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QUOTIENTS OF THE SPACE OF IRRATIONALS

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It is proved that every metric space which is a continuous image of the irrationals is also a quotient of the irrationals.

In this paper we are concerned with the class $\mathscr{A}$ of all those metric spaces which are continuous images of complete separable metric spaces. The members of $\mathscr{A}$ are generally called "(absolutely) analytic sets" or "A-sets" [9] or "Souslin spaces" [5], and are known to be precisely those metric spaces which are either empty or are continuous images of the space $\boldsymbol{P}$ of irrational numbers[1]. Suppose, then, that $Y \in \mathscr{A}$ and Y is nonempty. There exists a continuous surjection $f: \boldsymbol{P} \rightarrow Y$; how "nice" can f be taken to be? In general, f cannot be one-to-one (or Y would have to be absolutely Borel; see [9 p. 487]); nor can f be open or closed (as Y would then be an absolute G_δ; see 3.4 and 3.5 below). However, we shall see that f can always be chosen to be a quotient map. More precisely, we prove the following theorem.

THEOREM 1.1. *Every metrizable space Y which is a continuous image of $\boldsymbol{P}$ is also a quotient of $\boldsymbol{P}$ (under a different map, in general).*

Since the space $\boldsymbol{Q}$ of rational numbers is in $\mathscr{A}$, Theorem 1.1 has the following rather striking consequence:

COROLLARY 1.2. *The space of rationals is a quotient of the space of irrationals.*

The proof of Theorem 1.1 is given in the next section, after which we mention some generalizations, related results and open questions.

2. Proof of Theorem 1.1.

The proof depends on the following characterization of $\boldsymbol{P}$, due to Hausdorff [7].

LEMMA 2.1. *A space X is homeomorphic to $\boldsymbol{P}$ if and only if X is a separable metrizable 0-dimensional absolute G_δ such that no non-empty open subset of X is compact.*

Now let Y be a metrizable space which is a continuous image of $\boldsymbol{P}$, and let us show that Y is a quotient of $\boldsymbol{P}$. Since $\boldsymbol{P}$ is separable,

[1] While the reals are usually denoted by $\boldsymbol{R}$, and the rationals by $\boldsymbol{Q}$ (for quotient), there seems to be no standard symbol for the irrationals. The natural choice would be $\boldsymbol{I}$, but that has been pre-emptied by the unit interval. We therefore propose $\boldsymbol{P}$, which permits the equation $\boldsymbol{P} \cup \boldsymbol{Q} = \boldsymbol{R}$, and which may be thought of as standing for psychotic (=irrational).