THE INVARIANCE OF SYMMETRIC FUNCTIONS OF SINGULAR VALUES

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Let $M_{m,n}$ denote the vector space of all $m \times n$ matrices over the complex numbers. A general problem that has been considered in many forms is the following: suppose $\mathfrak{A}$ is a subset (usually subspace) of $M_{m,n}$ and let f be a scalar valued function defined on $\mathfrak{A}$. *Determine the structure of the set* $\mathfrak{A}_f$ *of all linear transformations* T *that satisfy*

$$(1) \qquad f(T(A)) = f(A) \text{ for all } A \in \mathfrak{A} .$$

The most interesting choices for f are the classical invariants such as rank [3, 4, 7] determinant [1, 2, 3, 5, 10] and more general symmetric functions of the characteristic roots [6, 8]. In case $\mathfrak{A}$ is the set of n-square real skew-symmetric matrices ($m = n$) and $f(A)$ is the Hilbert norm of A then Morita [9] proved the following interesting result: $\mathfrak{A}_f$ *consists of transformations* T *of the form*

$$T(A) = U'AU \text{ for } n \neq 4 ,$$
$$T(A) = U'AU \text{ or } T(A) = U'A^{+}U \text{ for } n = 4$$

where U *is a fixed real orthogonal matrix and* A^{+} *is the matrix obtained from* A *by interchanging its* $(1, 4)$ *and* $(2, 3)$ *elements.*

Recall that the Hilbert norm of A is just the largest *singular value* of A (i.e., the largest characteristic root of the nonnegative Hermitian square root of A^*A).

In the present paper we determine $\mathfrak{A}_f$ when $\mathfrak{A}$ is all of $M_{m,n}$ and f is some particular elementary symmetric function of the squares of the singular values. We first introduce a bit of notation to make this statement precise. If $A \in M_{n,n}$ then $\lambda(A) = (\lambda_1(A), \cdots, \lambda_n(A))$ will denote the n-tuple of characteristic roots of A in some order. The rth elementary symmetric function of the numbers $\lambda(A)$ will be denoted by $E_r[\lambda(A)]$; this is, of course, the same as the sum of all r-square principal subdeterminants of A. We also denote by $\rho(A)$ the rank of A.

THEOREM. *A linear transformation* T *of the space* $M_{m,n}$ *leaves invariant the rth elementary symmetric function of the squares of the singular values of each* $A \in M_{m,n}$, *for some fixed* r, $1 < r \leqq n$, *if and only if there exist unitary matrices* U *and* V *in* $M_{m,m}$ *and* $M_{n,n}$ *respectively such that*

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