PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 3, 1969

GAMES WITH UNIQUE SOLUTIONS THAT ARE NONCONVEX

W. F. LUCAS

In 1944 von Neumann and Morgenstern introduced a theory of solutions (stable sets) for n-person games in characteristic function form. This paper describes an eight-person game in their model which has a unique solution that is nonconvex. Former results in solution theory had not indicated that the set of all solutions for a game should be of this nature.

First, the essential definitions for an n-person game will be stated. Then, a particular eight-person game is described. Finally, there is a brief discussion on how to construct additional games with unique and nonconvex solutions.

The author [2] has subsequently used some variations of the techniques described in this paper to find a ten-person game which has no solution; thus providing a counterexample to the conjecture that every n-person game has a solution in the sense of von Neumann and Morgenstern.

2. Definitions. An *n-person game* is a pair (N, v) where $N = \{1, 2, \cdots, n\}$ and v is a real valued characteristic function on 2^N, that is, v assigns the real number $v(S)$ to each subset S of N and $v(\varphi) = 0$. The set of all *imputations* is

$$A = \left\{x\colon \sum_{i\in N} x_i = v(N) \text{ and } x_i \geqq v(\{i\}) \text{ for all } i \in N\right\}$$

where $x = (x_1, x_2, \cdots, x_n)$ is a vector with real components. If x and y are in A and S is a nonempty subset of N, then $x \operatorname{dom}_S y$ means $\sum_{i\in S} x_i \leqq v(S)$ and $x_i > y_i$ for all $i \in S$. For $B \subset A$ let $\operatorname{Dom}_S B = \{y \in A\colon$ there exists $x \in B$ such that $x \operatorname{dom}_S y\}$ and let $\operatorname{Dom} B = \bigcup_{S\subset N} \operatorname{Dom}_S B$. A subset K of A is a *solution* if $K \cap \operatorname{Dom} K = \varphi$ and $K \cup \operatorname{Dom} K = A$. The *core* of a game is

$$C = \left\{x \in A\colon \sum_{i\in S} x_i \geqq v(S) \text{ for all } S \subset N\right\}.$$

The core consists of those imputations which are maximal with respect to all of the relations dom_S, and hence it is contained in every solution.

3. Example. Consider the game (N, v) where $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and where v is given by: $v(N) = 4$, $v(\{1, 4, 6, 7\}) = 2$, $v(\{1, 2\}) =$