LENGTH AND AREA OF A CONVEX CURVE UNDER AFFINE TRANSFORMATION

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1. Introduction. We consider in the plane the class of all convex curves into which a given convex curve can be affinely transformed, and seek the minimum of L^2/A, where L denotes perimeter and A the area. This amounts to finding the minimum length for a fixed area, or, what is the same thing, to finding the minimum length under area-preserving affine transformations. In § 2 are found necessary conditions on the supporting function that a given curve yield the minimum of L^2/A, and in § 3 these are shown to be sufficient. In § 4 is derived a property of the minimizing curves; namely that if they are sufficiently smooth, they have at least six vertices. In § 5 is derived an integral representation of the supporting function of a convex curve, and another lemma to be used in §6. In 6 we study the problem of finding the maximum, over all convex curves, of the minimum over affine transformations of L^2/A; in other words, we seek that curve of given area, which when affinely transformed so as to minimize its length, gives the greatest length. We show that the extreme curve is a polygon of not more than five sides, but fail to show what is extremely likely, that the solution is a triangle.

For general facts about convex figures and their supporting functions which are used, see [3].

2. Necessary conditions. Consider a convex curve K and its area-preserving affine transforms. Since rigid motions can be ignored, any transformation in which we are interested can be written in the form

$$(1) \qquad T: \begin{cases} x = e^{\lambda} x', \\ y = \mu x' + e^{-\lambda} y'. \end{cases}$$

The length $L(\lambda, \mu)$ of the transformed curve $K(\lambda, \mu)$ is a continuous function of λ and μ, and tends to ∞ as $(\lambda^2 + \mu^2)^{1/2}$ becomes large. Thus $L(\lambda, \mu)$ has a minimum value, which we take for the moment to be at $\lambda = \mu = 0$.

Received June 2, 1952.

Pacific J. Math. 3 (1953), 393-402