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# A NOTE ON PRINCIPAL FUNCTIONS AND MULTIPLY-VALENT CANONICAL MAPPINGS

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**L. Sario has constructed principal analytic functions on planar bordered Riemann surfaces by applying the method of linear operators to certain sets of singularity functions. Weakly $\lambda$-valent principal functions result from a similar construction, starting with singularity functions having flux equal to integral multiples of $2\pi$. In fact, such $\lambda$-valent maps are characterized as integral powers of principal analytic functions already mentioned.**

L. Sario has used linear operators to establish the existence of certain canonical mappings of planar bordered Riemann surfaces $\bar{W}$ onto slit disks [4]. These mappings $F_0(z)$ and $F_1(z)$, called principal analytic functions, are formed from principal harmonic functions, themselves constructed by applying the linear operator method of [5] to systems of singularity functions defined near certain point sets of $\bar{W}$. In particular, near $\gamma$, the border of $\bar{W}$, the singularity function $s_\gamma(z)$, which is constant on $\gamma$ with flux $2\pi$ there, is chosen, while near $\zeta$, a point of the surface $W = \bar{W} - \gamma$, the singularity function $s_\zeta(z) = \log|z - \zeta|$ is selected. By exhausting the planar bordered surface $\bar{W}$, one constructs the mappings $F_0(z)$ and $F_1(z)$ of $\bar{W}$ onto a plane disk, with radial or circular slits, possibly degenerate. It is easily established that, for $i = 0, 1$, $\Delta_\gamma(\arg F_i(z))$ is $2\pi$, the flux on $\gamma$ of the singularity function $s_\gamma(z)$, and that each $F_i(z)$ has a first order zero at $z = \zeta$. These conditions are easily seen to be a consequence of the selection of the singularity functions $s_\gamma(z)$ and $s_\zeta(z)$.

In this note, we investigate the nature of "canonical" maps $F_0^\lambda(z)$ and $F_1^\lambda(z)$ which result from starting with singularity functions $s_\gamma^\lambda(z)$ near $\gamma$ and $s_\zeta^\lambda(z)$ near $\zeta$. Here, $s_\gamma^\lambda(z)$ is constant for $z \in \gamma$ with flux $\int_\gamma ds_\gamma^{\lambda*} = 2\pi\lambda$ while $s_\zeta^\lambda(z) = \lambda \log|z - \zeta|$. If an approximation process similar to that of [4] is applied, canonical maps $F_0^\lambda(z)$ and $F_1^\lambda(z)$ result. Because $\Delta_\gamma \arg F_i(z) = \int_\gamma ds_\gamma^{\lambda*} = 2\pi\lambda$, it follows that the mappings $F_0^\lambda(z)$ and $F_1^\lambda(z)$ are $\lambda$-valent, at least near $\gamma$. Also, at the point $\zeta$ of $W$, these mappings have a $\lambda$-th order zero, and hence are $\lambda$-valent near $\zeta$ as well. It is then reasonable to ask whether the functions $F_0^\lambda(z)$ and $F_1^\lambda(z)$, with radial and circular slit behavior near the ideal boundary, are *globally* $\lambda$-valent in some sense.

For a bordered Riemann surface $\bar{V}$ with two border components $\delta$ and $\gamma$, constructions similar to those of [3], starting with singularity