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ON THE UNION OF TWO STARSHAPED SETS

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Let S be a compact subset of a topological linear space. We shall say that S has the property φ if there exists a line segment R such that each triple of points x, y and z in S determines at least one point p of R (depending on x, y and z) such that at least two of the segments xp, yp and zp are in S. It is clear that if S is the union of two starshaped sets then S has the property φ, and the problem has been raised by F. A. Valentine [1] as to whether the property φ ensures that S is the union of two starshaped sets. We shall show that this is not so, in general, but we begin by giving a further constraint which ensures the result.

THEOREM. *If a compact set S, of a topological linear space, has the property φ, and, for any point q of S, the set of points of R which can be seen, via S, from q form an interval, then S is the union of two starshaped sets.*

Proof. Consider the collection of sets $\{T_q\}, q \in S$, where T_q denotes the set of points of R which can be seen, via S, from q. If every two intervals of this collection have a nonempty intersection, then it follows from Helly's Theorem that S is starshaped from a point of R. Suppose, therefore, that there exist points q_1, q_2 of S such that $T_{q_1} \cap T_{q_2} = \phi$. We partition the collection $\{T_q\}, q \in S$, into three collections $\{T_q\}_1, \{T_q\}_2, \{T_q\}_{12}$, so that T_q belongs to $\{T_q\}_1$ if T_q meets T_{q_1} but not T_{q_2}, T_q belongs to $\{T_q\}_2$ if T_q meets T_{q_2} but not T_{q_1}, T_q belongs to $\{T_q\}_{12}$ if T_q meets both T_{q_1} and T_{q_2}. If T_q, T_r are two sets of $\{T_q\}_i$ $(i = 1, 2)$ then it follows from φ applied to the points q, r and q_j $(j \neq i)$ that T_q meets T_r. If T_q, T_r are two sets of $\{T_q\}_{12}$, then, since both T_q and T_r span the gap between T_{q_1} and T_{q_2}, it follows that T_q meets T_r. Further, if T_q belongs to $\{T_q\}_{12}$, then it must meet every set of at least one of the collections $\{T_q\}_i$ $(i = 1, 2)$. For, otherwise, there exists sets T_{r_1}, T_{r_2}, belonging to $\{T_q\}_1, \{T_q\}_2$ respectively, which do not meet T_q. However, by property φ applied to r_1, r_2 and q, this implies that T_{r_1} meets T_{r_2} and hence that

$$T_{r_1} \cup T_{r_2}$$

spans the gap between T_{q_1} and T_{q_2}. But this implies that $T_{r_1} \cup T_{r_2}$ meets T_q; contradiction. We now form the collections $\{T_q\}_{12i}$ $(i = 1, 2)$ so that T_q belongs to $\{T_q\}_{12i}$ if either T_q is in the collection $\{T_q\}_i$ or T_q is in $\{T_q\}_{12}$ and meets every member of $\{T_q\}_i$. We note that