{(0,t)}\|_{L(n,D_{A,n})(0,\infty)}$, and (3.32) follows. □

Proof of Theorem 1.7. For any fixed $x \in \Omega$, any $r>0$ such that $B_r(x) \Subset \Omega$, and given any $v \in W^{1,A}(\Omega)$, we start as in the proof of Theorem 1.4. Making use of inequality (2.15) instead of (2.10) we get

$$\begin{aligned}
\|v\|_{L(n,D_{A,n})(B_r(x))} &\le \|v - v_{B_r(x)}\|_{L(n,D_{A,n})(B_r(x))} + \|v_{B_r(x)}\|_{L(n,D_{A,n})(B_r(x))} \\
&\le k_2(n) \|\,|\nabla v|\,\|_{L^A(B_r(x))} \\
&\quad + \frac{1}{|B_r(x)|} \int_{B_r(x)} |v(y)|\,dy \|1\|_{L(n,D_{A,n})(B_r(x))} \\
&\le k_2(n) \|\,|\nabla v|\,\|_{L^A(B_r(x))} \\
&\quad + \frac{2}{|B_r(x)|} \|v\|_{L^A(B_r(x))} \|1\|_{L^{\tilde{A}}(B_r(x))} \|1\|_{L(n,D_{A,n})(B_r(x))} \\
&= k_2(n) \|\,|\nabla v|\,\|_{L^A(B_r(x))} \\
&\quad + \frac{2}{|B_r(x)|} \|v\|_{L^A(B_r(x))} \frac{\|\chi_{(0,|B_r(x)|)}\|_{L(n,D_{A,n})(0,\infty)}}{\tilde{A}^{-1}\left(\dfrac{1}{|B_r(x)|}\right)}.
\end{aligned} \tag{3.38}$$

Inequalities (3.32) and (3.20) ensure that

$$\frac{\|\chi_{(0,t)}\|_{L(n,D_{A,n})(0,\infty)}}{t^{1/n'}\tilde{A}^{-1}\left(\frac{1}{t}\right)} \le c_1(n)k_4(n) \quad \text{for } t>0. \tag{3.39}$$

Combining (3.38) and (3.39) yields

(3.40)

$$\|v\|_{L(n,D_{A,n})(B_r(x))} \le k_4(n)\| \, |\nabla v| \, \|_{L^A(B_r(x))} + \frac{2}{\omega_n^{1/n}} c_1(n)k_4(n)\|v/r\|_{L^A(B_r(x))}.$$

Now, let M denote the quantity (3.22), let A_M be given by (3.12), and let D_M be an abbreviated notation for $D_{A_M,n}$, the function defined as in (1.19) with A replaced by A_M. Then an analogous argument as in the proof of Theorem 1.4 enables us to deduce from (3.40) that

$$\|v\|_{L(n,D_M)(B_r(x))} \le c_4(n), \tag{3.41}$$

where $c_4(n)=k_4(n)+2\omega_n^{-1/n}c_1(n)k_4(n)$. It is easily verified that

$$D_M(t) = \frac{D_{A,n}(t)}{M}$$

for $t \ge 0$. Thus, inequality (3.41) implies that

$$\fint_0^{|B_r(x)|} D_{A,n}\left(\frac{s^{-1/n}v^*(s)}{c_4(n)}\right) ds \le \fint_{B_r(x)} A(|\nabla v|)\,dy + \fint_{B_r(x)} A\left(\frac{|v|}{r}\right) dy. \tag{3.42}$$

By applying (3.42) with $v(\cdot)=(1/\lambda)(u(\cdot)-u(x)-\langle \nabla u(x), \cdot - x\rangle)$ and making use of Lemma 3.2, we get

$$\begin{aligned}
\fint_0^{|B_r(x)|} & D_{A,n}\left(\frac{s^{-1/n}(u(\cdot)-u(x)-\langle \nabla u(x), \cdot - x\rangle)^*(s)}{\lambda c_4(n)}\right) ds \\
&\le \fint_{B_r(x)} A\left(\frac{|\nabla u(y)-\nabla u(x)|}{\lambda}\right) dy \\
&\quad + \fint_{B_r(x)} A\left(\frac{|u(y)-u(x)-\langle \nabla u(x), y-x\rangle|}{\lambda r}\right) dy \\
&\le 2 \sup_{0\le s\le 1} \fint_{B_{sr}(x)} A\left(\frac{|\nabla u(y)-\nabla u(x)|}{\lambda}\right) dy.
\end{aligned} \tag{3.43}$$

By Lemma 3.1, a positive number λ exists such that the last expression converges to 0 as r goes to 0 for a.e. $x \in \Omega$. Hence (1.20) follows. □

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