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PURITY AND ALGEBRAIC COMPACTNESS FOR MODULES

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A submodule A of a left module B (over an associative ring with 1) is pure if for any right module F, the natural homomorphism $F \otimes A \to F \otimes B$ is injective. A module C is pure-injective if for any module B and pure submodule A, any homomorphism from A to C extends to B. The theory of this notion of purity and the corresponding class of pure-injectives is developed in this paper, with special attention to modules over commutative Noetherian rings and Prüfer rings. It is proved that pure-injective envelopes exist and the pure-injective modules are characterized as retracts of topologically compact modules. For this reason, the pure-injective modules are also called algebraically compact. For modules over Prüfer rings, certain simplifications occur, due essentially to the fact that a finitely presented module is a summand of a direct sum of cyclic modules. Complete sets of invariants are obtained for certain classes of algebraically compact modules over certain Prüfer rings.

This work is an extension of the theory of algebraically compact Abelian groups due to Kaplansky [8], Łoś [10], Maranda [12] and others. Our notion of algebraic compactness agrees with that introduced for general algebraic systems by Mycielski [14] and studied by Weglorz [20]. Related topics in module theory have been discussed by Fuchs [5], Fieldhouse [4], and Stenström [17], and there is some overlap between these papers and the results in the first and third sections below.

In the first section below, we discuss several notions of purity and identify the pure-projective modules in the cases which are of interest to us. In the second section we study finitely presented modules over a Prüfer ring, and use them to give a characterization of Prüfer rings. In the next two sections we consider the general theory of algebraically compact modules over arbitrary associative rings with 1. Sections five and six are devoted to the special results obtainable when the rings are commutative Noetherian rings and Prüfer rings respectively. All rings in this paper are associative with 1. We adopt the notation of [11] in using the arrow $\rightarrowtail$ for a monomorphism and $\twoheadrightarrow$ for an epimorphism.

1. Purity and pure-projectives.

Let S be a class of left R-modules. We say a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ is S-pure if for