CHARACTERIZATIONS OF CERTAIN LATTICES OF FUNCTIONS

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Introduction. The set $C(X, R)$ of all real-valued continuous functions on a compact Hausdorff space X has been characterized from a variety of points of view. We mention, in particular, those characterizations of $C(X, R)$ as a partially ordered system of some prescribed kind: namely, the characterizations of $C(X, R)$ by Stone as a partially ordered ring [14] and as a lattice-ordered group [15], those by Kakutani [7] and by M. and S. Krein [11] as a lattice-ordered Banach space, and those by Fan [4] and Fleischer [5] as a partially ordered group. The problem of characterizing $C(X, R)$ as a lattice alone was posed by Birkhoff [1, Problem 81] and by Kaplansky [9]. As a partial solution of this problem Kaplansky [9] characterized certain sublattices of $C(X, R)$ as "translation lattices". A solution of the general problem has recently been obtained by Heider [6], and, still more recently, another solution has been announced by Pinsker [12].

In the present paper we obtain, as corollaries of our main results, two new characterizations of the lattice $C(X, R)$. We shall actually solve, however, problems somewhat more general than that of Birkhoff and Kaplansky mentioned above. In the first place, we replace the real chain R by a conditionally complete dense-in-itself chain K which has neither a first nor a last element and which is equipped with its interval topology. In the second, we characterize not only $C(X, K)$ but also an extensive class of sublattices of $C(X, K)$.

We give next a more detailed summary of the results of this paper; following this, we pose some unsolved problems suggested by these results.

A sublattice L of $C(X, K)$ is characterizing (Definition 1.1) in case L separates points in X in a certain strong sense. The space X is K-normal in case $C(X, K)$ is itself characterizing. In Definition 2.10 the notion of an "S-lattice" is introduced. The main result (Theorem 2.16) of § 2 states that a characterizing sublattice of $C(X, K)$ is an S-lattice. (This usage of the term "S-lattice" is inexact but will suffice for the present; the concept itself is inspired by work of Shirota [13].) Section 3 is devoted to a further study of S-lattices and of "S-ideals" in S-lattices. The results of § 3, when applied (in § 4) to a characterizing sublattice L of $C(X, K)$, enable us to reconstruct X as a space of

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