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BOUNDEDNESS PRINCIPLES AND FOURIER THEORY

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A systematic application of simple functional analytic techniques (boundedness principles and the Hahn-Banach theorem) is made to establish a few results in harmonic analysis.

For the circle group, results are obtained about the possible misbehaviour of conjugate functions and related multiplier transforms.

For infinite compact Abelian groups, results are obtained about the possible misbehaviour of functions or pseudomeasures f for which the Fourier transform $\hat{f}$ is $o(\rho)$, where ρ is a preassigned nonnegative function on the character group.

1. Concerning conjugate functions. Except in Remark (e) at the end of this section, all functions and distributions appearing this section are assumed to have period 2π, and may thus be regarded as functions and distributions on the circle group. The n-th Fourier coefficient of any such (integrable) function or distribution f will be denoted by $\hat{f}(n)$, n here ranging over the set Z of integers.

It is known from the work of Lusin and Tolstov (see [1], Vol. 2, pp. 95–98) that

(i) There exists an absolutely continuous function f whose conjugate function $\tilde{f}$ is essentially unbounded on every nondegenerate interval;

(ii) The function f referred to in (i) may be so chosen that the Fourier series of f and of $\tilde{f}$ are each pointwise convergent a.e..

The Lusin-Tolstov approach is constructive and we have nothing to add to it. Instead, we present an existential proof of (i). Although the proof is nonexplicit, it sheds some light on the underlying reasons for the occurrence of the phenomena concerned.

In discussing (i) we use the fact that, at least for (measurable) functions f such that $f\cdot\log^+|f|\in L^1$, the traditionally-defined conjugate function $\tilde{f}$ may be identified distributionally with $H*f$, where H is the distribution

$$\sum_{n\in Z}(-i\cdot\operatorname{sgn} n)e^{inx}\ ,$$

the series being distributionally convergent. It will be convenient to refer to H as the (periodic) Hilbert distribution.

In view of the following theorem, the crucial property of H leading to the phenomenon (i) is seen to be expressed by the formula