INVARIANT EXTENSION OF LINEAR FUNCTIONALS

V. L. KLEE, JR.

1. Introduction. The expression $[L, \mathfrak{J}, f, p]$ will mean (i) L is a real linear space and $\mathfrak{J}$ a set of linear transformations of L into L; (ii) f is a linear functional on a linear subspace D_f of L; (iii) p is a positively homogeneous subadditive functional on L; (iv) $f \leqq p$; (v) $TD_f \subset D_f$ and $fT = f$ for each $T \in \mathfrak{J}$. If $[L, \mathfrak{J}, f, p]$, then $\langle L, \mathfrak{J}, f, p\rangle$ will denote the set of all F such that $[L, \mathfrak{J}, F, p]$, $D_F = L$, and $F \mid D_f = f$. With I denoting the identity transformation on L, the Hahn-Banach theorem [**2**, p.28] asserts that if $[L, \{I\}, f, p]$, then $\langle L, \{I\}, f, p\rangle$ is nonempty. More general conditions have been obtained by Agnew and Morse [**1**] and Woodbury [**10**] under which $\langle L, \mathfrak{J}, f, p\rangle$ is nonempty, and by Dunford [**3**] and Yood [**11**] under which $\langle L, \mathfrak{J}, f, mp\rangle$ includes for some $m \geqq 1$ an F which is not identically zero. Their results have applications to the extension and existence of measures [**1**; **3**; **10**; **11**], limits, and so on [**1**], and in proving the "normality" (as used in connection with the Banach-Tarski paradox) of certain sets [**6**]. We prove here a theorem whose corollaries include an extension of the results of Agnew and Morse and Woodbury, and also include the principal results of Dunford and Yood, although Yood's work with relaxed boundedness conditions is not covered here. In addition to the cases in which $\mathfrak{J}$ is a commutative semigroup or a finite or solvable group, we are able to handle the case in which $\mathfrak{J}$ is a compact group of bounded linear transformations.

2. The theorem. We shall use the following result.

(2.1) LEMMA. *Suppose L, $\mathfrak{J}$, f, and p satisfy conditions* (i)-(iii), *and for each $x \in L$ let*

$$q(x) = \inf \left\{ p\left(x + \sum_1^k T_i y_i\right) \middle| \, k \text{ a positive integer}, \; T_i \in \mathfrak{J}, \; y_i \in L \right\}.$$

Then $f \leq q$ on D_f if and only if there exists $F \in \langle L, \{I\}, f, p\rangle$ such that $FT \equiv 0$ on L for each $T \in \mathfrak{J}$.

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