THREE SPECTRAL THEOREMS FOR A PAIR OF SINGULAR FIRST-ORDER DIFFERENTIAL EQUATIONS

BERNARD W. ROOS AND WARD C. SANGREN*

1. Preliminaries. In the regular case the classical method of obtaining eigenvalues and eigenfunctions of the equation

$$y''(x) + [\lambda - q(x)]y = 0 \quad \left[' \equiv \frac{d}{dx}\right] \tag{1}$$

under Sturmian boundary conditions involves the use of asymptotic expansions. For the singular cases of (1) when the range of x is infinite or semi-infinite instead of finite, Titchmarsh [6] has shown that such asymptotic solutions are also necessary in obtaining spectral and expansion theorems by the method of complex variables. The objective of this paper is to generalize for a particular case these types of results to the following pair of equations

$$\begin{aligned} u'(x) - [\lambda a(x) + b(x)]v(x) &= 0 \,, \\ v'(x) + [\lambda c(x) + d(x)]u(x) &= 0 \,. \end{aligned} \tag{2}$$

Interest in this system arises from a consideration of the Dirac relativistic wave equations for a particle in a central field. The equations (2) correspond in this case to the radial wave equations. Conte and Sangren [2] and the authors [3] have shown that most of the results of Titchmarsh can be generalized for (2) over the interval $(0 \leqq x < \infty)$, under the restriction $a(x) = c(x) = 1$. Also, the spectral properties of (2) for $a(x) = c(x) = 1$ over the infinite interval $(-\infty, \infty)$ have been investigated [4]. In this paper a discussion of the system (2) for $a(x) = x^{2k}$, $c(x) = x^{-2k}$ over the interval $(0, \infty)$ is presented. It is assumed throughout, that k is a nonzero integer.

Let $\phi(x, \lambda) = [\phi_1(x, \lambda), \phi_2(x, \lambda)]$ and $\theta(x, \lambda) = [\theta_1(x, \lambda), \theta_2(x, \lambda)]$ be two solutions of system (2) over the interval $a \leqq x \leqq b$, where $a > 0$ and $b < \infty$, such that $\phi_1(l, \lambda) = 1$, $\phi_2(l, \lambda) = 0$, $\theta_1(l, \lambda) = 0$, $\theta_2(l, \lambda) = 1$, where $a \leqq l \leqq b$. It can be shown that the Wronskian $W_x[\phi, \theta] = \phi_1\theta_2 - \phi_2\theta_1$ is independent of x so that since $W_l(\phi, \theta] = 1$, $\phi(x, \lambda)$ and $\theta(x, \lambda)$ are linearly independent. For the singular case it can be shown that for complex values of λ the system (2) has a solution $\psi(x, \lambda) = [\psi_1, \psi_2] = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda)$. A limit circle case is determined separately at each of the end points, 0 and ∞, by the conditions that all functions $c|\psi|^2 + a|\psi_2|^2$ are integrable, that is, belong to the class $L(0, l)$ or $L(l, \infty)$. In the limit point case, at either end, there exist only one $m(\lambda)$ and

Received January 6, 1961. Presented to the American Mathematical Society, Pasadena California, November 19, 1960.

* Present address: Computer Applications, Inc., San Diego, California.