ASYMPTOTIC PROPERTIES OF DERIVATIVES OF STATIONARY MEASURES

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1. Introduction. Let X be a non-empty set and $\mathscr{S}$ be a σ-algebra of subsets of X. Consider the infinite product space $\Omega = \prod_{n=-\infty}^{\infty} X_n$ where $X_n = X$ for $n = 0, \pm 1, \pm 2, \cdots$ and the infinite product σ-algebra $\mathscr{F} = \prod_{n=-\infty}^{\infty} \mathscr{S}_n$ where $\mathscr{S}_n = \mathscr{S}$ for $n = 0, \pm 1, \pm 2, \cdots$. Elements of Ω are bilateral infinite sequences $\{\cdots, x_{-1}, x_0, x_1, \cdots\}$ with $x_n \in X$. Let us denote the elements of Ω by w. If $w = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$ x_n is called the nth coordinate of w and shall be considered as a function on Ω to X. Let T be the shift transformation on Ω to Ω: the nth coordinate of Tw is equal to the $n+1$th coordinate of w. For any function g on Ω, Tg is the function defined by $Tg(w) = g(Tw)$ so that $Tx_n = x_{n+1}$ for any integer n. We shall consider two probability measures μ, ν defined on $\mathscr{F}$. For $n = 1, 2, \cdots$ let $\Omega_n = \prod_{i=1}^{n} X_i$ where $X_i = X, i = 1, 2 \cdots, n$ and $\mathscr{F}_n = \prod_{i=1}^{n} \mathscr{S}_i$ where $\mathscr{S}_i = \mathscr{S}, i = 1, 2, \cdots, n$. Then $\Omega_1 = X$ and $\mathscr{F}_1 = \mathscr{S}$. Let $\mathscr{F}_{m\,n}, m \leqq n, n = 0, \pm 1, \pm 2, \cdots$, be the σ-algebra of subsets of Ω consisting of sets of the form

$$[w = \{\cdots, x_{-1}, x_0, x_1 \cdots\} \colon (x_m, x_{m+1}, \cdots, x_n) \in E]$$

Where $E \in \mathscr{F}_{n-m+1}$. Then $\mathscr{F}_{m\,n} \subset \mathscr{F}_{m\,n+1} \subset \mathscr{F}$. Let $\mu_{m\,n}, \nu_{m\,n}$ be the contractions of μ, ν, respectively to $\mathscr{F}_{m\,n}$. If $\nu_{m\,n}$ is absolutely continuous with respect to $\mu_{m\,n}$, the derivative of $\nu_{m\,n}$ with respect to $\mu_{m\,n}$ is a function of $x_m, \cdots, x_n$ and shall be designated by $f_{m\,n}(x_m, \cdots, x_n)$. Since $f_{m\,n}(x_m, \cdots, x_n)$ is positive with ν-probability one $1/f_{m\,n}(x_m, \cdots x_n)$ is well defined with ν-probability one. We shall let the function $1/f_{m\,n}(x_m, \cdots, x_n)$ take on the value 0 when $f_{m\,n}(x_m, \cdots, x_n) \leqq 0$. Thus $1/f_{m\,n}(x_m, \cdots, x_n)$ is well defined everywhere. In fact $1/f_{m\,n}(x_m, \cdots, x_n)$ is the derivative of $\nu_{m\,n}$-continuous part of $\mu_{m\,n}$ with respect to $\nu_{m\,n}$. According to the celebrated theorem of E. S. Anderson and B. Jessen [1] and J. L. Doob ([2]), pp. 343) $1/f_{m\,n}(x_m, \cdots, x_n)$ converges with ν-probability one as $n \to \infty$. If we assume that μ, ν are stationary, i.e., μ, ν are T invariant, more precise results may be expected. A fundamental theorem of Information Theory, first proved by C. Shannon for stationary Markovian measures [5] and later generalized to any stationary measure by B. McMillan [4], may be considered as a theorem of this sort. In their theorem X is assumed to be a finite set. In this paper we shall first treat Markovian stationary measures μ, ν with X being

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