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FINITE GROUPS WITH SMALL CHARACTER DEGREES AND LARGE PRIME DIVISORS II

I. M. ISAACS AND D. S. PASSMAN

In a previous paper one of the authors considered groups G with r. b. n (representation bound n) and $n < p^2$ for some prime p. Here we continue this study. We first offer a new proof of the fact that if $n = p - 1$ then G has a normal Sylow p-subgroup. Next we show that if $n = p^{3/2}$ then $p^2 \nmid |G/O_p(G)|$. Finally we consider $n = 2p - 1$ and with the help of the modular theory we obtain a fairly precise description of the structure of G. In particular we show that its composition factors are either p-solvable or isomorphic to $PSL(2, p)$, $PSL(2, p - 1)$ for p a Fermat prime or $PSL(2, p + 1)$ for p a Mersenne prime.

Now the irreducible characters of $PSL(2, \mathrm{p})$ have degrees (see [10] p. 128) $1, p, p \pm 1$, $(p \pm 1)/2$ for p odd and those of $PSL(2, 2^a)$ have degrees (see [10] p. 134) $1, 2^a, 2^a \pm 1$. Thus for $p > 2$ the linear groups of the preceding paragraph do in fact have r.b. $(2p - 1)$.

The notation here is standard. In addition, if χ is a character of G we let det χ denote the linear character which is the determinant of the representation associated with χ. Also $n_p(G)$ denotes the number of Sylow p-subgroups of G.

LEMMA 1. *Let G be a group with* r.b.n. *and let $N \neq G$* be a *subgroup. Suppose $G = \bigcup_{i=0}^{t} Nx_iN$ is the (N, N)-double coset decomposition of G with $x_0 = 1$. Set $a_i = |Nx_iN|/|N| = [N: N \cap N^{x_i}]$. Then $n \geqq (a_1 + a_2 + \cdots + a_t)/t$.*

Proof. Let $\theta = (1_N)^G$ be the character of the permutation representation of G on the cosets of N. Then $\theta(1) = [G: N]$, $[\theta, 1_G] = 1$ and $\|\theta\|^2 = 1 + t$. Since $[\theta, 1_G] = 1$ we can write

$$\theta = 1_G + b_1\chi_1 + \cdots + b_s\chi_s$$

where the χ_i are distinct nonprincipal irreducible characters of G. Thus since G has r.b.n we have

$$\begin{aligned} 1 + nt &= 1 + n(\|\theta\|^2 - 1) = 1 + n(b_1^2 + \cdots + b_s^2) \\ &\geqq 1 + n(b_1 + \cdots + b_s) \geqq 1 + b_1\chi_1(1) + \cdots + b_s\chi_s(1) \\ &= \theta(1) = 1 + (a_1 + a_2 + \cdots + a_t) \end{aligned}$$

and the result follows.