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THE ASYMPTOTIC BEHAVIOR OF THE KLEIN-GORDON EQUATION WITH EXTERNAL POTENTIAL, II

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Let $U_0(t)$ and $U(t)$ be the one-parameter groups governing the time development of solutions of the Klein-Gordon equation, $\Box\varphi = m^2\varphi$, and the perturbed equation, $\Box\varphi = m^2\varphi + V(\vec{x})\varphi$, respectively. In a previous work the author obtained sufficient conditions on the potential $V(\vec{x})$ which guaranteed the existence of the wave operators, $W_\pm := s - \lim U(-t)U_0(t)$ as $t \to \pm\infty$. Here it is shown that if, in addition, the associated (Schrödinger) wave operators, $W_\pm^S := s - \lim e^{i(m^2I+V-\Delta)t}e^{-i(m^2I-\Delta)t}$ as $t \to \infty$, are complete and the Invariance Theorem is valid then the $W_\pm$ are also complete and are isometries. Finally, these results are used to show that the scattering operator, $W_+^{-1}W_-$, is unitarily implemented in Fock space.

The similarity between the wave operators $W_\pm$ and $W_\pm^S$ observed in [1] as far as their existence theories are concerned, is clearly reaffirmed in their completeness theories. Indeed, the proof of the above results is based on the development of an explicit relationship between these wave operators. Connections of this sort were observed by Birman [3, p. 114, § 5] for abstract differential equations of the form $\varphi_{tt} + A\varphi = 0$. Sufficient conditions for such a relationship in this more general framework were obtained by Kato [4, §§ 9, 10] and used to study both potential and obstacle scattering for the wave equation [4, § 11].

In this investigation of the Klein-Gordon equation the argument will be directed so as to take best advantage of the above general results of Kato. However some generalizations will be necessary in order to establish the cited results on the Lorentz-invariant as well as the finite-energy solution spaces of the Klein-Gordon equation. Because a specific equation is being considered some simplification of Kato's arguments will also be possible.

1. Preliminaries.

In this section the concepts discussed above are given precise definitions. Some related results which are directly used in the proofs of the main theorems are also included in summarized form.

Suppose Δ is the Laplacian in three dimensions and A^2 is the self-adjoint realization of $m^2I - \Delta$ on $L^2(E^3)$. Throughout this paper V is taken to be a real-valued function of three (space) variables and in