ON A CLASSICAL THEOREM OF NOETHER IN IDEAL THEORY

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A classical result in the ideal theory of commutative rings is that an integral domain D with unit is a Dedekind domain if and only if D is noetherian, of dimension less than two, and integrally closed. [8; 275]. The statement of this theorem is due essentially to Noether [6; 53], though the present statement is a refined version of Noether's theorem. (See Cohen [1; 32] for the historical development of the theorem above.) Noether did not, in fact, require that the domain D contain a unit element. By imposing greater restrictions on the prime ideal factorization of each ideal, she showed that D must contain a unit element.

This paper considers an integral domain J with Property C: Every ideal of J may be expressed as a product of prime ideals.

In particular, it is shown that an integral domain J with property C need not contain a unit element. However, factorization of an ideal as a product of prime ideals is unique and J is noetherian, of dimension less than two, and integrally closed.[1] A domain without unit having these three properties need not have property C. If J does not contain a unit element, J is the maximal ideal of a discrete valuation ring V of rank one such that V is generated over J by the unit element e, and conversely. The structure of all such valuation rings V is known. [4; 62].

If J is an integral domain with quotient field k, then J^* will denote the subring of k generated by J and the unit element e of k. We will assume that all domains considered contain more than one element.

If D is an integral domain, not necessarily containing a unit, and if k is the quotient field of D, the definitions of fractionary ideals of D, of sums, products and quotients of fractionary ideals, and of the fractionary ideal $(u_1, u_2, \cdots, u_t)$ of D generated by finitely many elements $u_1, u_2, \cdots, u_t$ of k, are generalized in the obvious ways. In particular, D^* is a fractionary ideal of D and if $\mathscr{S}$ is the collection of all nonzero fractionary ideals of D, $\mathscr{S}$ is an abelian semigroup under multiplication with unit element D^*. A fractionary ideal F of

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[1] A domain D with quotient field k is integrally closed if D contains every element x of k with the following property: There exist elements $d_0, d_1, \cdots, d_n$ of D such that $x^{n+1} + d_n x^n + \cdots + d_1 x + d_0 = 0$.