# A SYSTEM OF QUADRATIC DIOPHANTINE EQUATIONS

W. H. MILLS

**1. Introduction.** In spite of the efforts of many mathematicians of the last 300 years, comparatively few general methods of solving nonlinear Diophantine equations are available, and much of the literature on the subject consists of isolated results. When it comes to systems of simultaneous nonlinear Diophantine equations, the results become even more fragmentary, and a complete solution of such a system is a rarity. In this paper we study a pair of simultaneous quadratic Diophantine equations that can be solved easily and completely by difference equation methods.

The system in question,

$$(1) \qquad x \mid y^2 + ay + 1, \quad y \mid x^2 + ax + 1,$$

where $a$ is a fixed integer, is essentially a pair of simultaneous quadratic equations in four unknowns. This system is equivalent to a nonlinear second order difference equation. Furthermore, every solution of this nonlinear difference equation is also a solution of a linear difference equation with constant coefficients. We can thus obtain the complete solution of (1) in integers. With some additional effort we can obtain all positive integral solutions.

The principal result is that if $a \neq \pm 2$, then there exists a finite number of sequences such that $x$ and $y$ satisfy (1) if and only if they are consecutive terms of one of these sequences. These sequences are similar to the Fibonacci sequence in that there is a linear relation connecting any three consecutive terms. For the special case $a = 0$, we obtain the following result: $x$ and $y$ are positive integers such that

$$x \mid y^2 + 1 \quad \text{and} \quad y \mid x^2 + 1$$

if and only if $x$ and $y$ are consecutive elements of the sequence 1, 1, 2, 5, 13, 34, $\cdots$ obtained from the classical Fibonacci sequence by striking out alternate terms. For $a = \pm 2$, the chief differences are that there is an infinite number of sequences and that 0 can be a term of a sequence.

Received July 10, 1951.
*Pacific J. Math.* 3 (1953), 209-220