PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 3, 1968

# SUBDIRECT DECOMPOSITIONS OF LATTICES OF WIDTH TWO

OSCAR TIVIS NELSON, JR.

**The class of nontrivial distributive lattices is the class of subdirect products of two-element chains. Lattices of width one are distributive and hence are subdirect products of two element chains. Below it is shown that lattices of width two are subdirect products of two element chains and nonmodular lattices of order five ($N_5$). (width = greatest number of pairwise incomparable elements.)**

The statement follows from several lemmas. Throughout we shall assume that $a$, $b$ are arbitrary noncomparable elements of a lattice $L$ of width two.

LEMMA 1. $x \cdot (a + b) + y \cdot (a + b) = (x + y) \cdot (a + b)$ *and*

$$(x + a \cdot b) \cdot (y + a \cdot b) = x \cdot y + a \cdot b$$

*for any* $x, y \in L$.

*Proof.* In any lattice

$$x \cdot (a + b) + y \cdot (a + b) \leqq (x + y) \cdot (a + b) . \tag{1}$$

Trivially, if $x$ and $y$ are related, the identity holds. Thus, assume that $x$ and $y$ are unrelated. There are three possibilities:

(i) Suppose $x \leqq a$ and $y \leqq b$. Then

$$x \cdot (a + b) + y \cdot (a + b) = x + y = (x + y) \cdot (a + b) .$$

(ii) In case $a \leqq x$ and $b \leqq y, a + b \leqq x + y$. If $a + b \leqq x$ or $y$, it is easy to verify that the identity holds. If $a + b \nleqq x$ or $y$, then $x$ or $y \leqq a + b$. Suppose $x \leqq a + b$. Then

$$(x + y) \cdot (a + b) = a + b \leqq x + y \cdot (a + b) = x \cdot (a + b) + y \cdot (a + b) .$$

This relation and (1) yield the equality.

(iii) Now suppose $a \leqq x$ and $y \leqq b$. $b \leqq x$ implies that $x$ and $y$ are comparable while $x \leqq b$ implies that $a$ and $b$ are comparable. Thus, $x$ and $b$ are unrelated. Since $L$ is of width two, $a + y$ is related to either $x$ or $b$, $a + y \leqq x$ and $a + y \leqq b$ imply that $y \leqq x$ and $a \leqq b$ respectively. Thus, either $x \leqq a + y$ or $b \leqq a + y$. In case $x \leqq a + y, x \leqq a + y \leqq a + b$ and $y \leqq b \leqq a + b$. Hence

$$x \cdot (a + b) + y \cdot (a + b) = x + y = (x + y)(a + b) .$$