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POLYNOMIALS IN LINEAR RELATIONS II

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The study of linear relations is continued in the setting of the theory of locally convex linear topological spaces. The investigation is limited to the polynomials in one fixed closed linear relation. Conditions both on the relation and on the locally convex space are discussed that are sufficient or necessary and sufficient for all the polynomials in this relation to be also closed.

The reader is referred to [1] for full details of the algebraic properties of linear relations, and to [4] for a summary. Since we are concerned here with a special class of linear relations in locally convex spaces, we present a compendium of definitions tailored to this case.

Let X be a locally convex space equipped with the Mackey-Arens topology. A *linear relation* in X is a linear subspace of $X \oplus X$. This concept generalizes that of an operator on X. If T is a linear relation in X, the definitions of the domain and range of $T, D(T)$ and $R(T)$ respectively, are obvious.

If S and T are linear relations in X,

$$S + T = \{(x, y + z) : (x, y) \in S \ (x, z) \in T\}$$
$$S \cdot T = \{(x, z) : \exists y \in X \ni (x, y) \in T, (y, z) \in S\}$$

are linear relations in X. If λ is a complex number, we may consider λ as the linear relation $\{(x, \lambda x) : x \in X\}$ in X. We write λT for $\lambda \cdot T$.

Combining the three operations defined above, we can arrive at a well-defined notion of a polynomial in a linear relation. To avoid the complications treated in [1], we shall assume that the coefficient of the highest power of the linear relation is always nonzero.

Finally a linear relation is closed if it is a closed subspace of $X \oplus X$.

2. Polyclosed relations.

If T is a closed linear relation, and if for every polynomial P, $P(T)$ is a closed linear relation, then we shall say that T is *polyclosed*. There exist closed linear relations in Hilbert space that are not *polyclosed*. This is demonstrated in the following example, originally due to Arens.

EXAMPLE 2.1. Let l^2 be the Hilbert space of square summable sequences of complex numbers, and let $X = l^2 \oplus l^2$. Let A be the operator on l^2 such that $A(x_1, x_2, \cdots) = (x_1/1, x_2/2, x_3/3, \cdots)$. The domain of $A = l^2$, A is continuous, and hence A is a closed operator.