# LOWER BOUNDS FOR HIGHER EIGENVALUES BY FINITE DIFFERENCE METHODS

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**1. Introduction.** This paper gives lower bounds for all the eigenvalues of an arbitrary second order self-adjoint elliptic differential operator on a bounded domain $R$ with zero boundary conditions in terms of the eigenvalues of an associated finite difference problem. When $R$ is sufficiently smooth, the lower bounds converge to the eigenvalues themselves as the mesh size approaches zero. A certain class of self-abjoint systems of elliptic differential equations containing no mixed derivatives is also treated.

Upper bounds for the eigenvalues of a differential operator can always be found by the Rayleigh-Ritz method. That is, one puts piecewise differentiable functions vanishing on the boundary into the Poincaré inequality [14]. It was pointed out by Courant [2] that in the case of second order operators one can reduce the problem of upper bounds to a finite difference eigenvalue problem by using piecewise linear functions (see § 6).

Lower bounds are more difficult to find. The only known method giving arbitrarily close lower bounds for the eigenvalues is that of A. Weinstein [20], which is usually quite difficult to apply. It was shown by G. E. Forsythe [5, 6, 7] that if the eigenvalues $\lambda_1 \leqq \lambda_2 \leqq \cdots$ of the two-dimensional problem

$$\Delta u + \lambda u = 0 \quad \text{in } R \tag{1.1}$$

with $u=0$ on the boundary are approximated by the eigenvalues $\lambda_1^{(h)} \leqq \lambda_2^{(h)} \leqq \cdots$ of a certain finite difference problem on a mesh of size $h$, then there exist constants $\gamma^{(1)}, \gamma^{(2)}, \cdots$ such that

$$\lambda_k^{(h)} \leqq \lambda_k - \gamma^{(k)} h^2 + o(h^2) . \tag{1.2}$$

The $\gamma^{(k)}$ cannot be computed, but are positive for convex $R$. However, the $o(h^2)$ term is completely unknown, so that this asymptotic formula cannot be used to bound $\lambda_k$ below.

It was shown independently by J. Hersch [8] and the author [18, 19] that if $\lambda_1$ is the lowest eigenvalue of (1.1) and if $\lambda_1^{(h)}$ is the lowest eigenvalue of a finite difference problem on a mesh that is slightly arger than $R$, then $\lambda_1^{(h)}$ and, in fact, a quantity slightly larger than $\lambda_1^{(h)}$ are lower bounds for $\lambda_1$.

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