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A NOTE ON HAUSDORFF'S SUMMATION METHODS

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If $\{a_n\}$ is a moment sequence and (Δa) is the difference matrix having base sequence $\{a_n\}$, then (Δa) is symmetric about the main diagonal if and only if the function $\alpha(x)$ such that $a_n = \int_0^1 x^n d\alpha(x)$, $n = 0, 1, 2, \cdots$, is symmetric in the sense that $\alpha(x) + \alpha(1 + x) = \alpha(1) + \alpha(0)$ except for at most countably many x in $[0, 1]$. This property is related to the "fixed points" of the matrix H, where HaH is the Hausdorff matrix determined by the moment sequence $\{a_n\}$.

In each of the papers [2], [3] and [5], there is reference to difference matrices of the form

$$(\Delta d) = \begin{bmatrix} \Delta^0 d_0 & \Delta^0 d_1 & \Delta^0 d_2 & \\ \Delta^1 d_0 & \Delta^1 d_1 & \Delta^1 d_2 & \vdots \\ \Delta^2 d_0 & \Delta^2 d_1 & \Delta^2 d_2 & \\ & \cdots & & \end{bmatrix}$$

where $\{d_n\}$ is a moment sequence, $\Delta^0 d_n = d_n$, $n = 0, 1, 2, \cdots$ and $\Delta^m d_n = \Delta^{m-1} d_n - \Delta^{m-1} d_{n+1}$, for $n = 0, 1, 2, \cdots$ and $m = 1, 2, 3, \cdots$. In [2], Garabedian and Wall discussed the importance of (Δd) having the property of being symmetric about the main diagonal, i.e. $\Delta^m d_n = \Delta^n d_m$. They also showed that if $\{d_n\}$ is a totally monotone sequence, then (Δd) is symmetric about the main diagonal if and only if the function $f(x)$ which generates $\{d_n\}$ has a certain type continued fraction expansion.

In this paper, the symmetry of (Δd) is investigated with the restriction of total monotonicity removed and a collection of necessary and sufficient conditions are given, Theorem 3, for moment sequences in general. A relation is established between the symmetry of (Δd) and the "fixed points" of the difference matrix

$$H = \begin{bmatrix} \binom{0}{0} & & \\ \binom{1}{0} & -\binom{1}{1} & \\ \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} \\ & \cdots & \end{bmatrix}. \tag{1}$$

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