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FUNCTION-THEORETIC DEGENERACY CRITERIA FOR RIEMANNIAN MANIFOLDS

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The modulus of a relatively compact set with border consisting of at least two components is a measure of its magnitude with regard to harmonic functions. A divergent modular sum associated with difference sets obtained from an exhaustion of a Riemannian manifold is characteristic of parabolicity. The existence of a divergent minimum modular sum implies that the manifold carries no nonconstant harmonic functions with finite Dirichlet integral.

The modular criteria presented in this paper are generalizations of criteria established for Riemann surfaces by Noshiro [6] and Sario [8], [9]. In the two-dimensional case function-theoretic degeneracy is completely determined by the conformal structure, whereas it has been shown by Nakai and Sario [5] that the type of a Riemannian manifold varies when the metric is replaced by a conformally equivalent one. The significance of our result stems from this fact.

For the completeness of the presentation it is shown that the various characterizations of parabolicity due to Ahlfors, Brelot, Nevanlinna and Ohtsuka remain equivalent in higher dimensions. This overlaps with the work of Itô [2] and Loeb [3] in different settings. A new proof for Riemannian manifolds of the relation, $O_{HD} = O_{HBD}$, established in [10] is also given.

1. Let R be an orientable noncompact Riemannian manifold. Let $A \subset R$ and denote by $H(A)$ the class of harmonic functions on A and by $H^c(A)$ the functions in $H(A)$ with continuous extensions to $\bar{A}$. For every parametric region V there exists a Green's function q_x^V with the property $-h(x) = \int_{\partial V} h^* dq_x^V$ for every $h \in H^c(V)$. It is well-known that the sheaf of harmonic functions over R satisfies the axioms of a harmonic space and we shall use the standard facts of the theory freely. These together with Green's formulas will serve as our main tools.

2. Consider a fixed parametric region $V \subset R$ and a point $a \in V$. Let F consist of the constant $+\infty$ and of all nonnegative superharmonic functions s on R such that $s \mid V - q_a^V$ is bounded. Clearly F is a Perron family on $R - a$ and its lower envelope is either $+\infty$ or a function g_a harmonic on $R - a$. If the function g_a exists it is called the *Green's function* for R.