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EXTENSIONS OF OPIAL'S INEQUALITY

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In this paper certain inequalities involving integrals of powers of a function and of its derivative are proved. The prototype of such inequalities is Opial's Inequality which states that $2\int_0^X |yy'|\,dx \leqq X\int_0^X y'^2 dx$ whenever y is absolutely continuous on $[0, X]$ with $y(0) = 0$. The extensions dealt with here are all integral inequalities of the form

$$\int_a^b s\,|\,y\,|^p\,|\,y'\,|^q\,dx \leqq K(p, q)\int_a^b r\,|\,y'\,|^{p+q}\,dx\ ,$$

(or with $\leqq$ replaced by $\geqq$), where r, s are nonnegative, measurable functions on $I = [a, b]$, and y is absolutely continuous on I with either $y(a) = 0$, or $y(b) = 0$, or both. In some cases y may be complex-valued, while in other cases y' must not change sign on I. The inequality (as stated) is obtained in case $pq > 0$ and either $p + q \geqq 1$ or $p + q < 0$, while the opposite inequality is obtained in case $p < 0, q \geqq 1, p + q < 0$, or $p > 0, p + q < 0$. In all cases, necessary and sufficient conditions are obtained for equality to hold.

1. In a recent paper [11], G. S. Yang proved the following generalization of an inequality of Z. Opial [7]:

If y *is absolutely continuous on* $[a, X]$ *with* $y(a) = 0$, *and if* $p, q \geqq 1$, *then*

$$\int_a^X |\,y\,|^p\,|\,y'\,|^q\,dx \leqq \frac{q}{p+q}(X-a)^p\int_a^X |\,y'\,|^{p+q}\,dx\ . \tag{1}$$

Yang's proof is actually valid for $p \geqq 0, q \geqq 1$. For $p = q = 1, a = 0$, (1) is Opial's result. (See also Olech [6], Beesack [1], Levinson [4], Mallows [5], and Pederson [8] for successively simpler proofs of Opial's inequality; as well as Redheffer [9] for other generalizations of this inequality.) The case $q = 1, p$ a positive integer, was proved by Hua [3], and the result for $q = 1, p \geqq 0$ is included in a generalization of Calvert [2]; a short, direct proof of the latter case was also given by Wong [10]. If $q = 1$ the inequality (1) is sharp, but it is not sharp for $q > 1$.

2. The purpose of this paper is to obtain sharp generalizations of (1), and to consider other values of the parameters p, q; the method of proof is a modification of that of Yang [11]. To this end, *we suppose first that* y *is absolutely continuous on* $[a, X]$, *where* $-\infty \leqq a < X \leqq \infty$, *and that* y' *does not change sign on* (a, X), *so that*