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ON THE BOUNDARY CORRESPONDENCE OF QUASICONFORMAL MAPPINGS OF DOMAINS BOUNDED BY QUASICIRCLES

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A Jordan curve $\mathscr{L}$ is a quasicircle if there exists a constant $C(1 \leqq C < \infty)$ such that the cross ratio (z_1, z_2, z_3, z_4) of any four points $z_1, \cdots, z_4$ in order on $\mathscr{L}$ satisfies $|(z_1, z_2, z_3, z_4)| \leqq C$. It is shown that the boundary correspondence f induced by a quasiconformal homeomorphism of two Jordan domains bounded by quasicircles satisfies $|(w_1, w_2, w_3, w_4)| \leqq A\,|(z_1, z_2, z_3, z_4)|^\alpha$, $(A \geqq 1, 0 < \alpha \leqq 1)$ where $w_k = f(z_k)$ and the points are in order on the boundary. A converse to this result is proved and estimates are computed in each direction.

DEFINITION 1. A Jordan curve $\mathscr{L}$ is called a C-quasicircle $(1 \leqq C < \infty)$ if

$$|(z_1, z_2, z_3, z_4)| < C \tag{1}$$

for any four points $z_1, \cdots, z_4$ in order on $\mathscr{L}$.

Since cross ratios are invariant under Möbius transformations we will assume without loss of generality from now on that $\mathscr{L}$ contains ∞ whence we may assume $z_4 = \infty$ so that (1) becomes $|(z_1, z_2, z_3)| \leqq C$ or more graphically,

$$|z_1 - z_2| \leqq C\,|z_1 - z_3|\ . \tag{1'}$$

This definition and the observation of the importance of these curves to the theory of quasiconformal mappings are due to Ahlfors [1].

DEFINITION 2. (a) We will say that a homeomorphism f of a C-quasicircle $\mathscr{L}$ onto a Jordan curve $\mathscr{L}'$ is (A, α)-upper quasisymmetric, $A \geqq 1$, $0 < \alpha \leqq 1$ $(A, \alpha$ constants) if

$$|(w_1, w_2, w_3, w_4)| \leqq A\,|(z_1, z_2, z_3, z_4)|^\alpha \tag{2}$$

for any four points z_1, z_2, z_3, z_4 in order on $\mathscr{L}$ and where $w_k = f(z_k)$, $k = 1, 2, 3, 4$.

(b) If we replace (2) by

$$B\,|(z_1, z_2, z_3, z_4)|^\beta \leqq |\,w_1, w_2, w_3, w_4)\,| \tag{3}$$

under the same conditions and for some constants B and β, $\beta \geqq 1$, $0 < B \leqq 1$ then we will call $f\,(B, \alpha)$-lower quasisymmetric.